Complementary Cycles in Almost Regular Multipartite Tournaments *, †

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Abstract: A c-partite or multipartite tournament is an orientation of a complete c-partite graph. A digraph D is cycle complementary if there exist two vertex-disjoint directed cycles C and C' such that $V(D) = V(C) \cup V(C')$. The global irregularity of a digraph D is defined by

$$i_g(D) = \max\{\max(d^+(x), d^-(x)) - \min(d^+(y), d^-(y)) | x, y \in V(D)\}.$$

If $i_g(D) = 0$, then D is regular, and if $i_g(D) \leq 1$, then D is almost regular. We prove in this paper that every almost regular c-partite tournament with $c \geq 3$ such that all partite sets have the same cardinality $r \geq 4$ contain two complementary directed cycles of length 3 and $|V(D)| - 3$.

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1 Introduction

We will assume that the reader is familiar with the standard terminology on directed graphs. In this paper all digraphs are finite without loops or multiple arcs. Let $V(D)$ and $E(D)$ denote the vertex set and arc set of a digraph D, respectively. For a subset X of $V(D)$, we use $D[X]$ to denote the subdigraph induced by X. If xy is an arc of a digraph D, then we write $x \rightarrow y$ and say x *dominates* y. If X and Y are two disjoint subsets of $V(D)$ or subdigraphs of D such that every vertex of X dominates every vertex of Y, then X *dominates* Y, denoted by $X \rightarrow Y$. We write $X \Rightarrow Y$ if there is no arc from Y to X.

The *out-neighborhood* $N_D^+(x) = N^+(x)$ of a vertex x is the set of vertices dominated by x, and the *in-neighborhood* $N_D^-(x) = N^-(x)$ is the set of vertices dominating x. The numbers $d_D^+(x) = d^+(x) = |N^+(x)|$ and $d_D^-(x) = d^-(x) = |N^-(x)|$ are the *outdegree* and *indegree* of x, respectively. The *minimum outdegree* and the *minimum indegree* of D are denoted by $\delta^+(D) = \delta^+$ and $\delta^-(D) = \delta^-$ and $\delta(D) = \delta = \min\{\delta^+, \delta^-\}$. The *local irregularity* is defined by $i_l(D) = \max |d^+(x) - d^-(x)|$ over all vertices x of D and the *global irregularity* is defined by $i_g(D) = \max\{\max(d^+(x), d^-(x)) - \min(d^+(y), d^-(y)) | x, y \in V(D)\}$. Clearly, $i_l(D) \leq i_g(D)$. If $i_g(D) = 0$, then D is *regular* and if $i_g(D) \leq 1$, then D is *almost regular.* If $i_l(D) \leq 1$, then D is *locally almost regular.*

A digraph D is *strong* if, for each pair of vertices u and v, there is a path from u to v in D. A digraph D with at least $k+1$ vertices is k-connected if for any set A of at most $k-1$ vertices, the subdigraph $D - A$ obtained by deleting A is strong. The *connectivity* of D, denoted by $\kappa(D)$, is defined to be the largest value of k such that D is k-connected. A set S of vertices of a digraph D is a *separating set* if $D - S$ is not strong.

A *cycle-factor* is a spanning subdigraph consisting of vertex-disjoint cycles. A cycle-factor with the minimum number of cycles is called a *minimal cycle-factor.* If x is a vertex of a cycle C, then the *predecessor* and the *successor* of x on C are denoted by x^- and x^+, respectively. If we replace every arc xy of D by yx, then we call the resulting digraph, denoted by D^{-1}, the *converse digraph* of D.

A *c-partite* or *multipartite tournament* is an orientation of a complete c-partite graph. Let $V_1, V_2, \cdots, V_c$ be the partite sets of the c-partite tournament D. If the vertex x of D belongs to the partite set V_i, then we define $V(x) = V_i$.

A digraph D is *cycle complementary* if there exist two vertex-disjoint cycles C and C' such that $V(D) = V(C) \cup V(C')$. The problem of complementary cycles in tournaments was almost completely solved by Reid [6] in 1985 and by Song [7] in 1993. The authors proved that every 2-connected tournament D on at least 8 vertices has complementary cycles of length t and $|V(D)| - t$ for all $t \in \{3, 4, \cdots, |V(D)| - 3\}$. Later, Guo and Volkmann [2, 3] extended this result to locally semicomplete digraphs. For c-partite tournaments, there exist the following two conjectures.

Conjecture 1.1 *(Yeo [17] 1999). A regular c-partite tournament D with* $c \geq 4$ *and* $|V(D)| \geq 8$ *has a pair of vertex-disjoint cycles of length t and* $|V(D)| - t$ *for all* $t \in \{3, 4, \cdots, |V(D)| - 3\}$.

Conjecture 1.2 *(Volkmann [9] 2002). Let D be a multipartite tournament. If* $\kappa(D) \geq \alpha(D) + 1$*, then D is cycle complementary, unless D is a member of a finite family of multipartite tournaments.*

In 2004 and 2005, Volkmann [10, 12] confirmed the first conjecture for $t = 4$ and $t = 3$, unless D is a regular 4-partite tournament with two vertices in each partite set. In 2009, He, Korneffel, Meierling, Volkmann and Winzen [5] showed that Conjecture 1.1 is valid for $t = 5$ and $|V(D)| \geq 10$. For more information on complementary cycles in multipartite tournaments, we refer the reader to the survey articles [9, 13] by Volkmann.

As a supplement to the results in [11, 12], we will prove in this paper that every almost regular c-partite tournament with $c \geq 3$ such that all partite sets have the same cardinality $r \geq 4$ contain two complementary cycles of length 3 and $|V(D)| - 3$.

2 Preliminary results

The following results play an important role in the proof of our main theorem (Theorem 3.1 below).

Lemma 2.1 *(Yeo [16] 1998). If D is a multipartite tournament, then*

$$\kappa(D) \geq \frac{|V(D)| - 2i_l(D) - \alpha(D)}{3}.$$

Lemma 2.2 *(Bondy [1] 1976). Each strong c-partite tournament with $c \geq 3$ contains an m-cycle for each $m \in \{3, 4, \cdots, c\}$.*

Lemma 2.3 *(Yeo [18] 1999). Let D be a c-partite tournament with the partite sets $V_1, V_2, \cdots, V_c$ such that $k = |V_1| \leq |V_2| \leq \cdots \leq |V_c| = k + i$ for some $i \in \{0, 1\}$. If*

$$i_l(D) \leq \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 2}{2},$$

then D is Hamiltonian.

Lemma 2.4 (Volkmann [11] 2004). Let D be a regular 3-partite tournament such that $|V(D)| \geq 12$. Then D contains two complementary cycles of length 3 and $|V(D)| - 3$.

Lemma 2.5 (Volkmann [12] 2005). Let D be a regular c-partite tournament with $c \geq 4$ and $|V(D)| \geq 12$. Then D contains two complementary cycles of length 3 and $|V(D)| - 3$.

Lemma 2.6 (Tewes, Volkmann, Yeo [8] 2002). If D is a multipartite tournament, then

$$\delta(D) \geq \frac{|V(D)| - i_l(D) - \alpha(D)}{2} \geq \frac{|V(D)| - i_g(D) - \alpha(D)}{2}.$$

Lemma 2.7 (Yeo [15] 1997). Let D be a $(\lfloor q/2 \rfloor + 1)$-connected multipartite tournament such that $\alpha(D) \leq q$. If D has a cycle-factor, then D is Hamiltonian.

Lemma 2.8 *(Yeo [18] 1999 and Gutin, Yeo [4] 2000). A digraph D has no cycle-factor if and only if its vertex set $V(D)$ can be partitioned into four subsets Y, Z, R_1, and R_2 such that*

$$R_1 \Rightarrow Y,\ (R_1 \cup Y) \Rightarrow R_2 \ \text{ and } \ |Y| > |Z|, \tag{1}$$

where Y is an independent set.

Lemma 2.9 *(Yeo [15] 1997). Let D be a multipartite tournament having a cycle-factor but no Hamiltonian cycle. Then there exists a partite set V^* of D and an indexing $C_1, C_2, \cdots, C_t$ of the cycles of some minimal cycle-factor of D such that for all arcs yx from C_j to C_1 for $2 \leq j \leq t$, it holds that $\{y^+, x^-\} \subseteq V^*$.*

Lemma 2.10 *(Volkmann, Winzen [14] 2004). Let $V_1, V_2, \cdots, V_c$ be the partite sets of a c-partite tournament D with no cycle-factor such that $|V_1| \leq |V_2| \leq \cdots \leq |V_c|$. According to Lemma 2.8, the vertex set $V(D)$ can be partitioned into subsets Y, Z, R_1 and R_2 satisfying (1) such that $|Z| + k + 1 \leq |Y| \leq |V_c| - t$ with integers $k, t \geq 0$. Let V_i be the partite set with the property that $Y \subseteq V_i$. If $Q = V(D) - Z - V_i$, $Q_1 = Q \cap R_1$ and $Q_2 = Q \cap R_2$, then*

$$i_l(D) \geq |V(D)| - 3|V_c| + 2t + 2k + 2 \text{ and}$$

$$i_g(D) \geq \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 3k + 3}{2},$$

if $Q_1 = \emptyset$ or $Q_2 = \emptyset$ case part (1) of lemma and

$$i_g(D) \geq i_l(D) \geq \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 3k + 3 + t}{2},$$

if $Q_1 \neq \emptyset$ and $Q_2 \neq \emptyset$ case part (2) of lemma.

3 Main Result

Theorem 3.1 *If D is an almost regular c-partite tournament with $c \geq 3$ such that all partite sets have the same cardinality $r \geq 4$, then D contains two complementary cycles of length 3 and $|V(D)| - 3$.*

Proof. First of all, notice that the condition that all partite sets have the same cardinality shows that $i_l(D) = i_g(D)$. Let $V_1, V_2, \cdots, V_c$ be the partite sets of the almost regular c-partite tournament D such that $|V_1| = |V_2| = \cdots = |V_c| = r \geq 4$. If $|V(D)| - r$ is even, then it is easy to see that D is regular. Thus, if $c = 3$, then D is a regular 3-partite tournament, and by Lemma 2.4, D contains two complementary cycles of length 3 and $|V(D)| - 3$.

Assume next that $c \geq 4$. According to Lemma 2.1, we have

$$\kappa(D) \geq \frac{|V(D)| - 2i_l(D) - \alpha(D)}{3}. \quad (2)$$

Inequality (2) shows that D is strongly connected, and Lemma 2.2 implies that there exists a 3-cycle C_3 in D. If we define in the c-partite tournament H by $H = D - V(C_3)$, then $i_l(H) \leq i_g(H) \leq 4$ and $|V(H)| = cr - 3$. If $V_1', V_2', \cdots, V_c'$ are the partite sets of H such that $|V_1'| \leq |V_2'| \leq \cdots \leq |V_c'|$, then $|V_1'| = r - 1$ and $|V_c'| = r$. With exception of the cases that $c = 4$ and $r = 4, 5, 6, 7, 8$ or $c = 5$ and $r = 4$, this leads to

$$i_l(H) \leq 4 \leq \frac{|V(H)| - |V_{c-1}'| - 2|V_c'| + 2}{2}.$$

Applying Lemma 2.3, we conclude that H has a Hamiltonian cycle C, and we obtain the desired result that $V(D) = V(C_3) \cup V(C)$. Since D is a regular c-partite tournament for $c = 4$ and $r = 4, 6, 8$ and $c = 5$ and $r = 4$, by Lemma 2.5, D contains two complementary cycles of length 3 and $|V(D)| - 3$. Thus, the remaining cases are $c = 4$ and $r = 5$ or $r = 7$.

Case 1. Assume that $c = 4$ and $r = 7$. In that case $\alpha(D) = 7$ and $10 \leq d_D^+(x), d_D^-(x) \leq 11$ for each $x \in V(D)$. Lemma 2.1 implies that $\kappa(H) \geq 4$ and Lemma 2.6 yields $\delta(H) \geq 7$.

Assume that H has a cycle-factor. Applying Lemma 2.7 with $q = 7$, we deduce that H has a Hamiltonian cycle C and so $V(D) = V(C_3) \cup V(C)$.

Next assume that H has no cycle-factor. Then, with respect to Lemma 2.8, the vertex set $V(H)$ can be partitioned into subsets Y, Z, R_1, R_2 such that $R_1 \Rightarrow Y$, $(R_1 \cup Y) \Rightarrow R_2$, $|Y| > |Z|$, and Y is an independent set. Since $\kappa(H) \geq 4$ and $\alpha(H) = 7$, we see that $4 \leq |Z| < |Y| \leq 7$. Let, WLOG, $|R_1| \leq |R_2|$. If $|R_1| > |R_2|$, then we consider D^{-1} instead of D. In reference to Lemma 2.10, $Q_1, Q_2 \neq \emptyset$ (in particular, $R_1, R_2 \neq \emptyset$) and that by giving appropriate values to k and t, we get $i_l(H) \geq 5$, a contradiction, except for the case $|Z| = 6$ and $|Y| = 7$. Therefore, let $|Z| = 6$ and $|Y| = 7$. If $R_1 = \emptyset$, then we obtain the contradiction $d_D^+(y) \geq 12$ for each $y \in Y$. So assume next that $R_1 \neq \emptyset$.

Let V_1', V_2', V_3', V_4' be the partite sets of H such that $|V_1'| = |V_2'| = |V_3'| = 6$ and $|V_4'| = 7$. Then $8 \leq d_H^+(x), d_H^-(x) \leq 11$ for $x \in V_1' \cup V_2' \cup V_3'$. Since $|Y| = 7$, we observe that $Y = V_4'$. Because of $d_H^-(x) \geq 8$ and $|Z| = 6$ for $x \in R_1$, it follows that $d_{D[R_1]}^-(x) \geq 2$ for each $x \in R_1$. Therefore $|R_1| = 6$

and $D[R_1]$ is 3-partite and 2-regular. Since $|Z| = 6$, we arrive at the contradiction $d_H^-(w) \le 7$ for at least one vertex $w \in R_1$.

Case 2. Assume that $c = 4$ and $r = 5$. Then $\alpha(D) = 5$ and $7 \le d_D^+(x), d_D^-(x) \le 8$ for each $x \in V(D)$. By Lemma 2.1, $\kappa(D) \ge 5$ and thus $\kappa(H) \ge 2$. In addition, $4 \le d_H^+(x), d_H^-(x) \le 8$ for each $x \in V(H)$.

Subcase 2.1. Assume that H has a cycle-factor. If H is Hamiltonian, then we are done. If not, then let $C_1', C_2', \dots, C_t'$ be a minimal cycle-factor of H with the properties described in Lemma 2.9. Since $|V^*| \le 5$, it follows from Lemma 2.9 that there are at most 6 arcs from $H - V(C_1')$ to C_1'. The bound $\kappa(H) \ge 2$ implies that $|V^* \cap V(C_1')| \ge 2$ and $|V^* \cap (V(H) - V(C_1'))| \ge 2$.

If $|V(C_1')| \le 6$, then it follows from Lemma 2.9 that

$$d_H^+(w) \ge |V(H) - V^* - V(C_1')| + |N_{C_1'}^+(w)|$$

$$\ge |V(H)| - |V^*| - |V(C_1')| + |V^* \cap V(C_1')| + |N_{C_1'}^+(w)|$$

$$\ge 17 - 5 - 6 + 2 + 1 = 9$$

for $w \in V^* \cap V(C_1')$, a contradiction to $d_H^+(w) \le 8$. If C_1' is a 7-cycle, then it is straightforward to verify that

$$\sum_{x \in V(C_1')} d_H^+(x) = \sum_{x \in V(C_1')} d_{D[V(C_1')]}^+(x) + d^+(C_1', H - V(C_1'))$$

$$= |E(H)| - |E(D[V(H) - V(C_1')])| - |E(H - V(C_1'), C_1')|$$

$$\ge 108 - 37 - 6 = 65,$$

where $|E(H - V(C_1'), C_1')|$ is the number of arcs from $H - V(C_1')$ to C_1'. This is a contradiction to $56 \ge \sum_{x \in V(C_1')} d_H^+(x)$. If C_1' is a 8-cycle, then we obtain the contradiction $64 \ge \sum_{x \in V(C_1')} d_H^+(x) \ge 72$. If C_1' is a 9-cycle, then we obtain the contradiction $72 \ge \sum_{x \in V(C_1')} d_H^+(x) \ge 78$. If C_1' is a 10-cycle, then we obtain the contradiction $80 \ge \sum_{x \in V(C_1')} d_H^+(x) \ge 84$. If $|V(C_1')| \ge 11$, then it follows from Lemma 2.9 that $d_H^-(w) \ge 9$ for $w \in V^* \cap (V(H) - V(C_1'))$, a contradiction.

Subcase 2.2. Assume that H has no cycle-factor. Then, in view of Lemma 2.8, the vertex set $V(H)$ can be partitioned into subsets Y, Z, R_1

and R_2 such that $R_1 \Rightarrow Y$, $(R_1 \cup Y) \Rightarrow R_2$, $|Y| > |Z|$, and Y is an independent set. Since $\kappa(H) \geq 2$ and $\alpha(H) = 5$, we deduce that $2 \leq |Z| < |Y| \leq 5$. We assume, WLOG, that $|R_1| \leq |R_2|$. Let V_1', V_2', V_3', V_4' be the partite sets of H such that $4 = |V_1'| = |V_2'| = |V_3'| < |V_4'| = 5$.

Subcase 2.2.1. Assume that $|Z| = 2$.

If $R_1 = \emptyset$, then $d_H^+(y) \geq 10$ for each $y \in Y$, a contradiction. If $1 \leq |R_1| \leq 5$, then there exists a vertex $x \in R_1$ such that $d_{D[R_1]}^-(x) \leq 1$ and thus $d_H^-(x) \leq 3$, a contradiction.

Assume now that $|R_1| = |R_2| = 6$. Then $|Y| = 3$. If $D[R_1]$ is bipartite, then it is easy to see that $d_H^-(x) \leq 3$ for at least one vertex $x \in R_1$, a contradiction. If $D[R_1]$ is 3-partite, then we obtain the contradiction $d_H^-(x) \leq 3$ for at least one vertex $x \in R_1$ or $D[R_1]$ is 2-regular. In the case that $D[R_1]$ is 2-regular, R_1 contains a vertex $x \in V_1' \cup V_2'$, and this yields the contradiction $d_H^+(x) \geq 9$. Finally, assume that $D[R_1]$ is 4-partite. Assume that there exists an index $i_0 \in \{1,2,3,4\}$ with $|R_1 \cap V_{i_0}'| = 3$ and $|R_1 \cap V_j'| = 1$ $(j \neq i_0)$. Thus $D[R_1]$ is not 2-regular. Now we obtain the contradiction $d_H^-(x) \leq 3$ for at least one vertex $x \in R_1$. So assume that $|R_1 \cup V_i'| \leq 2$ for each $i \in \{1,2,3,4\}$. Then we obtain the contradiction $48 \geq \sum_{x \in R_1} d_H^+(x) \geq 49$.

Subcase 2.2.2. Assume that $|Z| = 3$. If $R_1 = \emptyset$, then $d_H^+(y) \geq 9$ for each $y \in Y$, a contradiction. If there is a vertex $x \in R_1$ with $d_{D[R_1]}^-(x) = 0$, then $d_H^-(x) \leq 3$, a contradiction. Hence we assume in the following that $d_{D[R_1]}^-(x) \geq 1$ for each $x \in R_1$ and thus $|R_1| \geq 3$. If $3 \leq |R_1| \leq 4$, then there exists a vertex $y \in R_1$ such that $d_H^+(y) \geq 9$, a contradiction. Hence assume that $|R_1| = |R_2| = 5$. If $D[R_1]$ is bipartite, then we obtain the contradiction $d_H^-(u) \leq 3$ for at least one vertex $u \in R_1$ or $D[R_1]$ has partite sets of size 2 and 3. In the last case it is easy to see that $40 \geq \sum_{x \in R_1} d_H^+(x) \geq 41$, a contradiction. If $D[R_1]$ is 3-partite, then we deduce that $40 \geq \sum_{x \in R_1} d_H^+(x) \geq 41$, a contradiction. In the case that $D[R_1]$ is 4-partite, we obtain the contradiction $40 \geq \sum_{x \in R_1} d_H^+(x) \geq 43$.

Subcase 2.2.3. Assume that $|Z| = 4$ and $|Y| = 5$. This implies that $Y = V_4'$.

Subcase 2.2.3.1. Assume that $R_1 \neq \emptyset$. This implies $R_1 \rightarrow Y \rightarrow R_2$. If R_1 is an independent set or there is a vertex $x \in R_1$ with $d_{D[R_1]}^+(x) \geq 2$, then there exists a vertex $y \in R_1$ such that $d^+_H(y) \geq 9$, a contradiction. Assume next that R_1 is not independent and there is no vertex $x \in R_1$ with $d^+_{D[R_1]}(x) \geq 2$.

If $2 \leq |R_1| \leq 3$, then there exists a vertex $x \in R_1$ such that $d^+_H(x) \geq 9$, a contradiction. Hence, R_1 is bipartite and it is 4-cycle. Since $i_l(D) \leq 1$, R_2 is also a bipartite 4-cycle and Z is an independent set. Then we have $R_2 \Rightarrow (C_3 \cup Z) \Rightarrow R_1$.

Assume that $Y = \{x_1, x_2, x_3, x_4, x_5\}$, $R_1 = \{u_1, u_2, u_3, u_4\}$, $R_2 = \{v_1, v_2, v_3, v_4\}$, $Z = \{w_1, w_2, w_3, w_4\}$ and $C_3 = w_5yzw_5$. In addition, let $V_1 = Z \cup \{w_5\}$, $V_2 = \{u_1, u_3, v_2, v_4, y\}$, $V_3 = \{u_2, u_4, v_1, v_3, z\}$ and $V_4 = Y$. If $Z \rightarrow Y$, then we arrive at the contradiction $d^+_H(v) \geq 9$ for each $v \in Z$. Hence we assume that there exists at least one arc from Y to Z, say $x_3 \rightarrow w_4$ and at least one arc from w_5 to Y, say $w_5 \rightarrow x_5$. Now we define a path P from v_3 to u_1 as follows:

if there is an arc from z to $\{w_1, w_2, w_3\}$, say $z \rightarrow w_3$, then $P = v_3yzw_3u_1$;

if $\{w_1, w_2, w_3\} \rightarrow z$ and there is an arc $\{w_1, w_2, w_3\}$ to y, say $w_3 \rightarrow y$, then $P = v_3w_3yzu_1$; if $\{w_1, w_2, w_3\} \rightarrow z$ and $y \rightarrow \{w_1, w_2, w_3\}$, then $P = v_3yw_3zu_1$. Then we obtain cycles $C_3^* = x_5v_4w_5x_5$ and $C_{17} = u_1x_1v_1w_1u_2x_2 v_2w_2u_3x_3w_4u_4x_4v_3Pu_1$.

Subcase 2.2.3.2. Assume that $R_1 = \emptyset$. This implies that $(C_3 \cup Z) \rightarrow Y$. Now let $V_1 = \{u_1, u_2, u_3, u_4, u_5\}$, $V_2 = \{v_1, v_2, v_3, v_4, v_5\}$, $V_3 = \{w_1, w_2, w_3, w_4, w_5\}$ and $V_4 = \{x_1, x_2, x_3, x_4, x_5\}$ be the partite sets of the multipartite tournament D such that $C_3 = u_1v_1w_1u_1$ and $Y = \{x_1, x_2, x_3, x_4, x_5\}$.

Subcase 2.2.3.2.1. Assume that $D[R_2]$ is bipartite. Let, WLOG, $R_2 = \{u_2, u_3, u_4, u_5\} \cup \{v_2, v_3, v_4, v_5\}$ and $Z = \{w_2, w_3, w_4, w_5\}$. Since $i_l(D) \leq 1$, every vertex of Z is dominated by at least five vertices of R_2, and $d^+_{D[R_2]}(x) \geq 1$ for each vertex $x \in R_2$. Therefore we can assume, WLOG, that $u_2 \rightarrow w_2$, $v_2 \rightarrow w_3$ and $u_3 \rightarrow w_4$. Then we obtain the 3-cycle $C_3^* = x_1u_2w_2x_1$. Next we find the complementary cycle C_{17}.

Subcase 2.2.3.2.1.1. Assume that there is a vertex $y \in \{v_3, v_4, v_5\}$ such that $y \rightarrow w_5$ and let, without loss of generality, $y = v_3$. Now we assume, WLOG, that $u_5 \rightarrow v_5$.

First assume that $v_5 \rightarrow u_4$ and $u_4 \rightarrow v_2$ or $u_4 \rightarrow v_3$, say $u_4 \rightarrow v_3$.

If $v_4 \to u_1$ or $v_4 \to w_1$, say $v_4 \to u_1$, then we obtain the complementary cycle $C_{17} = x_2v_4u_1v_1w_1x_3u_5v_5\ u_4v_3w_5x_4v_2w_3x_5u_3w_4x_2$.

Otherwise $\{u_1, w_1\} \to v_4$. This implies that $v_4 \to u_2$ or $v_4 \to u_3$, say $v_4 \to u_3$. If $u_5 \to w_1$ or $u_5 \to v_1$, say $u_5 \to w_1$, then we obtain the complementary cycle $C_{17} = x_2u_5w_1u_1v_1x_3v_5u_4v_3w_5x_4v_4u_3w_4x_5v_2w_3x_2$. However, in the case $\{v_1, w_1\} \to u_5$, we see that $v_5 \to w_1$, and we obtain $C_{17} = x_2u_5v_5w_1u_1v_1x_3u_4v_3w_5x_4\ v_4u_3w_4x_5v_2w_3x_2$.

Second assume that $v_5 \to u_4$ and $\{v_2, v_3\} \to u_4$. It follows that $u_4 \to (Z \cup \{v_4\})$. If $v_4 \to u_5$, then two vertices of the set $\{u_2, u_3, v_2, v_3\}$ dominate w_1, say $v_2 \to w_1$. This leads to the the complementary cycle $C_{17} = x_2v_2w_1u_1v_1x_3v_3w_5x_4u_3w_4x_5v_4u_5v_5u_4w_3x_2$. However, if $u_5 \to v_4$, then $v_4 \to w_1$ or $v_4 \to u_1$, say $v_4 \to u_1$. If we assume, WLOG, that $u_3 \to v_3$, then we obtain $C_{17} = x_2v_4u_1v_1w_1x_3u_3v_3w_5x_4v_2w_3\ x_5u_5v_5u_4w_4x_2$.

Third assume that $u_4 \to v_5$. This implies that $v_5 \to u_3$ or $v_5 \to u_2$, say $v_5 \to u_3$ (in the case $v_5 \to u_2$ take the 3-cycle $x_1u_3w_4x_1$ instead of C_3^*).

Assume next that $v_4 \to u_4$ or $v_4 \to u_5$, say $v_4 \to u_5$. If $u_4 \to w_1$ or $u_4 \to v_1$, say $u_4 \to w_1$, then we obtain the complementary cycle $C_{17} = x_2u_4w_1u_1v_1x_3v_4u_5v_5u_3w_4x_4\ v_3w_5x_5v_2w_3x_2$. However, in the case $\{v_1, w_1\} \to u_4$, we see that $u_4 \to v_2$ or $u_4 \to v_3$, say $u_4 \to v_3$. If $u_5 \to w_1$ or $u_5 \to v_1$, say $u_5 \to w_1$, then we obtain $C_{17} = x_2v_4u_5w_1u_1v_1x_3v_5u_3w_4x_4u_4v_3\ w_5x_5v_2w_3x_2$. But if $\{v_1, w_1\} \to u_5$, then $u_5 \to Z$ and $v_3 \to w_1$ and so $C_{17} = x_2v_3w_1u_1v_1x_3u_4v_5u_3w_4x_4v_2w_3x_5v_4u_5w_5\ x_2$ is a complementary cycle. In the remaining case that $\{u_4, u_5\} \to v_4$, we observe that $v_4 \to w_1$ or $v_4 \to u_1$, say $v_4 \to w_1$. Now we obtain $C_{17} = x_2u_5v_4w_1u_1v_1x_3u_4v_5u_3w_4x_4v_3w_5x_5v_2w_3\ x_2$.

Subcase 2.2.3.2.1.2. Assume that $w_5 \to \{v_3, v_4, v_5\}$. This implies that $\{u_3, u_4, u_5, v_2\} \to w_5$.

(1). Assume first that there is an arc from $\{u_4, u_5\}$ to $\{v_3, v_4, v_5\}$, say $u_5 \to v_5$.

(1.1). Assume that $v_5 \to u_3$.

(1.1.1). Assume that $v_3 \to u_4$ or $v_4 \to u_4$, say $v_3 \to u_4$. If $v_4 \to u_1$ or $v_4 \to w_1$, say $v_4 \to w_1$, then we obtain the complementary cycle $C_{17} = x_2v_4w_1u_1v_1x_3v_3u_4w_5x_4u_5v_5u_3\ w_4x_5v_2w_3x_2$. Otherwise $\{u_1, w_1\} \to v_4$ and we have $v_4 \to \{u_3, u_4, u_5, w_3, w_4\}$. If $v_3 \to u_1$ or $v_3 \to w_1$, say $v_3 \to w_1$, then $C_{17} = x_2v_3w_1u_1v_1x_3v_4u_5v_5u_3w_4x_4v_2w_3x_5u_4w_5x_2$ is a complementary cycle. So assume that $\{u_1, w_1\} \to v_3$. Then $v_2 \to w_1$ and we have $C_{17} = x_2v_2w_1u_1v_1x_3v_4w_3x_4v_3u_4w_5x_5u_5v_5u_3\, w_4x_2$.

(1.1.2). Assume that $u_4 \to \{v_3, v_4\}$. Furthermore, assume that $v_3 \to u_5$ or $v_4 \to u_5$, say $v_3 \to u_5$. We see that $v_4 \to u_1$ or $v_4 \to w_1$, say $v_4 \to w_1$. Then we obtain the complementary cycle $C_{17} = x_2v_4w_1u_1v_1x_3v_3u_5v_5u_3w_4\, x_4v_2w_3x_5u_4w_5x_2$. Otherwise $u_5 \to \{v_3, v_4\}$. We deduce that $v_3 \to w_1$ and $v_3 \to w_3$ and we arrive at $C_{17} = x_2v_3w_1u_1v_1x_3u_5v_5\, u_3w_4x_4u_4v_4w_3x_5v_2w_5x_2$.

(1.2). Assume that $u_3 \to v_5$. This implies that $v_5 \to \{u_1, u_4, w_1, w_3, w_4\}$.

(1.2.1). Assume that $v_3 \to u_5$ or $v_4 \to u_5$, say $v_4 \to u_5$. If $v_3 \to u_1$ or $v_3 \to w_1$, say $v_3 \to w_1$, then $C_{17} = x_2v_3w_1u_1v_1x_3v_2w_3x_4u_3w_4x_5v_4u_5v_5u_4w_5\, x_2$ is a complementary cycle. Otherwise $\{u_1, w_1\} \to v_3$, and we have $v_3 \to \{u_3, u_4, u_5, w_3, w_4\}$. If $v_4 \to u_1$ or $v_4 \to w_1$, say $v_4 \to w_1$, then we have $C_{17} = x_2v_4w_1u_1v_1x_3v_3u_3w_4x_4v_2w_3x_5u_5v_5u_4w_5x_2$. So assume that $\{u_1, w_1\} \to v_4$. Then $v_4 \to u_4$ and $v_5 \to w_1$, and $C_{17} = x_2u_5v_5w_1u_1v_1x_3v_2\, w_3x_4v_3u_3w_4x_5v_4u_4w_5x_2$ is a complementary cycle.

(1.2.2). Assume that $u_5 \to \{v_3, v_4\}$. Furthermore, assume that $v_3 \to u_3$ or $v_4 \to u_3$, say $v_3 \to u_3$. Then $v_4 \to u_1$ or $v_4 \to w_1$, say $v_4 \to w_1$, and we obtain the complementary cycle $C_{17} = x_2v_4w_1u_1v_1x_3v_3u_3w_4x_4v_2w_3x_5u_5v_5\, u_4w_5x_2$. Otherwise $u_3 \to \{v_3, v_4\}$. We deduce that $v_4 \to w_4$ and $v_3 \to w_1$, and we arrive at the complementary cycle $C_{17} = x_2v_3w_1u_1v_1x_3v_2w_3x_4u_3v_4\, w_4x_5u_5v_5u_4w_5x_2$.

Subcase 2.2.3.2.2. Assume that $D[R_2]$ is 3-partite. We distinguish three different cases.

Subcase 2.2.3.2.2.1. Assume that the partite sets have size 1,3 and 4. Let, WLOG, $R_2 = \{u_2, v_2, v_3, v_4, w_2, w_3, w_4, w_5\}$ and $Z = \{u_3, u_4, u_5, v_5\}$. Since $i_l(D) \leq 1$, there is an arc from $\{w_2, w_3, w_4, w_5\}$ to v_5, say $w_2 \to v_5$. Then we have the 3-cycle $C_3^* = x_1w_2v_5x_1$. If $\{u_3, u_4, u_5\} \to \{v_2, v_3, v_4\}$, then we obtain $\{v_2, v_3, v_4\} \Rightarrow (C_3 \cup \{u_2, w_2, w_3, w_4, w_5\})$ and $u_2 \Rightarrow (C_3 \cup \{w_2, w_3, w_4, w_5, v_5\})$. This leads to the contradiction $d_D^-(w_i) \geq 9$ for $i \in \{2, 3, 4, 5\}$. Therefore, there is an arc from $\{v_2, v_3, v_4\}$ to $\{u_3, u_4, u_5\}$, say $v_2 \to u_3$.

(1.1). Assume that there is an arc from $\{w_3, w_4, w_5\}$ to $\{u_4, u_5\}$. WLOG, let $w_3 \to u_4$.

(1.1.1). Assume that there is an arc from $\{v_3, v_4\}$ to u_5, say $v_3 \to u_5$.

(1.1.1.1). If there is an arc from $\{w_4, w_5\}$ to v_3, then, let, WLOG, $w_4 \to v_3$.

If $C_3 \Rightarrow \{u_2, v_4, w_5\}$, then $((Z \cup R_2) \setminus \{u_2, v_4, w_5\}) \Rightarrow C_3$. WLOG, let $v_4 \to w_5$. Then $w_5 \to \{v_2, v_3, v_5, u_2, u_3, u_4, u_5\}$, $u_2 \to \{w_2, w_3, w_4, v_2, v_3, v_4, v_5\}$, and we have $C_{17} = x_2u_2v_2u_3x_3v_4w_5u_4x_4w_4v_3u_5\, x_5w_3u_1v_1w_1x_2$. Therefore, there is an arc from $\{u_2, v_4, w_5\}$ to C_3.

(a) Assume that there is an arc from u_2 to C_3, say $u_2 \to v_1$.

If $v_4 \to w_5$ and $w_5 \to u_2$, then we have $C_{17} = x_2v_2u_3x_3w_3u_4x_4w_4v_3u_5x_5\, v_4w_5u_2v_1w_1u_1x_2$.

If $v_4 \to w_5$ and $u_2 \to w_5$, then $w_5 \to u_1$ or $w_5 \to v_1$, say $w_5 \to u_1$. If $u_2 \to v_4$, then we have $C_{17} = x_2v_2u_3x_3w_3u_4x_4w_4v_3\; u_5x_5u_2v_4w_5u_1v_1w_1x_2$. If $v_4 \to u_2$, then we have $C_{17} = x_2v_2u_3x_3w_3u_4x_4w_4v_3u_5x_5v_4\; u_2w_5u_1v_1w_1x_2$.

If $w_5 \to v_4$ and $v_4 \to u_2$, then we have $C_{17} = x_2v_2u_3x_3w_3u_4x_4w_4v_3u_5x_5\, w_5v_4u_2v_1w_1u_1x_2$.

If $w_5 \to v_4$ and $u_2 \to v_4$, then $v_4 \to u_1$ or $v_4 \to w_1$, say $v_4 \to u_1$. Let, WLOG, $u_2 \to w_5$. Then we have $C_{17} = x_2v_2u_3x_3w_3u_4x_4w_4v_3u_5x_5u_2w_5v_4u_1\, v_1w_1x_2$.

(b) Assume that $C_3 \Rightarrow u_2$ and there is an arc form v_4 to C_3, say $v_4 \to u_1$.

If $w_5 \to u_2$, then $u_2 \to \{v_2, v_3, v_4, v_5, w_2, w_3, w_4\}$ and we have $C_{17} = x_2v_2u_3x_3w_3u_4x_4w_4v_3u_5x_5w_5u_2v_4\; u_1v_1w_1x_2$.

If $u_2 \to w_5$ and $w_5 \to v_4$, then we have $C_{17} = x_2v_2u_3x_3w_3u_4x_4w_4v_3u_5x_5\, u_2w_5v_4u_1v_1w_1x_2$.

If $u_2 \to w_5$ and $v_4 \to w_5$, then $w_5 \to u_1$ or $w_5 \to v_1$, say $w_5 \to u_1$. Then we have $C_{17} = x_2v_2u_3x_3w_3u_4x_4w_4v_3u_5x_5u_2v_4w_5u_1v_1w_1x_2$ if $u_2 \to v_4$ or $C_{17} = x_2v_2u_3x_3w_3u_4x_4w_4v_3u_5x_5v_4u_2w_5u_1v_1w_1x_2$ if $v_4 \to u_2$.

(c) Assume that $C_3 \Rightarrow \{u_2, v_4\}$ and there is an arc from w_5 to C_3, say $w_5 \to u_1$. Let, WLOG, $u_2 \to v_4$. Then $v_4 \to w_5$ and we we have the complementary cycle $C_{17} = x_2v_2u_3x_3w_3u_4x_4w_4v_3u_5x_5u_2v_4w_5u_1v_1w_1x_2$.

(1.1.1.2). Assume that $v_3 \to \{w_4, w_5\}$.

First we assume that there is an arc from $\{w_4, w_5\}$ to v_2, say $w_4 \to v_2$.

If $v_4 \to w_5$, then $w_5 \to u_1$ or $w_5 \to v_1$, say $w_5 \to u_1$, and there is an arc from u_2 to $\{w_3, w_4, v_3, v_4\}$. Then we have $C_{17} = x_2w_4v_2u_3x_3u_2w_3u_4x_4v_3\, u_5x_5v_4w_5u_1v_1w_1x_2$ when $u_2 \to w_3$ or $C_{17} = x_2u_2w_4v_2u_3x_3w_3u_4x_4v_3u_5x_5v_4\, w_5u_1v_1w_1x_2$ when $u_2 \to w_4$ or $C_{17} = x_2w_4v_2u_3x_3w_3u_4x_4u_2v_3u_5x_5v_4w_5u_1v_1\, w_1x_2$ when $u_2 \to v_3$ or $C_{17} = x_2w_4v_2u_3x_3w_3u_4x_4v_3u_5x_5u_2v_4w_5u_1v_1w_1x_2$ when $u_2 \to v_4$.

If $w_5 \to v_4$ and $u_2 \to v_4$, then $v_4 \to u_1$ or $v_4 \to w_1$, say $v_4 \to u_1$. Then we have $C_{17} = x_2w_4v_2u_3x_3w_3u_4x_4v_3u_5x_5w_5u_2v_4u_1v_1w_1x_2$ if $w_5 \to u_2$ or $C_{17} = x_2w_4v_2u_3x_3w_3u_4x_4v_3u_5x_5u_2w_5v_4u_1v_1w_1x_2$ if $u_2 \to w_5$.

If $w_5 \to v_4$, $v_4 \to u_2$ and $u_2 \to w_5$, then $w_5 \to u_1$ or $w_5 \to v_1$, say $w_5 \to u_1$. Thus $C_{17} = x_2w_4v_2u_3x_3w_3u_4x_4v_3u_5x_5v_4u_2w_5u_1v_1w_1x_2$ is a complementary cycle.

If $w_5 \to v_4$, $v_4 \to u_2$ and $w_5 \to u_2$, then $u_2 \to v_1$ or $u_2 \to w_1$, say $u_2 \to v_1$. Therefore $C_{17} = x_2w_4v_2u_3x_3w_3u_4x_4v_3u_5x_5w_5v_4u_2v_1w_1u_1x_2$ is a complementary cycle.

Now assume that $v_2 \to \{w_4, w_5\}$. Then w_4 dominates one vertex of $\{u_3, u_5\}$, say $w_4 \to u_3$.

If $\{v_2, v_3, w_3\} \to u_2$, then we see that $u_2 \Rightarrow (\{w_2, w_4, w_5, v_4, v_5\} \cup C_3)$, $\{w_4, w_5\} \Rightarrow (C_3 \cup \{v_4, v_5, u_3, u_4, u_5\})$, and $v_4 \Rightarrow C_3$. Hence, we obtain the complementary cycle $C_{17} = x_2v_2w_4u_3x_3w_3u_4x_4v_3u_5x_5u_2w_5v_4u_1v_1w_1x_2$.

Therefore, there is an arc from u_2 to $\{v_2, v_3, w_3\}$.

(i) Assume that there is an arc from u_2 to $\{v_2, v_3\}$, say $u_2 \to v_2$.

If $v_4 \to w_5$, then $w_5 \Rightarrow C_3$ and $C_{17} = x_2u_2v_2w_4u_3x_3w_3u_4x_4v_3u_5 x_5v_4w_5u_1v_1w_1x_2$.

If $w_5 \to v_4$ and v_4 dominates one vertex of C_3, say $v_4 \to u_1$, then we have $C_{17} = x_2u_2v_2w_4u_3x_3w_3u_4x_4v_3u_5\ x_5w_5v_4u_1v_1w_1x_2$.

If $(C_3 \cup \{w_5\}) \Rightarrow v_4$, then $v_4 \Rightarrow (\{u_2, w_2, w_3, w_4\} \cup Z)$. Since $w_5 \to u_1$ or $w_5 \to v_1$, say $w_5 \to u_1$, we have $C_{17} = x_2v_4u_2v_2w_4u_3x_3w_3u_4x_4v_3u_5x_5w_5u_1v_1 w_1x_2$.

(ii) Assume that $\{v_2, v_3\} \to u_2$ and $u_2 \to w_3$. WLOG, let $u_2 \to w_5$. Then $w_5 \Rightarrow (C_3 \cup \{v_4\})$.

If there is an arc from v_4 to C_3, say $v_4 \to u_1$, then we have $C_{17} = x_2v_2w_4u_3x_3u_2w_3u_4x_4v_3u_5x_5w_5v_4u_1v_1w_1x_2$.

If $C_3 \Rightarrow v_4$, then $v_4 \Rightarrow (\{u_2, w_2, w_3, w_4\} \cup Z)$ and we have $C_{17} = x_2v_2w_4u_3x_3v_4u_2w_3u_4x_4v_3u_5x_5w_5u_1v_1w_1x_2$.

(1.1.2). Assume that $u_5 \to \{v_3, v_4\}$. Then there is an arc from $\{w_4, w_5\}$ to u_5, say $w_4 \to u_5$.

(1.1.2.1). Assume that there is an arc from $\{v_3, v_4\}$ to $\{w_3, w_4\}$, say $v_3 \to w_3$.

If $u_2 \to w_5$ and $w_5 \to v_4$, then v_4 dominates one vertex of C_3, say $v_4 \to u_1$, and we have $C_{17} = x_2v_2u_3x_3v_3w_3u_4x_4w_4u_5x_5u_2w_5v_4u_1v_1w_1x_2$.

If $u_2 \to w_5$ and $v_4 \to w_5$, then w_5 dominates one vertex of C_3, say $w_5 \to u_1$. Then we have $C_{17} = x_2v_2u_3x_3v_3w_3u_4x_4w_4u_5x_5u_2v_4w_5u_1v_1w_1x_2$ if $u_2 \to v_4$ or $C_{17} = x_2v_2u_3x_3v_3w_3u_4x_4w_4u_5x_5v_4u_2w_5u_1v_1w_1x_2$ if $v_4 \to u_2$.

If $w_5 \to u_2$ and $u_2 \to v_4$, then v_4 dominates one vertex of C_3, say $v_4 \to u_1$, and we have $C_{17} = x_2v_2u_3x_3v_3w_3u_4x_4w_4u_5x_5w_5u_2v_4u_1v_1w_1x_2$.

If $w_5 \to u_2$ and $v_4 \to u_2$, then u_2 dominates one vertex of C_3, say $u_2 \to v_1$. Then we have $C_{17} = x_2v_2u_3x_3v_3w_3u_4x_4w_4u_5x_5w_5v_4u_2v_1w_1u_1x_2$ if $w_5 \to v_4$ or $C_{17} = x_2v_2u_3x_3v_3w_3u_4x_4w_4u_5x_5v_4w_5u_2v_1w_1u_1x_2$ if $v_4 \to w_5$.

(1.1.2.2). Assume that $\{w_3, w_4\} \to \{v_3, v_4\}$. Then $\{v_3, v_4\} \Rightarrow (\{w_2, w_5, u_2, u_3, u_4\} \cup C_3)$.

If $u_2 \to w_5$, then $w_5 \Rightarrow C_3$ and we have $C_{17} = x_2v_2u_3x_3w_3v_3u_4x_4w_4u_5 x_5v_4u_2w_5u_1v_1w_1x_2$.

If $w_5 \to u_2$, then $u_2 \Rightarrow C_3$, and we have $C_{17} = x_2v_2u_3x_3w_3v_3u_4x_4w_4u_5 x_5v_4w_5u_2v_1w_1\ u_1x_2$.

(1.2). Assume that $\{u_4, u_5\} \to \{w_3, w_4, w_5\}$. Then $(\{v_2, v_3, v_4, v_5, w_2\} \cup C_3) \Rightarrow \{u_4, u_5\}$ and there is an arc from w_3 to $\{v_2, v_3\}$, say $w_3 \to v_2$.

(1.2.1). Assume that $w_4 \to v_3$ or $w_5 \to v_3$, say $w_4 \to v_3$. Since $i_l(D) \leq 1$, w_5 dominates one vertex of C_3, say $w_5 \to u_1$ and u_2 dominates one vertex of $\{w_3, w_4, w_5, v_4\}$. Then we have $C_{17} = x_2u_2w_3v_2u_3x_3w_4v_3u_4x_4v_4u_5x_5w_5u_1 v_1w_1x_2$ if $u_2 \to w_3$ or $C_{17} = x_2w_3v_2u_3x_3u_2w_4v_3u_4x_4v_4u_5x_5w_5u_1v_1w_1x_2$ if $u_2 \to w_4$ or $C_{17} = x_2w_3v_2u_3x_3w_4v_3u_4x_4v_4u_5x_5u_2w_5u_1v_1w_1x_2$ if $u_2 \to w_5$ or $C_{17} = x_2w_3v_2u_3x_3w_4v_3u_4x_4u_2\ v_4\ u_5x_5w_5u_1v_1w_1x_2$ if $u_2 \to v_4$.

(1.2.2). Assume that $v_3 \to \{w_4, w_5\}$. Then we have $\{w_4, w_5\} \Rightarrow (\{u_2, u_3, v_2, v_4, v_5\} \cup C_3)$, $v_2 \Rightarrow (\{w_2, u_2, u_3, u_4, u_5\} \cup C_3)$, $u_2 \Rightarrow (\{w_2, w_3, v_3, v_4, v_5\} \cup C_3)$, and $v_4 \Rightarrow (\{w_2, w_3, u_3, u_4, u_5\} \cup C_3)$. This leads to $d^-(w_3) \geq 9$, a contradiction.

Subcase 2.2.3.2.2.2. Assume that the partite sets have size 2,2 and 4. Let, WLOG, $R_2 = \{u_4, u_5, v_4, v_5, w_2, w_3, w_4, w_5\}$ and $Z = \{u_2, u_3, v_2, v_3\}$. Since $i_l(D) \leq 1$, there is an arc from $\{w_2, w_3, w_4, w_5\}$ to u_2, say $w_2 \to u_2$. Then we obtain the 3-cycle $C_3^* = x_1w_2u_2x_1$.

(2.1). If there is an arc from $\{w_3, w_4, w_5\}$ to $\{v_2, v_3\}$, then, let, WLOG, $w_3 \to v_2$.

(2.1.1). If there is an arc from $\{w_4, w_5\}$ to v_3, then, let, WLOG, $w_4 \to v_3$.

(2.1.1.1). If there is an arc from $\{v_4, v_5\}$ to u_3, then, let, WLOG, $v_4 \to u_3$. WLOG, let $u_5 \to v_5$.

(2.1.1.1.1). Assume that $w_5 \to v_5$. Then v_5 dominates one vertex of C_3, say $v_5 \to u_1$. WLOG, let $u_5 \to w_5$.

If u_4 dominates one vertex of $\{w_3, w_4, v_4\}$, then we have $C_{17} = x_2u_4v_4u_3 x_3w_3v_2x_4w_4v_3x_5u_5w_5v_5u_1v_1w_1x_2$ if $u_4 \to v_4$ or $C_{17} = x_2v_4u_3x_3u_4w_3v_2x_4w_4 v_3x_5u_5w_5v_5u_1v_1w_1x_2$ if $u_4 \to w_3$ or $C_{17} = x_2v_4u_3x_3w_3v_2x_4u_4w_4v_3x_5u_5w_5v_5 u_1v_1w_1x_2$ if $u_4 \to w_4$.

If $\{w_3, w_4, v_4\} \to u_4$, then $u_4 \Rightarrow (\{w_2, w_5, v_2, v_3, v_5\} \cup C_3)$, $v_5 \Rightarrow (\{w_2, w_3, w_4, u_2, u_3\} \cup C_3)$, and w_5 dominates one vertex of C_3, say $w_5 \to u_1$. This leads to the complementary cycle $C_{17} = x_2v_4u_4v_5u_3x_3w_3v_2x_4w_4v_3x_5u_5 w_5u_1v_1w_1x_2$.

(2.1.1.1.2). Assume that $v_5 \to w_5$.

First assume that there is an arc from $\{u_4, u_5\}$ to w_5, say $u_4 \to w_5$, then w_5 dominates one vertex of C_3, say $w_5 \to u_1$. WLOG, let $u_4 \to v_5$.

If u_5 dominates one vertex of $\{w_3, w_4, v_4\}$, then we have $C_{17} = x_2u_5v_4u_3 x_3w_3v_2x_4w_4v_3x_5u_4v_5w_5u_1v_1w_1x_2$ if $u_5 \to v_4$ or $C_{17} = x_2v_4u_3x_3u_5w_3v_2x_4w_4 v_3x_5u_4v_5w_5u_1v_1w_1x_2$ if $u_5 \to w_3$ or $C_{17} = x_2v_4u_3x_3w_3v_2x_4u_5w_4v_3x_5u_4v_5w_5 u_1v_1w_1x_2$ if $u_5 \to w_4$.

If $\{w_3, w_4, v_4\} \to u_5$, then $u_5 \Rightarrow (\{w_2, w_5, v_2, v_3, v_5\} \cup C_3)$ and $w_5 \Rightarrow (\{u_2, u_3, v_2, v_3, v_4\} \cup C_3)$. This leads to the complementary cycle $C_{17} = x_2v_4u_3x_3w_3v_2x_4w_4u_5v_3x_5u_4v_5w_5u_1 v_1w_1x_2$.

Second assume that $w_5 \to \{u_4, u_5\}$. Assume now that $C_3 \Rightarrow v_5$. Then $v_5 \to u_4$ and thus $u_4 \to w_1$ or $u_4 \to v_1$, say $u_4 \to v_1$. This leads to the complementary cycle $C_{17} = x_2v_4u_3x_3w_3v_2x_4w_4v_3x_5u_5v_5w_5u_4v_1w_1u_1x_2$. Next assume that v_5 dominates one vertex of C_3, say $v_5 \to u_1$. We observe that u_4 dominates one vertex of $\{w_3, w_4, v_4\}$. Then we obtain the $C_{17} = x_2v_4u_3x_3u_4w_3v_2x_4w_4v_3x_5w_5u_5v_5u_1v_1w_1x_2$ if $u_4 \to w_3$ or $C_{17} = x_2v_4u_3x_3w_3 v_2x_4u_4w_4v_3x_5w_5u_5v_5u_1v_1w_1x_2$ if $u_4 \to w_4$ or $C_{17} = x_2u_4v_4u_3x_3w_3v_2x_4w_4v_3 x_5w_5u_5v_5u_1v_1w_1x_2$ if $u_4 \to v_4$.

(2.1.1.2). Assume that $u_3 \to \{v_4, v_5\}$.

(2.1.1.2.1). Assume that $w_5 \to u_3$. Then there is an arc from v_i to $\{w_3, w_4, w_5\}$ for $i = 4, 5$, let, without loss of generality, $v_4 \to w_3$.

Assume that there is an arc from u_4 to $\{v_4, w_4, w_5\}$.

If $u_5 \to v_5$, then v_5 dominates one vertex of C_3, say $v_5 \to u_1$, and we have $C_{17} = x_2w_5u_3x_3u_4v_4w_3v_2x_4w_4v_3x_5u_5v_5u_1v_1w_1x_2$ if $u_4 \to v_4$ or

$C_{17} = x_2w_5u_3x_3v_4w_3v_2x_4u_4w_4v_3\ x_5u_5v_5u_1v_1w_1x_2$ if $u_4 \to w_4$ or $C_{17} = x_2u_4w_5u_3x_3v_4w_3v_2x_4w_4v_3x_5u_5v_5u_1v_1w_1x_2$ if $u_4 \to w_5$.

If $v_5 \to u_5$ and u_5 dominates one vertex of C_3, say $u_5 \to v_1$, and we have $C_{17} = x_2w_5u_3x_3v_4w_3v_2x_4u_4w_4v_3x_5v_5u_5v_1w_1u_1x_2$ if $u_4 \to w_4$ or $C_{17} = x_2u_4w_5u_3x_3v_4w_3v_2x_4w_4\ v_3x_5v_5u_5v_1w_1u_1x_2$ if $u_4 \to w_5$ or $C_{17} = x_2w_5u_3x_3u_4v_4w_3v_2x_4w_4v_3x_5v_5u_5v_1w_1u_1x_2$ if $u_4 \to v_4$.

If $(C_3 \cup \{v_5\}) \Rightarrow u_5$, then $u_5 \to \{v_2, v_3, v_4, w_2, w_3, w_4, w_5\}$, and thus v_4 dominates one vertex of C_3, say $v_4 \to u_1$. We obtain $C_{17} = x_2w_5u_3x_3v_5u_5w_3\ v_2x_4u_4w_4v_3x_5v_4u_1v_1w_1x_2$ if $u_4 \to w_4$ or $C_{17} = x_2u_4w_5u_3x_3v_5u_5w_3v_2x_4w_4v_3\ x_5v_4u_1v_1w_1x_2$ if $u_4 \to w_5$ or $C_{17} = x_2w_5u_3x_3v_5u_5w_3v_2x_4w_4v_3x_5u_4v_4u_1v_1w_1\ x_2$ if $u_4 \to v_4$.

Now assume that $\{v_4, w_4, w_5\} \to u_4$. Then $u_4 \Rightarrow (\{v_2, v_3, v_5, w_2, w_3\} \cup C_3)$. If $u_5 \to v_5$, then v_5 dominates one vertex of C_3, say $v_5 \to u_1$, and we have $C_{17} = x_2w_5u_3x_3w_3v_2x_4w_4v_4u_4v_3x_5u_5v_5u_1v_1\ w_1x_2$ if $w_4 \to v_4$ or $C_{17} = x_2w_5u_3x_3w_3v_2x_4v_4w_4u_4v_3x_5u_5v_5u_1v_1w_1x_2$ if $v_4 \to w_4$.

Now assume that $v_5 \to u_5$ and $C_3 \Rightarrow u_5$. Then $u_5 \to \{v_2, v_3, w_3, w_4, w_5, v_4\}$ and so $v_4 \to u_1$ or $v_4 \to w_1$, say $v_4 \to w_1$. This leads to the complementary cycle $C_{17} = x_2u_4w_3v_2x_3w_4v_3x_4w_5u_3x_5v_5u_5v_4w_1u_1v_1x_2$.

Next assume that $v_5 \to u_5$ and u_5 dominates one vertex of C_3, say $u_5 \to v_1$. Then we have $C_{17} = x_2v_4u_4w_3v_2x_3w_4v_3x_4w_5u_3\ x_5v_5u_5v_1w_1u_1x_2$.

(2.1.1.2.2). Assume that $u_3 \to w_5$. Then $(\{w_2, w_3, w_4, v_2, v_3\} \cup C_3) \Rightarrow u_3$.

First assume that there is an arc from $\{v_4, v_5\}$ to w_5, say $v_5 \to w_5$. Then w_5 dominates one vertex of $\{v_2, v_3\}$, say $w_5 \to v_2$.

If there is an arc from $\{u_4, u_5\}$ to v_4, say $u_4 \to v_4$, then v_4 dominates one vertex of C_3, say $v_4 \to u_1$.

If u_5 dominates one vertex of $\{w_3, w_4, v_5\}$, then we have $C_{17} = x_2u_5w_3u_3\ x_3v_5w_5v_2x_4w_4v_3x_5u_4v_4u_1v_1w_1x_2$ if $u_5 \to w_3$ or $C_{17} = x_2w_3u_3x_3v_5w_5v_2x_4u_5\ w_4v_3x_5u_4v_4u_1v_1w_1x_2$ if $u_5 \to w_4$ or $C_{17} = x_2w_3u_3x_3u_5v_5w_5v_2x_4w_4v_3x_5u_4v_4\ u_1v_1w_1x_2$ if $u_5 \to v_5$.

If $\{w_3, w_4, v_5\} \to u_5$, then $u_5 \Rightarrow (\{v_2, v_3, v_4, w_2, w_5\} \cup C_3)$, $v_4 \Rightarrow (\{u_2, w_2, w_3, w_4, w_5\} \cup C_3)$, and we have $C_{17} = x_2u_4v_4w_3u_3x_3v_5w_5v_2x_4w_4v_3\ x_5u_5v_1w_1u_1x_2$.

If $v_4 \to \{u_4, u_5\}$, then there is an arc from u_4 to $\{w_3, w_4, w_5\}$. The next four lines show that we can assume, without loss of generality, that $u_5 \to v_5$. Then there is an arc from v_5 to C_3, say $v_5 \to u_1$, and we have $C_{17} = x_2v_4u_4w_3u_3x_3w_5v_2x_4w_4v_3x_5u_5v_5u_1v_1w_1x_2$ when $u_4 \to w_3$ or $C_{17} = x_2v_4u_4w_4v_3x_3w_5v_2x_4w_3u_3x_5u_5v_5u_1v_1w_1x_2$ when $u_4 \to w_4$ $C_{17} = x_2v_4u_4w_5v_2x_3w_3u_3x_4w_4v_3x_5u_5v_5u_1v_1w_1x_2$ when $u_4 \to w_5$.

Now assume that $w_5 \to \{v_4, v_5\}$. Then v_4 dominates one vertex of $\{u_4, u_5\}$, say $v_4 \to u_4$, and v_5 dominates one vertex of C_3, say $v_5 \to u_1$.

If there is an arc from $\{w_2, w_3, w_4, w_5, v_5\}$ to u_4, then $u_4 \to v_2$ or $u_4 \to v_3$, say $u_4 \to v_2$.

If $u_5 \to w_5$, then $C_{17} = x_2v_4u_4v_2x_3w_3u_3x_4w_4v_3x_5u_5w_5v_5u_1v_1w_1x_2$.

If $w_5 \to u_5$ and $u_5 \to v_5$, then we have $C_{17} = x_2v_4u_4v_2x_3w_3u_3x_4w_4v_3x_5 w_5u_5v_5u_1v_1w_1x_2$.

If $\{w_5, v_5\} \to u_5$, then u_5 dominates one vertex of C_3, say $u_5 \to v_1$. Then we have $C_{17} = x_2v_4u_4v_2x_3w_3u_3x_4w_4v_3x_5w_5v_5u_5v_1w_1u_1x_2$.

If $u_4 \to \{w_2, w_3, w_4, w_5, v_5\}$, then $v_5 \Rightarrow (\{w_2, w_3, w_4, u_2, u_5\} \cup C_3)$ and w_5 dominates one vertex of $\{v_2, v_3\}$, say $w_5 \to v_2$.

If there is an arc from u_5 to C_3, say $u_5 \to v_1$, then we have $C_{17} = x_2w_3u_3x_3v_4u_4w_5v_2\ x_4v_5w_4v_3x_5u_5v_1w_1u_1x_2$.

If $C_3 \Rightarrow u_5$, then $u_5 \to \{w_2, w_3, w_4, w_5, v_2, v_3, v_4\}$ and $w_5 \Rightarrow C_3$. Then we have $C_{17} = x_2w_3u_3x_3u_5v_2x_4v_5w_4v_3x_5v_4u_4w_5u_1v_1w_1x_2$.

(2.1.2). If $v_3 \to \{w_4, w_5\}$, then there is an arc from $\{u_4, u_5\}$ to v_3, say $u_4 \to v_3$.

(2.1.2.1). Assume that there is an arc from $\{v_4, v_5\}$ to u_3, let, WLOG, $v_4 \to u_3$.

Assume that w_4 dominates one vertex of $\{u_4, v_4\}$, let, WLOG, $w_4 \to u_4$.

If there is an arc from $\{v_5, u_5\}$ to w_5, say $v_5 \to w_5$, then w_5 dominates one vertex of C_3, say $w_5 \to u_1$.

If $u_5 \to v_5$, then $C_{17} = x_2v_4u_3x_3w_3v_2x_4w_4u_4v_3x_5u_5v_5w_5u_1v_1w_1x_2$.

If $v_5 \to u_5$ and $u_5 \to w_5$, then we have $C_{17} = x_2v_4u_3x_3w_3v_2x_4w_4u_4v_3x_5 v_5u_5w_5u_1v_1\ w_1x_2$.

If $v_5 \to u_5$ and $w_5 \to u_5$, then u_5 dominates one vertex of C_3, say $u_5 \to v_1$ and we have $C_{17} = x_2v_4u_3x_3w_3v_2x_4w_4u_4v_3x_5v_5w_5u_5v_1w_1u_1x_2$.

Next assume that $w_5 \to \{u_5, v_5\}$. WLOG, let $v_5 \to u_5$. It follows that there is an arc from u_5 to C_3, say $u_5 \to v_1$, and we have $C_{17} = x_2v_4u_3x_3w_3v_2x_4w_4u_4v_3x_5w_5v_5u_5v_1w_1u_1x_2$.

Assume now that $\{u_4, v_4\} \to w_4$. Then $w_4 \Rightarrow (\{v_2, v_5, u_2, u_3, u_5\} \cup C_3)$, and w_5 dominates one vertex of $\{u_4, v_4, u_5\}$. WLOG, let $u_5 \to v_5$. Then there is an arc from v_5 to C_3, say $v_5 \to u_1$, and we have

$C_{17} = x_2v_4w_4u_3x_3w_3v_2x_4w_5u_4v_3x_5u_5v_5u_1v_1w_1\ x_2$ if $w_5 \to u_4$ or $C_{17} = x_2v_4w_4u_3x_3w_3v_2x_4u_4v_3x_5w_5u_5v_5u_1v_1w_1x_2$ if $w_5 \to u_5$ or $C_{17} = x_2w_5v_4w_4u_3\ x_3w_3v_2x_4u_4v_3x_5u_5v_5u_1v_1w_1x_2$ if $w_5 \to v_4$.

(2.1.2.2). If $u_3 \to \{v_4, v_5\}$, then there is an arc from $\{w_4, w_5\}$ to u_3, say $w_4 \to u_3$, and there is an arc from v_i to $\{w_3, w_4, u_4\}$ for $i = 4, 5$. Let, without loss of generality, $v_5 \to w_5$. Then there is an arc from w_5 to C_3, say $w_5 \to u_1$.

If $u_5 \to v_5$, then we have $C_{17} = x_2w_4u_3x_3v_4w_3v_2x_4u_4v_3x_5u_5v_5w_5u_1v_1w_1\ x_2$ if $v_4 \to w_3$ or $C_{17} = x_2v_4w_4u_3x_3w_3v_2x_4u_4v_3x_5u_5v_5w_5u_1v_1w_1x_2$ if $v_4 \to w_4$ or $C_{17} = x_2w_4u_3x_3w_3\ v_2x_4v_4u_4v_3x_5u_5v_5w_5u_1v_1w_1x_2$ if $v_4 \to u_4$.

If $v_5 \to u_5$ and $u_5 \to w_5$, then we have $C_{17} = x_2w_4u_3x_3v_4w_3v_2x_4u_4v_3x_5\ v_5u_5w_5u_1v_1w_1x_2$ if $v_4 \to w_3$ or $C_{17} = x_2v_4w_4u_3x_3w_3v_2x_4u_4v_3x_5v_5u_5w_5u_1v_1\ w_1x_2$ if $v_4 \to w_4$ or $C_{17} = x_2w_4u_3x_3w_3v_2x_4v_4u_4v_3x_5v_5u_5w_5u_1v_1w_1x_2$ if $v_4 \to u_4$.

If $\{w_5, v_5\} \to u_5$, then there is an arc from u_5 to C_3, say $u_5 \to v_1$, and we have $C_{17} = x_2w_4u_3x_3v_4w_3v_2x_4u_4v_3x_5v_5w_5u_5v_1w_1u_1x_2$ if $v_4 \to w_3$ or $C_{17} = x_2v_4w_4u_3x_3w_3v_2x_4\ u_4v_3x_5v_5w_5u_5v_1w_1u_1x_2$ if $v_4 \to w_4$ or $C_{17} = x_2w_4u_3x_3w_3v_2x_4v_4u_4v_3x_5v_5w_5u_5v_1w_1u_1x_2$ if $v_4 \to u_4$.

(2.2). If $\{v_2, v_3\} \to \{w_3, w_4, w_5\}$, then $\{w_2, u_2, u_3, u_4, u_5\} \cup C_3 \Rightarrow \{v_2, v_3\}$.

Assume that there is an arc from $\{w_3, w_4, w_5\}$ to u_3. WLOG, let $w_3 \to u_3$. Then there is an arc from $\{w_4, w_5\}$ to $\{u_4, u_5\}$, say $w_4 \to u_4$, w_5 dominates one vertex of $\{v_4, v_5\}$, say $w_5 \to v_5$, and v_4 dominates one vertex of $\{u_5, w_3, w_4, w_5\}$.

If there is an arc from v_5 to C_3, say $v_5 \to u_1$, then there exist the cycle $C_{17} = x_2w_3u_3x_3w_4u_4v_2x_4v_4u_5v_3x_5w_5v_5u_1v_1w_1x_2$ when $v_4 \to u_5$ or $C_{17} = x_2v_4w_3u_3x_3w_4u_4v_2x_4u_5v_3x_5w_5v_5u_1v_1w_1x_2$ when $v_4 \to w_3$ or $C_{17} = x_2w_3u_3x_3v_4w_4u_4v_2x_4u_5v_3x_5w_5v_5u_1v_1w_1x_2$ when $v_4 \to w_4$ or $C_{17} = x_2w_3u_3\ x_3w_4u_4v_2x_4u_5v_3x_5v_4w_5v_5u_1v_1w_1x_2$ when $v_4 \to w_5$.

If $C_3 \Rightarrow v_5$, then $v_5 \to \{w_2, w_3, w_4, u_2, u_3, u_4, u_5\}$ and $w_3 \Rightarrow (\{v_4, u_2, u_3, u_4, u_5\} \cup C_3)$. Then we have $C_{17} = x_2w_5v_5u_3x_3w_4u_4v_2x_4v_4u_5v_3x_5w_3u_1v_1w_1\ x_2$ when $v_4 \to u_5$ or $C_{17} = x_2w_5v_5u_3x_3w_4u_4v_2x_4u_5v_3x_5v_4w_3u_1v_1w_1x_2$ when $v_4 \to w_3$ or $C_{17} = x_2w_5v_5u_3x_3v_4w_4u_4v_2x_4u_5v_3x_5w_3u_1v_1w_1x_2$ when $v_4 \to w_4$ or $C_{17} = x_2v_4w_5v_5u_3x_3w_4u_4v_2x_4u_5v_3x_5w_3u_1v_1w_1x_2$ when $v_4 \to w_5$.

Assume now that $u_3 \to \{w_3, w_4, w_5\}$. Then $\{w_3, w_4, w_5\} \Rightarrow (\{v_4, v_5, u_2, u_4, u_5\} \cup C_3)$. If, WLOG, $v_5 \to u_5$, then we obtain the contradiction $d_D^-(u_5) \geq 9$.

Subcase 2.2.3.2.2.3. Assume that the partite sets have size 2,3 and 3. WLOG, let $R_2 = \{u_4, u_5, v_3, v_4, v_5, w_3, w_4, w_5\}$ and $Z = \{u_2, u_3, v_2, w_2\}$. Since $i_l(D) \leq 1$, there is an arc from $\{v_3, v_4, v_5, w_3, w_4, w_5\}$ to u_2, say $v_3 \to u_2$. Then we obtain the 3-cycle $C_3^* = x_1 v_3 u_2 x_1$.

(3.1). Assume that there is an arc from $\{w_3, w_4, w_5\}$ to u_3, say $w_3 \to u_3$.

(3.1.1). If there is an arc from $\{w_4, w_5\}$ to v_2, then, WLOG, let $w_4 \to v_2$.

(3.1.1.1). Assume that there is an arc from $\{u_4, u_5\}$ to w_2, say $u_4 \to w_2$.

Assume that there is an arc from v_4 to $\{u_4, w_3, w_4\}$. WLOG, let $u_5 \to v_5$.

If $v_5 \to w_5$ and w_5 dominates one vertex of C_3, say $w_5 \to u_1$, then we have $C_{17} = x_2 w_3 u_3 x_3 w_4 v_2 x_4 v_4 u_4 w_2 x_5 u_5 v_5 w_5 u_1 v_1 w_1 x_2$ if $v_4 \to u_4$ or $C_{17} = x_2 v_4 w_3 u_3 x_3 w_4 v_2 x_4 u_4 w_2\ x_5 u_5 v_5 w_5 u_1 v_1 w_1 x_2$ if $v_4 \to w_3$ or $C_{17} = x_2 w_3 u_3 x_3 v_4 w_4 v_2 x_4 u_4 w_2 x_5 u_5 v_5 w_5 u_1 v_1 w_1 x_2$ if $v_4 \to w_4$.

If $(\{v_5\} \cup C_3) \Rightarrow w_5$, then $w_5 \to \{v_2, v_3, v_4, u_2, u_3, u_4, u_5\}$.

If there is an arc from $\{u_5, v_5\}$ to C_3, say $u_5 \to v_1$, then we have $C_{17} = x_2 w_3 u_3 x_3 w_4 v_2\ x_4 v_4 u_4 w_2 x_5 v_5 w_5 u_5 v_1 w_1 u_1 x_2$ if $v_4 \to u_4$ or $C_{17} = x_2 v_4 w_3 u_3 x_3 w_4 v_2 x_4 u_4 w_2 x_5 v_5 w_5 u_5 v_1 w_1 u_1\ x_2$ if $v_4 \to w_3$ or $C_{17} = x_2 w_3 u_3 x_3 v_4 w_4 v_2 x_4 u_4 w_2 x_5 v_5 w_5 u_5 v_1 w_1 u_1 x_2$ if $v_4 \to w_4$.

If $C_3 \Rightarrow \{u_5, v_5\}$, then $\{u_2, u_3, u_4, v_2, v_3, v_4, w_2, w_3, w_4\} \Rightarrow C_3$, $u_5 \to \{w_2, w_3, w_4, v_2, v_3, v_4, v_5\}$, and $v_5 \to \{w_2, w_3, w_4, w_5, u_2, u_3, u_4\}$. This leads to the cycle $C_{17} = x_2 w_5 u_5 v_5 w_3 u_3 x_3 w_4 v_2 x_4 u_4 w_2 x_5 v_4 u_1 v_1 w_1\ x_2$.

If $w_5 \to v_5$, then v_5 dominates one vertex of C_3, say $v_5 \to u_1$. WLOG, let $u_5 \to w_5$. Then we have $C_{17} = x_2 w_3 u_3 x_3 w_4 v_2 x_4 v_4 u_4 w_2 x_5 u_5 w_5 v_5 u_1 v_1 w_1\ x_2$ if $v_4 \to u_4$ or $C_{17} = x_2 v_4 w_3 u_3 x_3 w_4 v_2 x_4 u_4 w_2 x_5 u_5 w_5 v_5 u_1 v_1 w_1 x_2$ if $v_4 \to w_3$ or $C_{17} = x_2 w_3 u_3 x_3 v_4 w_4 v_2 x_4 u_4 w_2 x_5 u_5 w_5 v_5 u_1 v_1 w_1 x_2$ if $v_4 \to w_4$.

Assume next that $\{u_4, w_3, w_4\} \to v_4$. Then $v_4 \Rightarrow (\{w_2, w_5, u_2, u_3, u_5\} \cup C_3)$. WLOG, let $u_5 \to w_5$. Then w_5 dominates one vertex of C_3, say $w_5 \to u_1$.

If v_5 dominates one vertex of $\{u_4, w_3, w_4\}$, then we have $C_{17} = x_2 w_3 u_3 x_3\ w_4 v_2 x_4 v_5 u_4 w_2 x_5 v_4 u_5 w_5 u_1 v_1 w_1 x_2$ if $v_5 \to u_4$ or $C_{17} = x_2 v_5 w_3 u_3 x_3 w_4 v_2 x_4 u_4\ w_2 x_5 v_4 u_5 w_5 u_1 v_1 w_1 x_2$ if $v_5 \to w_3$ or $C_{17} = x_2 w_3 u_3 x_3 v_5 w_4 v_2 x_4 u_4\ w_2 x_5 v_4 u_5\ w_5 u_1 v_1 w_1 x_2$ if $v_5 \to w_4$.

If $\{u_4, w_3, w_4\} \to v_5$, then $v_5 \Rightarrow \{w_2, w_5, u_2, u_3, u_5\}$. WLOG, let $u_5 \to w_5$. Then $w_5 \Rightarrow (\{v_2, v_3, u_2, u_3, u_4\} \cup C_3)$ and we have $C_{17} = x_2w_3v_4u_3x_3w_4 v_2x_4u_4w_2x_5v_5u_5w_5u_1v_1w_1x_2$.

(3.1.1.2). Assume now that $w_2 \to \{u_4, u_5\}$. Then $v_4 \to w_2$ or $v_5 \to w_2$, say $v_4 \to w_2$.

Assume that there is an arc from $\{u_4, u_5\}$ to v_4. Then, without loss of generality, let $u_4 \to v_4$.

If $v_5 \to w_5$ and $w_5 \to u_5$, then there is an arc from u_5 to C_3, say $u_5 \to v_1$, and we have $C_{17} = x_2w_3u_3x_3u_4v_4w_2x_4w_4v_2x_5v_5w_5u_5v_1w_1u_1x_2$.

If $v_5 \to w_5$ and $u_5 \to w_5$, then there is an arc from w_5 to C_3, say $w_5 \to u_1$. Then we obtain $C_{17} = x_2w_3u_3x_3u_4v_4w_2x_4w_4v_2x_5v_5u_5w_5u_1v_1w_1x_2$ if $v_5 \to u_5$ or $C_{17} = x_2w_3u_3x_3u_4v_4w_2x_4w_4v_2x_5u_5v_5w_5u_1v_1w_1x_2$ if $u_5 \to v_5$.

If $w_5 \to v_5$ and $v_5 \to u_5$, then u_5 dominates one vertex of C_3, say $u_5 \to v_1$, and we obtain $C_{17} = x_2w_3u_3x_3u_4v_4w_2x_4w_4v_2x_5w_5v_5u_5v_1w_1u_1x_2$.

If $w_5 \to v_5$ and $u_5 \to v_5$, then v_5 dominates one vertex of C_3, say $v_5 \to u_1$. Then we obtain the cycles $C_{17} = x_2w_3u_3x_3u_4v_4w_2x_4w_4v_2x_5u_5w_5v_5u_1v_1 w_1x_2$ when $u_5 \to w_5$ or $C_{17} = x_2w_3u_3x_3u_4v_4w_2x_4w_4v_2x_5w_5u_5v_5u_1v_1w_1x_2$ when $w_5 \to u_5$.

Now assume that $v_4 \to \{u_4, u_5\}$. Then u_4 dominates one vertex of $\{w_3, w_4\}$, say $u_4 \to w_3$. Without loss of generality, let $v_5 \to w_5$.

If $w_5 \to u_5$, then $u_5 \Rightarrow C_3$ and we have $C_{17} = x_2u_4w_3u_3x_3w_4v_2x_4v_4w_2x_5 v_5w_5u_5v_1w_1\ u_1x_2$.

If $u_5 \to w_5$, then w_5 dominates one vertex of C_3, say $w_5 \to u_1$, and we have $C_{17} = x_2u_4w_3u_3x_3w_4v_2x_4v_4w_2x_5v_5u_5w_5u_1v_1w_1x_2$ if $v_5 \to u_5$ or $C_{17} = x_2u_4w_3u_3x_3w_4v_2x_4v_4w_2\ x_5u_5v_5w_5u_1v_1w_1x_2$ if $u_5 \to v_5$.

(3.1.2). If $v_2 \to \{w_4, w_5\}$, then $u_4 \to v_2$ or $u_5 \to v_2$, say $u_4 \to v_2$.

(3.1.2.1). Assume that there is an arc from $\{v_4, v_5\}$ to w_2, say $v_4 \to w_2$.

Assume that there is an arc from w_4 to $\{v_4, u_4\}$, say $w_4 \to u_4$. WLOG, let $v_5 \to u_5$.

If $u_5 \to w_5$, then w_5 dominates one vertex of C_3, say $w_5 \to u_1$, and we have $C_{17} = x_2w_3u_3x_3w_4u_4v_2x_4v_4w_2x_5v_5u_5w_5u_1v_1w_1x_2$.

If $w_5 \to u_5$, then u_5 dominates one vertex of C_3, say $u_5 \to v_1$. This leads to the complementary cycle $C_{17} = x_2w_3u_3x_3w_4u_4v_2x_4v_4w_2x_5v_5w_5u_5v_1w_1u_1 x_2$ if $v_5 \to w_5$ or $C_{17} = x_2w_3u_3x_3w_4u_4v_2x_4v_4w_2x_5w_5v_5u_5v_1w_1u_1x_2$ if $w_5 \to v_5$.

Assume next that $\{v_4, u_4\} \to w_4$. Then $w_4 \Rightarrow (\{v_3, v_5, u_2, u_3, u_5\} \cup C_3)$. WLOG, let $u_5 \to v_5$. Therefore v_5 dominates one vertex of C_3, say $v_5 \to u_1$.

If w_5 dominates one vertex of $\{u_4, v_4\}$, then we have $C_{17} = x_2w_3u_3x_3w_5 u_4v_2x_4v_4w_2x_5w_4u_5v_5u_1v_1w_1x_2$ if $w_5 \to u_4$ or $C_{17} = x_2w_3u_3x_3u_4v_2x_4w_5v_4w_2 x_5w_4u_5v_5u_1v_1w_1x_2$ if $w_5 \to v_4$.

If $\{u_4, v_4\} \to w_5$, then $w_5 \Rightarrow (\{v_3, v_5, u_2, u_3, u_5\} \cup C_3)$ and $v_5 \Rightarrow (\{w_2, w_3, u_2, u_3, u_4\} \cup C_3)$. This yields the cycle $C_{17} = x_2w_3u_3x_3u_4v_2x_4w_4u_5 v_5w_2x_5v_4w_5u_1v_1w_1x_2$.

(3.1.2.2). Assume that $w_2 \to \{v_4, v_5\}$.

First, assume that $u_5 \to w_2$. WLOG, let $v_4 \to w_4$. Then w_4 dominates one vertex of $\{u_4, u_5\}$.

If $v_5 \to w_5$, then w_5 dominates one vertex of C_3, say $w_5 \to u_1$, and we have $C_{17} = x_2w_3u_3x_3v_4w_4u_4v_2x_4u_5w_2x_5v_5w_5u_1v_1w_1x_2$ if $w_4 \to u_4$ or $C_{17} = x_2w_3u_3x_3u_4v_2x_4v_4w_4u_5\ w_2x_5v_5w_5u_1v_1w_1x_2$ if $w_4 \to u_5$.

If $w_5 \to v_5$, then v_5 dominates one vertex of C_3, say $v_5 \to u_1$, and we have $C_{17} = x_2w_3u_3x_3v_4w_4u_4v_2x_4u_5w_2x_5w_5v_5u_1v_1w_1x_2$ if $w_4 \to u_4$ or $C_{17} = x_2w_3u_3x_3u_4v_2x_4v_4w_4u_5\ w_2x_5v_5w_5u_1v_1w_1x_2$ if $w_4 \to u_5$.

Now assume that $w_2 \to u_5$. Then $(\{v_2, v_3, u_2, u_3, u_4\} \cup C_3) \Rightarrow w_2$ and thus $u_5 \to v_2$.

WLOG, let $v_4 \to w_4$. Then w_4 dominates one vertex of $\{u_4, u_5\}$. WLOG, let $v_5 \to w_5$. Then w_5 dominates one vertex of C_3, say $w_5 \to u_1$. Then we obtain $C_{17} = x_2w_3u_3x_3v_4w_4u_4w_2x_4u_5v_2x_5v_5w_5u_1v_1w_1x_2$ if $w_4 \to u_4$ or $C_{17} = x_2w_3u_3x_3u_4w_2x_4v_4w_4u_5v_2x_5v_5w_5u_1v_1w_1x_2$ if $w_4 \to u_5$.

(3.2). If $u_3 \to \{w_3, w_4, w_5\}$, then $(\{v_2, v_3, v_4, v_5, w_2\} \cup C_3) \Rightarrow u_3$, and there is an arc from $\{w_3, w_4, w_5\}$ to v_2, say $w_3 \to v_2$.

(3.2.1). If there is an arc from $\{w_4, w_5\}$ to $\{u_4, u_5\}$, then, let, WLOG, $w_4 \to u_4$.

(3.2.1.1). Assume that $u_4 \to w_2$.

Assume that there is an arc from $\{v_5, u_5\}$ to w_5, say $v_5 \to w_5$. Then w_5 dominates one vertex of C_3, say $w_5 \to u_1$.

If $u_5 \to v_5$, then $C_{17} = x_2v_4u_3x_3w_3v_2x_4w_4u_4w_2x_5u_5v_5w_5u_1v_1w_1x_2$.

If $v_5 \to u_5$ and $u_5 \to w_5$, then we have $C_{17} = x_2v_4u_3x_3w_3v_2x_4w_4u_4w_2x_5 v_5u_5w_5u_1v_1w_1x_2$.

If $\{v_5, w_5\} \to u_5$, then u_5 dominates one vertex of C_3, say $u_5 \to v_1$ and we have $C_{17} = x_2v_4u_3x_3w_3v_2x_4w_4u_4w_2x_5v_5w_5u_5v_1w_1u_1x_2$.

Now assume that $w_5 \to \{v_5, u_5\}$. Without loss of generality, let $v_5 \to u_5$. Then u_5 dominates one vertex of C_3, say $u_5 \to v_1$ and we have $C_{17} = x_2v_4u_3x_3w_3v_2x_4w_4u_4w_2x_5\ w_5v_5u_5v_1w_1u_1x_2$.

(3.2.1.2). Assume now that $w_2 \to u_4$. Then there is an arc from $\{v_4, v_5\}$ to w_2, say $v_4 \to w_2$, and u_4 dominates one vertex of $\{v_4, v_5\}$.

If $u_5 \to w_5$, then w_5 dominates one vertex of C_3, say $w_5 \to u_1$, and we have $C_{17} = x_2v_5u_3x_3w_4u_4v_4w_2x_4w_3v_2x_5u_5w_5u_1v_1w_1x_2$ if $u_4 \to v_4$ or $C_{17} = x_2w_4u_4v_5u_3x_3v_4w_2x_4w_3\ v_2x_5u_5w_5u_1v_1w_1x_2$ if $u_4 \to v_5$.

If $w_5 \to u_5$ and there is an arc from u_5 to C_3, say $u_5 \to v_1$, then we have $C_{17} = x_2v_5u_3x_3w_4u_4v_4w_2x_4w_3v_2x_5w_5u_5v_1w_1u_1x_2$ if $u_4 \to v_4$ or $C_{17} = x_2w_4u_4v_5u_3x_3v_4w_2x_4w_3\ v_2x_5w_5u_5v_1w_1u_1x_2$ if $u_4 \to v_5$.

If $(\{w_5\} \cup C_3) \Rightarrow u_5$, then $u_5 \to \{v_2, v_3, v_4, v_5, w_2, w_3, w_4\}$. Since u_4 dominates one vertex of C_3, say $u_4 \to v_1$, we have $C_{17} = x_2v_5u_3x_3v_4w_2x_4w_3\ v_2x_5w_5u_5w_4u_4v_1w_1u_1x_2$.

(3.2.2). If $\{u_4, u_5\} \to \{w_4, w_5\}$, then $\{w_4, w_5\} \Rightarrow (\{u_2, v_2, v_3, v_4, v_5\} \cup C_3)$.

If there is an arc from $\{v_4, v_5\}$ to w_2, say $v_5 \to w_2$, then we have $C_{17} = x_2v_4u_3x_3u_4\ w_4v_5w_2x_4w_3v_2x_5u_5w_5u_1v_1w_1x_2$ if $v_5 \to w_2$ or $C_{17} = x_2v_5u_3x_3u_4w_4v_4w_2x_4w_3v_2x_5u_5w_5u_1\ v_1w_1x_2$ if $v_4 \to w_2$.

If $w_2 \to \{v_4, v_5\}$, then $(\{u_2, u_4, u_5, v_2, v_3\} \cup C_3) \Rightarrow w_2$, $\{v_4, v_5\} \Rightarrow (\{w_3, u_2, u_3, u_4, u_5\} \cup C_3)$, and we obtain the cycle $C_{17} = x_2w_4v_4u_3x_3v_5w_3v_2\ x_4u_4w_2x_5u_5w_5u_1v_1w_1x_2$.

□

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