A neighborhood union condition for fractional (k, m)-deleted graphs

Wei Gao[1], Weifan Wang[2*]

1. School of Information Science and Technology, Yunnan Normal University, Kunming 650500, China

2.Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China

Abstract A graph G is called a fractional (k, m)-deleted graph if any m edges are removed from G then the resulting graph admits a fractional k-factor. In this paper, we prove that for integers $k \geq 2$, $m \geq 0$, $n \geq 8k + 4m - 7$, and $\delta(G) \geq k + m$, if

$$|N_G(x) \cup N_G(y)| \geq \frac{n}{2}$$

for each pair of non-adjacent vertices x, y of G, then G is a fractional (k, m)-deleted graph. The bounds for neighborhood union condition, order and the minimum degree of G are all sharp.

Key words: graph, fractional factor, fractional (k, m)-deleted graph, neighborhood union condition

1 Introduction

All graphs considered in this paper are finite, loopless, and without multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$.

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For any $x \in V(G)$, the degree and the neighborhood of x in G are denoted by $d_G(x)$ and $N_G(x)$, respectively. For $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S, and $G - S = G[V(G) \setminus S]$. For two vertex-disjoint subsets S and T of G, we use $e_G(S,T)$ to denote the number of edges with one end in S and the other end in T. We denote the minimum degree and the maximum degree of G by $\delta(G)$ and $\Delta(G)$, respectively. The *distance* $d_G(x,y)$ between two vertices x and y is defined to be the length of a shortest path connecting them. The notation and terminology used but undefined in this paper can be found in [1].

Let $k \geq 1$ be an integer. A spanning subgraph F of G is called a *k-factor* if $d_F(x) = k$ for each $x \in V(G)$. Let $h : E(G) \rightarrow [0,1]$ be a function. If $\sum_{x\in e} h(e) = k$ for any $x \in V(G)$, then we call $G[F_h]$ a *fractional k-factor* of G with indicator function h where $F_h = \{e \in E(G) : h(e) > 0\}$. Zhou [6] introduced the definition of a fractional (k,m)-deleted graph, that is, a graph G is called a *fractional (k,m)-deleted graph* if removing any m edges from G, the resulting graph has a fractional k-factor. A fractional (k,m)-deleted graph is simply called a fractional k-deleted graph if $m = 1$.

In what follows, we always assume that n is order of G, i.e., $n = |V(G)|$. Yu showed a degree condition for the existence of a fractional k-factor.

Theorem 1 (Yu et al [4]). *Let k be an integer with $k \geq 1$, and let G be a connected graph of order n with $n \geq 4k-3$, $\delta(G) \geq k$. If*

$$\max\{d_G(x), d_G(y)\} \geq \frac{n}{2}$$

for each pair of nonadjacent vertices x, y of G, then G has a fractional k-factor.

Let $\omega(G-S)$ denote the number of components of $G-S$. The *toughness* $t(G)$ of a graph G is defined as follows: $t(G) = +\infty$ if G is a complete graph; otherwise, $t(G) = \min\{\frac{|S|}{\omega(G-S)} : S \subseteq V(G), \omega(G-S) \geq 2\}$. The toughness condition for graphs to have fractional k-factors obtained by Liu and Zhang.

Theorem 2 (Liu and Zhang [3]). *Let $k \geq 2$ be an integer. A graph G of order n with $n \geq k+1$ has a fractional k-factor if $t(G) \geq k - \frac{1}{k}$.*

For fractional (k,m)-deleted graphs, we have the following known results.

Theorem 3 (Zhou [6]). *Let $k \geq 2$ and $m \geq 0$ be two integers. Let G be a connected graph of order n with $n \geq 9k - 1 - \sqrt{2(k-1)^2+2} + 2(2k+1)m$, $\delta(G) \geq k + m + \frac{(m+1)^2-1}{4k}$. If*

$$|N_G(x) \cup N_G(y)| \geq \frac{1}{2}(n+k-2)$$

for each pair of non-adjacent vertices x, y of G, then G is a fractional (k,m)-deleted graph.

Theorem 4 (Zhou [5]). *Let $k \geq 1$ and $m \geq 1$ be two integers. Let G be a graph of order n with $n \geq 4k - 5 + 2(2k+1)m$. If $\delta(G) \geq \frac{n}{2}$, then G is a fractional (k,m)-deleted graph.*

More sufficient conditions for graphs to have fractional factor can be found in [7]. In this paper, we give the following result:

Theorem 5 *Let $k \geq 2$ and $m \geq 0$ be two integers, and let G be a graph of order n with $n \geq 8k + 4m - 7$, $\delta(G) \geq k + m$. If*

$$|N_G(x) \cup N_G(y)| \geq \frac{n}{2}$$

for each pair of non-adjacent vertices x, y of G, then G is a fractional (k,m)-deleted graph.

We will show that the bounds for neighborhood union condition, order and the minimum degree of G are all sharp. In order to prove our main results, we need the following lemma which is Lemma 2.2 in [6].

Lemma 6 (Zhou [6]). *Let $k \geq 1$ and $m \geq 0$ be two integers, and let G be a graph and H a subgraph of G with m edges. Then G is a fractional (k,m)-deleted graph if and only if*

$$\delta_G(S,T) = k|S| + \sum_{x\in T} d_{G-S}(x) - k|T| \geq \sum_{x\in T} d_H(x) - e_H(S,T),$$

for all disjoint subsets S and T of $V(G)$.

2 Proof of Theorem 5

Suppose that G satisfies the conditions of Theorem 5, but is not a fractional (k,m)-deleted graph. According to Lemma 6 there exist disjoint subsets S and T of $V(G)$ such that

$$\delta_G(S,T) = k|S| + \sum_{x\in T} d_{G-S}(x) - k|T| \leq 2m - 1, \tag{1}$$

We choose subsets S and T such that $|T|$ is minimum. Obviously, $T \neq \emptyset$.

Claim 1 $d_{G-S}(x) \leq k-1$ *for any $x \in T$.*

Proof. If $d_{G-S}(x) \geq k$ for some $x \in T$, then the subsets S and $T \setminus \{x\}$ satisfy (1). This contradicts the choice of S and T. □

Let $d_1 = \min\{d_{G-S}(x) : x \in T\}$ and choose $x_1 \in T$ such that $d_{G-S}(x_1) = d_1$. If $T - N_T[x_1] \neq \emptyset$, let $d_2 = \min\{d_{G-S}(x) : x \in T - N_T[x_1]\}$ and choose $x_2 \in T - N_T[x_1]$ such that $d_{G-S}(x_2) = d_2$. So, $d_1 \leq d_2$. Let $|S| = s$, $|T| = t$, $|N_T[x_1]| = p$. Then, $p \leq d_1 + 1$, $d_{G-S}(T) \geq d_1 p + d_2(t-p)$, and $ks - kt + d_1 p + d_2(t-p) - 2m \leq k|S| - k|T| + d_{G-S}(T) - 2m < 0$.

We have

$$\begin{aligned} |S| &\leq \frac{k|T| - d_{G-S}(T) + (\sum_{x\in T} d_H(x) - e_H(S,T) - 1)}{k} \\ &\leq \frac{k|T| + (2m-1)}{k}. \end{aligned}$$

Thus, $|S| \le |T| + \frac{2m-1}{k}$. If $|S| \le m$, then $|T| = 0$ by $\delta(G) \ge k+m$, which is a contradiction. So, $m+1 \le s \le t + \frac{2m-1}{k}$. We consider following two cases:

Case 1. $T = N_T[x_1]$. In this case, $t \le d_1 + 1$ and $d_2 = 0$. If $d_1 = k-1$, then $t \le k$, $k|S| - k|T| + d_{G-S}(T) - 2m \ge ks - kt + d_1 p - 2m = ks - kt + (k-1)t - 2m \ge ks - k - 2m \ge k(m+1) - k - 2m \ge 0$. If $0 \le d_1 \le k-2$, then $t \le d_1 + 1 \le k-1$. By $\delta(G) \ge k+m$ and $d_G(x_1) \le s + d_1$, we have $s \ge k + m - d_1$. Thus, $k|S| - k|T| + d_{G-S}(T) - 2m \ge ks - kt + d_1 p - 2m \ge k(k+m-d_1) + (d_1 - k)t - 2m = (k-d_1)(k-t) + km - 2m > 0$. It is a contradiction.

Case 2. $T - N_T[x_1] \ne \emptyset$. We consider following three subcases.

Case 2.1. $d_1 = d_2 = k-1$. In this subcase, $k|S| - k|T| + d_{G-S}(T) - 2m \ge ks - kt + d_1 p + d_2(t-p) - 2m = ks - kt + (k-1)p + (k-1)(t-p) - 2m = ks - t - 2m \ge 0$, which is a contradiction. In fact, if $ks \le t + 2m - 1$, then $s \le \frac{t+2m-1}{k}$, $s + ks - 2m + 1 \le s + t \le n$. Thus, $s + 2k - 2 \ge |N_G(x_1) \cup N_G(x_2)| \ge \frac{n}{2} \ge \frac{s+sk-2m+1}{2}$ since $x_1x_2 \notin E(G)$. Then $4k \ge (k-1)s + 2(2-m) + 1 \ge (k-1)s + 2(3-s) + 1 = (k-3)s + 7$, i.e., $s \le \frac{4k-7}{k-3}$ if $k \ge 5$. Then $\frac{4k-7}{k-3} + 2k - 2 \ge s + 2k - 2 \ge |N_G(x_1) \cup N_G(x_2)| \ge \frac{n}{2}$, i.e. $\frac{4k-7}{k-3} + 2k - 2 \ge 4k + 2m - 3$ since $s + 2k - 2$ is an integer. Then we have $\frac{4k-7}{k-3} \ge 2k + 2m - 1 \ge 2k - 1$, which contradicts to $k \ge 5$.

If $k = 4$, we have $s \ge \frac{n}{2} - 6$ and $t \le n - s \le \frac{n}{2} + 6$ sine $s + 2k - 2 \ge |N_G(x_1) \cup N_G(x_2)| \ge \frac{n}{2}$. Thus, $k|S| - k|T| + d_{G-S}(T) - 2m \ge k(\frac{n}{2} - 6) - k(\frac{n}{2} + 6) + 3(\frac{n}{2} + 6) - 2m > 0$, a contradiction.

If $k = 3$, we have $s \ge \frac{n}{2} - 4$ and $t \le n - s \le \frac{n}{2} + 4$. Thus, $k|S| - k|T| + d_{G-S}(T) - 2m \ge k(\frac{n}{2} - 4) - k(\frac{n}{2} + 4) + 2(\frac{n}{2} + 4) - 2m > 0$, a contradiction.

If $k = 2$, we have $s \ge \frac{n}{2} - 2$ and $t \le n - s \le \frac{n}{2} + 2$. If $n \ge 8k + 4m - 4$, then $k|S| - k|T| + d_{G-S}(T) - 2m \ge k(\frac{n}{2} - 2) - k(\frac{n}{2} + 2) + (\frac{n}{2} + 2) - 2m > 0$, a contradiction. If $n = 8k + 4m - 5$, then $s \ge \frac{n}{2} - 2$ implies $s \ge 4k + 2m - 4$ and $t \le 4k + 2m - 1$. Thus, $k|S| - k|T| + d_{G-S}(T) - 2m \ge k(4k + 2m -$

$4) - (k-1)(4k+2m-1) - 2m = k-1 > 0$, which is a contradiction. If $n = 8k+4m-6$, then $s \geq \frac{n}{2}-2$ implies $s \geq 4k+2m-5$ and $t \leq 4k+2m-1$. If $s \geq 4k+2m-4$ or $t \leq 4k+2m-2$, then $k|S|-k|T|+d_{G-S}(T)-2m \geq 0$. If $s = 4k+2m-5$ and $t = 4k+2m-1$, then at least one vertex in T has degree at least 2 since $n = s+t$ and t is odd. Thus, $k|S|-k|T|+d_{G-S}(T)-2m \geq k(4k+2m-5) - k(4k+2m-1) + (4k+2m-1) + 1 - 2m = 0$, which is a contradiction. If $n = 8k+4m-7$, then $s \geq \frac{n}{2}-2$ implies $s \geq 4k+2m-5$ and $t \leq 4k+2m-2$. Thus, $k|S|-k|T|+d_{G-S}(T)-2m \geq k(4k+2m-5) - k(4k+2m-2) + (4k+2m-2) - 2m = k-2 \geq 0$, which is a contradiction.

Case 2.2. $0 \leq d_1 \leq k-2$ **and** $d_2 = k-1$. In this subcase, $p \leq d_1+1 \leq k-1$. $s+k-1+d_1 \geq |N_G(x_1) \cup N_G(x_2)| \geq \frac{n}{2} \geq 4k+2m-3$ since $s+k+d_1-1$ is integer, i.e., $n \leq 2s+2k-2+2d_1$ and $s \geq 3k+2m-d_1-2$. Thus,

$$\begin{aligned}
& k|S| - k|T| + d_{G-S}(T) - 2m \\
\geq\ & ks - kt + d_1 p + d_2(t-p) - 2m \\
\geq\ & ks - k(n-s) + (d_1-k+1)(d_1+1) + (k-1)(n-s) - 2m \\
=\ & (k+1)s - n - 2m - k + 1 + d_1^2 + (2-k)d_1 \\
\geq\ & (k+1)s - (2k+2s-2+2d_1) - 2m - k + 1 + d_1^2 + (2-k)d_1 \\
=\ & (k-1)s - 3k - 2m + 3 + d_1^2 - kd_1 \\
\geq\ & (k-1)(3k+2m-d_1-2) - 3k - 2m + 3 + d_1^2 - kd_1 \\
=\ & 3k^2 + (2k-4)m - 8k + 5 + d_1^2 - (2k-1)d_1 \\
\geq\ & 3k^2 - 8k + 5 + (k-2)^2 - (2k-1)(k-2) \\
=\ & 2k^2 - 7k + 7 > 0,
\end{aligned}$$

which is a contradiction.

Case 2.3. $0 \le d_1 \le d_2 \le k-2$. In this subcase, $k-1-d_2 \ge 1$, $n-s-t \ge 0$. So, $(k-1-d_2)(n-s-t) > ks-kt+d_1p+d_2(t-p)-2m$. Thus, $(k-d_2)(n-s)-ks > (d_1-d_2)p+(n-s-t)-2m \ge (d_1-d_2)(d_1+1)+(n-s-t)-2m \ge (d_1-d_2)(d_1+1)-2m$, i.e.,

$$(k-d_2)(n-s)-ks \ge (d_1-d_2)(d_1+1)-2m+1. \tag{2}$$

Since $n \ge 8k+4m-7$, we have

$$d_2\frac{n}{2} \ge d_2(4k+2m-\frac{7}{2}). \tag{3}$$

By $s+d_1+d_2 \ge \frac{n}{2}$, we have

$$(s-\frac{n}{2})(2k-d_2) \ge -(d_1+d_2)(2k-d_2). \tag{4}$$

Adding (2),(3) and (4), we get

$$\begin{aligned} 0 &\ge d_1^2+d_2^2+2kd_2-2kd_1+d_1-\frac{9}{2}d_2-2m+1+2md_2 \\ &\ge d_1^2+d_2^2+d_1-\frac{9}{2}d_2-2m+1+2md_2 \\ &= (d_1+\frac{1}{2})^2+(d_2-(\frac{9}{4}-m))^2-m^2+\frac{5m}{2}-\frac{69}{16}. \end{aligned}$$

So,

$$(d_2-(\frac{9}{4}-m))^2 \le m^2-\frac{5m}{2}+\frac{69}{16}-\frac{1}{4},$$

which implies

$$d_2 \le \sqrt{m^2-\frac{5m}{2}+\frac{65}{16}}+(\frac{9}{4}-m).$$

Let

$$f(m) = \sqrt{m^2-\frac{5m}{2}+\frac{65}{16}}+(\frac{9}{4}-m).$$

Then

$$f'(m) = \frac{m-\frac{5}{4}}{\sqrt{m^2-\frac{5m}{2}+\frac{65}{16}}}-1 < 0.$$

That is to say, $f(m)$ is a monotonically decreasing function, $d_2 \le f(0) = \frac{\sqrt{65}+9}{4} = 4$. Therefore, $0 \le d_1 \le d_2 \le 4$.

If (d_1, d_2) =(0,4), (1,4), (2,4), (3,4), (4,4), (3,3), (2,3), (1,3), (0,3), (2,2), (1,2), (0,2), (0,1), we can check that $d_1^2 + d_2^2 + 2kd_2 - 2kd_1 + d_1 - \frac{9}{2}d_2 - 2m + 1 + 2md_2 > 0$ since $k \geq 2$.

If $d_1 = d_2 = 1$, then a contradiction can be found by a discussion similar to that in Case 2.1 for k=2.

If $d_1 = d_2 = 0$, then $s \geq \frac{n}{2} - d_1 - d_2 = \frac{n}{2}$ and $t \leq \frac{n}{2}$. Thus, $k|S| - k|T| + d_{G-S}(T) - (\sum_{x \in T} d_H(x) - e_H(S,T)) \geq 0$, which is a contradiction.

Thus, we complete the proof of Theorem 5. □

Remark 1. We construct some graphs to show that the bounds in the Theorem 5 are best possible.

First, $\delta(G) \geq k+m$ cannot be replaced by $k+m-1$. Otherwise, choose a vertex v such that $d(v) = k+m-1$. Delete m edges incident to v, then the resulting graph has $\delta(G) = k-1$, which has no fractional k-factor by the definition.

Let $G = K_{4k+2m-4} \vee (4k+2m-3)K_1$. Then $n = 8k+4m-7$, $\delta(G) = 4k+2m-4 > k+m$, but $|N_G(x_1) \cup N_G(x_2)| = 4k+2m-4 < \frac{n}{2}$ for each non-adjacent vertex x_1 and x_2 in $(4k+2m-3)K_1$. Let $S = K_{4k+2m-4}$ and $T = (4k+2m-3)K_1$. Then $d_{G-S}(T) = 0$ and $\sum_{x \in T} d_H(x) - e_H(S,T) = 0$. We have $k|S| + \sum_{x \in T} d_{G-S}(x) - k|T| - (\sum_{x \in T} d_H(x) - e_H(S,T)) = -k < 0$. Thus, G is not a fractional (k,m)-deleted graph, and the condition $|N_G(x_1) \cup N_G(x_2)| \geq \frac{n}{2}$ is sharp.

Let $G = K_{4k+2m-6} \vee (2k+m-1)K_2$. Then $n = 8k+4m-8$, $\delta(G) = 4k+2m-6 \geq k+m$ and $|N_G(x_1) \cup N_G(x_2)| = 4k+2m-4 = \frac{n}{2}$ for any non-adjacent vertices x_1 and x_2 in G. Let $S = K_{4k+2m-6}$ and $T = (2k+m-1)K_2$. Let H be the set of m edges such that $H \subseteq (2k+m-1)K_2$, then $\sum_{x \in T} d_H(x) - e_H(S,T) = 2m$ and $\sum_{x \in T} d_{G-S}(x) = 4k+2m-2$. We have, $k|S| + \sum_{x \in T} d_{G-S}(x) - k|T| - (\sum_{x \in T} d_H(x) - e_H(S,T)) = k(4k+2m-6) - k(4k+2m-2) + (4k+2m-2) - 2m = -2 < 0$. Thus, G is not fractional (k,m)-deleted graph. Therefore, the condition $n \geq 8k+4m-7$

is sharp.

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