On a Class of Cyclic Bursts in Array Codes

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Abstract. In this paper, the notion of cyclic bursts in array codes equipped with a non-Hamming metric [13] as a generalization of classical cyclic bursts [5] is introduced and some bounds are obtained on the parameters of array codes for the detection and correction of cyclic burst array errors.

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1. Introduction

In a classical coding setting, codes are subsets (or subspaces) of ambient space F_q^n and are investigated with respect to the Hamming metric [11]. Also, array codes having 2-dimensional arrays as code vectors have been studied by many authors [1, 6, 14, 17] etc. Recently in [13], m-metric array codes which are subsets (or subspaces) of linear space of all m by s matrices $\mathrm{Mat}_{m\times s}(F_q)$ with entries from a finite field F_q endowed with a non-Hamming metric (viz. m-metric) were introduced and some bounds on code parameters were obtained. This newly defined non-Hamming metric gained the attention of several mathematicians as a result of which there has been a recent growth of interest and research in m-metric array codes (e.g. [3, 4, 7-10, 15-16]).

Here is a model of an information transmission for which array coding is useful and the non-Hamming metric defined in [13] is the natural quality characteristic of a code. Suppose that a sender transmits messages, each being an s-tuple of m-tuples of q-ary symbols over m parallel channels. We assume that there is an interfering noise in the channels which creates errors in the transmitted message. An important and practical situation is when errors are not scattered randomly in the code array but are in cluster form and are confined to a subarray of the code array. These errors arise, for example, due to lightning and thunder in deep space and satellite communications. Motivated by this idea, the author introduced the notion of bursts [7] in array coding. In this paper, we introduce a wider class of bursts viz. cyclic bursts (or end-around bursts) in array codes equipped with m-metric and obtain some bounds on the parameters of array codes for the detection and correction of cyclic burst array errors. The study of cyclic bursts is useful in situations when the same message is repeated a number of times.

2. Definitions and Notations

Let F_q be a finite field of q elements. Let $\mathrm{Mat}_{m\times s}(F_q)$ denote the linear space of all $m \times s$ matrices with entries from F_q. An m-metric array code is a subset of $\mathrm{Mat}_{m\times s}(F_q)$ and a linear m-metric array code is an F_q−linear subspace of $\mathrm{Mat}_{m\times s}(F_q)$. Note that the space $\mathrm{Mat}_{m\times s}(F_q)$ is identifiable with the space F_q^{ms}. Every matrix in $\mathrm{Mat}_{m\times s}(F_q)$ can be represented as a $1 \times ms$ vector by writing the first row of matrix followed by second row and so on. Similarly, every vector in F_q^{ms} can be represented as an $m \times s$ matrix in $\mathrm{Mat}_{m\times s}(F_q)$ by separating the co-ordinates of the vector into m groups of s-coordinates.

There are two equivalent ways of defining the non-Hamming weight and metric on the space $\mathrm{Mat}_{m\times s}(F_q)$ viz. row weight and column weight [4, 13]. We consider the row weight definition which runs as follows:

Let $Y \in \mathrm{Mat}_{1\times s}(F_q)$ with $Y = (y_1, y_2, \cdots, y_s)$. Define the row weight (or weight) of Y as

$$wt_\rho(Y) = \begin{cases} \max\ \{\ i \mid y_i \neq 0\} & \text{if } Y \neq 0 \\ 0 & \text{if } Y = 0. \end{cases}$$

Extending the definitions of wt_ρ to the class of $m \times s$ matrices as

$$wt_\rho(A) = \sum_{i=1}^{m} wt_\rho(R_i)$$

where $A = \begin{bmatrix} R_1 \\ R_2 \\ \cdots \\ R_m \end{bmatrix} \in \text{Mat}_{m\times s}(F_q)$ and R_i denotes the i^{th} row of A. Then wt_ρ satisfies $0 \leq wt_\rho(A) \leq n(= ms)\ \forall\ A \in \text{Mat}_{m\times s}(F_q)$ and determines a metric on $\text{Mat}_{m\times s}(F_q)$ known as row-metric or m-metric or RT metric.

In this paper, we take distance and weight in the sense of row-metric. Also, we define a number s^* in $\text{Mat}_{m\times s}(F_q)$ as follows:

$$s^* = \begin{cases} \frac{s}{2} & \text{when } s \text{ is even} \\ \frac{s+1}{2} & \text{when } s \text{ is odd.} \end{cases}$$

3. Cyclic Bursts in m-Metric Array Codes

We now define cyclic bursts in m-metric array codes:

Definition 3.1. A cyclic burst of order pr(or $p \times r$)$(1 \leq p \leq m, 1 \leq r \leq s)$ in the space $\text{Mat}_{m\times s}(F_q)$ is an $m \times s$ matrix in which all the nonzero entries are confined to some cyclic $p \times r$ submatrix which has non-zero first and last rows as well as non-zero first and last columns, where by a cyclic $p \times r$ submatrix, we mean a $p \times r$ submatrix such that number of starting column positions for the $p \times r$ submatrix in the $m \times s$ matrix is s i.e. it is possible to comeback at the beginning while enumerating the r columns of the $p \times r$ cyclic submatrix, or equivalently, a cyclic submatrix of a matrix is a submatrix which can be considered as wrapping around a cylinder.

Remark 3.1.

(1) For $m = p = 1$, Definition 3.1 reduces to the definition of cyclic burst for classical codes [5].

(2) The class of usual bursts [7] in $\text{Mat}_{m\times s}(F_q)$ is a subclass of the class of cyclic bursts.

Definition 3.2. A cyclic burst of order pr or less $(1 \leq p \leq m, 1 \leq r \leq s)$ in the space $\text{Mat}_{m\times s}(F_q)$ is a burst of order cd(or $c \times d$) where $1 \leq c \leq p \leq m$ and $1 \leq d \leq r \leq s$.

Example 3.1. Consider the linear space $\mathrm{Mat}_{3\times 3}(F_2)$. Then all cyclic bursts of order 2×2 are given by:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We now obtain a bound for the correction of cyclic burst errors in linear m-metric array codes.

Theorem 3.1. *An (n, k) linear m-metric array code $V \subseteq Mat_{m\times s}(F_q)$ where $n = ms$ that corrects all cyclic bursts of order $pr(1 \leq p \leq m, 1 \leq r \leq s^*)$ must satisfy*

$$q^{n-k} \geq 1 + E^{p\times r}_{m\times s}(F_q), \tag{1}$$

where $E^{p\times r}_{m\times s}(F_q)$ *is the number of cyclic bursts of order* $pr(1 \le p \le m, 1 \le r \le s^*)$ *in* $Mat_{m\times s}(F_q)$ *and is given by*

$$E^{p\times r}_{m\times s}(F_q) = \begin{cases} ms(q-1) & \text{if } p=1,\ r=1, \\ ms(q-1)^2q^{r-2} & \text{if } p=1, r\ge 2, \\ (m-p+1)s(q-1)^2q^{p-2} & \text{if } p\ge 2, r=1 \\ (m-p+1)sq^{r(p-2)}\times & \\ \times\Big[(q^r-1)^2-2(q^{r-1}-1)^2q^{2-p} & \\ +(q^{r-2}-1)^2q^{4-2p}\Big] & \text{if } \ p\ge 2,\ r\ge 2. \end{cases} \tag{2}$$

Proof. Consider a cyclic burst $A \in \text{Mat}_{m\times s}(F_q)$ of order $pr(1 \le p \le m, 1 \le r \le s^*)$. Let B be the $p\times r$ cyclic nonzero submatrix of A such that all the nonzero entries of A are confined to B with first and last rows as well as first and last columns of B nonzero. There are four cases depending upon the values of p and r.

Case 1. When $p=1,\ r=1$.

In this case, number of starting positions for the 1×1 nonzero cyclic submatrix B in $m\times s$ matrix A is ms and these ms positions can be filled by $(q-1)$ nonzero elements from F_q. Therefore, number of cyclic bursts of order 1×1 in $\text{Mat}_{m\times s}(F_q)$ is given by

$$E^{1\times 1}_{m\times s}(F_q) = ms(q-1).$$

Case 2. When $p=1,\ r\ge 2$.

In this case, number of starting positions for the $1\times r$ nonzero cyclic submatrix B in $m\times s$ matrix A is ms and entries in the $1\times r$ submatrix B can be selected in $(q-1)^2q^{r-2}$ ways. Therefore, number of cyclic bursts of order $1\times r$ in $\text{Mat}_{m\times s}(F_q)$ is given by

$$E^{1\times r}_{m\times s}(F_q) = ms(q-1)^2q^{r-2}.$$

Case 3. When $p\ge 2,\ r=1$.

In this case, number of starting positions for the $p \times 1$ nonzero column submatrix B in $m \times s$ matrix A is $(m-p+1)s$ and entries in the $p \times 1$ submtarix B can be selected in $(q-1)^2 q^{p-2}$ ways. Therefore, number of cyclic bursts of order $p \times 1$ in $\mathrm{Mat}_{m\times s}(F_q)$ is given by

$$E^{p\times 1}_{m\times s}(F_q) = (m-p+1)s(q-1)^2 q^{p-2}.$$

Case 4. When $p \geq 2,\ r \geq 2$.

In this case, we first compute the number of ways in which rows of B can be selected with the partial constraint of burst that is first and last rows of B to be nonzero. This can be done in

$$(q^r-1)^2 q^{r(p-2)} \tag{3}$$

ways as each of first and last rows of B can be selected in (q^r-1) ways and each of the intermediate $(p-2)$ rows can be selected in q^r ways. To take care of the fact that the first and last columns of B are also nonzero, we compute the number of ways enumerated in (3) which give rise either first or last (or both) columns as zero and this number is given by

$$2(q^{r-1}-1)^2 q^{(r-1)(p-2)} - (q^{r-2}-1)^2 q^{(r-2)(p-2)}. \tag{4}$$

Now subtracting (4) from (3) gives the number of ways in which rows of B can be selected with the full constraint of burst and is given by

$$\begin{aligned} & (3)-(4) \\ = \ & q^{r(p-2)}\Big[(q^r-1)^2 - 2(q^{r-1}-1)^2 q^{2-p} + (q^{r-2}-1)^2 q^{4-2p}\Big]. \end{aligned} \tag{5}$$

Since number of starting positions for the cyclic submatrix B of order $pr(2 \leq p \leq m, 2 \leq r \leq s^*)$ in the matrix $A(A \in \mathrm{Mat}_{m\times s}(F_q))$ are $(m-p+1)s$, therefore, number of cyclic bursts of order $pr(2 \leq p \leq m, 2 \leq r \leq s^*)$ in $\mathrm{Mat}_{m\times s}(F_q)$ is obtained by multiplying (5) with $(m-p+1)s$ and is given by

$$\begin{aligned} E^{p\times r}_{m\times s}(F_q) &= (m-p+1)sq^{r(p-2)} \times \\ &\times \Big[(q^r-1)^2 - 2(q^{r-1}-1)^2 q^{2-p} + (q^{r-2}-1)^2 q^{4-2p}\Big]. \end{aligned}$$

Combining the four cases, we get (2).

Now, since the linear m-metric array code $V \subseteq \text{Mat}_{m\times s}(F_q)$ corrects all cyclic bursts of order $pr(1 \leq p \leq m, 1 \leq r \leq s^*)$, therefore, all the cyclic bursts of order $pr(1 \leq p \leq m, 1 \leq r \leq s^*)$ including the null $m \times s$ matrix must belong to different cosets of the standard array. Since number of available cosets $= q^{n-k}$. Therefore, we must have

$$q^{n-k} \geq 1 + E_{m\times s}^{p\times r}(F_q)$$

where $E_{m\times s}^{p\times r}(F_q)$ is given by (2) and we get (1). □

Remark 3.2.

(i) Take $m = s = 3, p = r = 2$ and $q = 2$ in $E_{m\times s}^{p\times r}(F_q)$ computed in (2). We get $E_{3\times 3}^{2\times 2}(F_2) = 6 \times 7 = 42$ and these 42 cyclic bursts of order 2×2 in $\text{Mat}_{3\times 3}(F_2)$ are listed in Example 3.1.

(ii) Take $m = s = 3, p = 1, r = 2$ and $q = 2$ in $E_{m\times s}^{p\times r}(F_q)$ computed in (2). We get $E_{3\times 3}^{1\times 2}(F_2) = 3 \times 3 = 9$ and these 9 cyclic bursts of order 1×2 in $\text{Mat}_{3\times 3}(F_2)$ are listed below:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Remark 3.3. If $s^* < r \leq s$ then Theorem 3.1 holds good subject to that the bursts are counted with their multiplicities i.e. if an $m \times s$ array is a burst of order $p \times r$ starting at the i^{th} column and is also a burst of same order starting at the j^{th} column where $i \neq j$, then such an array will be of burst of order $p \times r$ of multiplicity 2. For example, if we take $m = p = 1, s = 4, r = 3$ and $q = 2$, then the vector 0101 is a burst of order 1×3 of multiplicity 2 since it is burst of order 1×3 which starts at the second position and is also a burst of same order starting from the fourth position. Similarly, 1010 is a burst of order 1×3 of multiplicity 2.

Now, we prove Fire's bound in linear m-metric array codes for cyclic burst error correction.

Theorem 3.2. *(Fire's bound) The number of parity check digits required for an (n, k) linear m-metric array code $V \subseteq Mat_{m\times s}(F_q)$ where $n = ms$, that corrects all cyclic bursts of order pr or less $(1 \le p \le m, 1 \le r \le s^*)$ is at least*

$$log_q\left[1 + \sum_{c=1}^{p}\sum_{d=1}^{r} E_{m\times s}^{c\times d}(F_q)\right], \tag{6}$$

where $E_{m\times s}^{c\times d}(F_q)$ is given by (2).

Proof. Follows directly from Theorem 3.1 and Definition 3.2. □

4. Cyclic Bursts with Weight Constraint in m-Metric Array Codes

In this section, we obtain a lower bound on the number of parity check digits required to correct all cyclic bursts of order pr or less $(1 \le p \le m, 1 \le r \le s^*)$ in $\text{Mat}_{m\times s}(F_q)$ having weight (or row weight) w or less $(1 \le w \le ms)$.

The bound obtained is analogous to the Hamming bound for random error correction [13]. To prove the bound, we first give a definition.

Definition 4.1. Let a, b and s be positive integers. Then $a + b(mod\ s)$ is defined as

$$a + b(mod\ s) = \begin{cases} a + b & \text{if } \ a + b \le s \\ (a + b) - s & \text{if } \ a + b > s. \end{cases}$$

Note. The mod function as defined in Definition 4.1 coincides with the well known standard mod function for $0 \le a + b < 2s$.

Throughout this section, all addition operations pertaining to column numbers of an $m \times s$ matrix are taken with respect to Definition 4.1.

Definition 4.2. Define a function χ; $\mathbf{N} \times \mathbf{N} \rightarrow \{0, 1\}$ as

$$\chi(w, j) = \begin{cases} 0 & \text{if } \ w < j \\ 1 & \text{if } \ w \ge j. \end{cases}$$

We now prove a lemma that enumerates the number of cyclic bursts of order $pr(1 \leq p \leq m, 1 \leq r \leq s^*)$ having row weight w or less.

Lemma 4.1. *The number of cyclic bursts of order* $pr(1 \leq p \leq m, 1 \leq r \leq s^*)$ *in* $Mat_{m\times s}(F_q)$ *having row weight* w *or less* $(1 \leq w \leq ms)$ *is given by*

$$E_{m\times s}^{p\times r}(F_q, w) = \begin{cases} m \times \min(w,s) \times (q-1) \quad \text{if } p = r = 1, \\ m \times \Big[\max\{0, \min(w-r+1, s-r+1)\} \\ + \sum_{j=s-r+2}^{s} \chi(w,j)\Big](q-1)^2 q^{r-2} \quad \text{if } p = 1, r \geq 2, \\ (m-p+1) \sum_{j=1}^{\min(\lfloor \frac{w}{2} \rfloor, s)} \sum_{\substack{\eta=0: \\ \eta j \leq w-2j}}^{p-2} (q-1)^2 \times \\ \times \binom{p-2}{\eta}(q-1)^{\eta} \qquad \text{if } p \geq 2, r = 1, \\ (m-p+1)\sum_{j=1}^{s}\Big[(Q_{j,j+r-1}^{p} - 2Q_{j,j+r-1}^{p-1} + Q_{j,j+r-1}^{p-2}) \\ -(Q_{j,j+r-2}^{p} - 2Q_{j,j+r-2}^{p-1} + Q_{j,j+r-2}^{p-2}) \\ -(Q_{j+1,j+r-1}^{p} - 2Q_{j+1,j+r-1}^{p-1} + Q_{j+1,j+r-1}^{p-2})\Big] \\ \qquad \text{if } p \geq 2, r \geq 2, \end{cases} \tag{7}$$

where $Q_{j,j+r-1}^{p}$, *for* $1 \leq j \leq s-r+1$, *is given by*

$$Q_{j,j+r-1}^{p} = \sum_{k_j, k_{j+1}, \cdots, k_{j+r-1}} \frac{p!}{\prod_{l=0}^{r-1} k_{j+l}! \left(p - \sum_{l=0}^{r-1} k_{j+l}\right)!} \times \\ \times \left(\frac{q-1}{q}\right)^{\sum_{l=0}^{r-1} k_{j+l}} (q)^{\sum_{l=0}^{r-1}(l+1)k_{j+l}}, \tag{8}$$

satisfying

$$k_j, k_{j+1}, \cdots, k_{j+r-1} \geq 0,$$

$$\sum_{l=0}^{r-1} k_{j+l} \leq p,$$

$$\sum_{l=0}^{r-1} (j+l) k_{j+l} \leq w, \tag{9}$$

and for $s-r+1 < j \leq s, Q^p_{j,j+r-1}$ *(here* $j+r-1$ *is taken according to Definition 4.1) is given by*

$$Q^p_{j,j+r-1} = \sum_{k_u,k_v} \frac{p!}{\prod_{u=1}^{(r-1)-(s-j)} k_u! \prod_{v=j}^{s} k_v! \left(p - \sum_{u=1}^{(r-1)-(s-j)} k_u + \sum_{v=j}^{s} k_v\right)!} \times$$
$$\times \left(\frac{q-1}{q}\right)^{\sum_{u=1}^{(r-1)-(s-j)} k_u} \times q^{\sum_{u=1}^{(r-1)-(s-j)} u k_u} \times$$
$$\times \left(q-1\right)^{\sum_{v=j}^{s} k_v} \times (q)^{\sum_{v=j}^{s} [r-(s-v+1)] k_v} \tag{10}$$

satisfying

$$k_u, k_v \geq 0 \quad \forall \;\; 1 \leq u \leq (r-1)-(s-j) \;\; \textit{and} \;\; j \leq v \leq s,$$

$$\sum_{u=1}^{(r-1)-(s-j)} k_u + \sum_{v=j}^{s} k_v \leq p,$$
$$\sum_{u=1}^{(r-1)-(s-j)} u k_u + \sum_{v=j}^{s} v k_v \leq w. \tag{11}$$

(Note that $1 \leq w \leq ms$ is feasible only for the case $p = r = 1$, otherwise we must have $2 \leq w \leq ms$.)

Proof. Consider a cyclic burst $A = \begin{bmatrix} A_1 \\ A_2 \\ \cdots \\ A_m \end{bmatrix}$ where $A_i = (a_{i_1}, a_{i_2}, \cdots, a_{i_s})$, of order $pr(1 \leq p \leq m, 1 \leq r \leq s^*)$ having row weight w or less $(1 \leq w \leq ms)$. Let B be the cyclic $p \times r$ nonzero submatrix of A such that all the nonzero entries of A are confined to B with first and last rows as well as first and last columns of B nonzero. There are four cases depending upon the values of p and r.

Case 1. When $p = 1,\ r = 1$.

In this case, number of starting positions for the 1×1 nonzero submatrix B in $m \times s$ matrix A is $m \times \min(w, s)$ and these $m \times \min(w, s)$ positions can be filled by $(q-1)$ nonzero elements from F_q. Therefore, number of cyclic bursts of order 1×1 having row weight w or less in $\mathrm{Mat}_{m\times s}(F_q)$ is given by

$$E^{1\times 1}_{m\times s}(F_q, w) = m \times \min(w, s) \times (q-1).$$

Case 2. When $p = 1,\ r \geq 2$.

In this case, number of starting positions for the $1 \times r$ nonzero cyclic submatrix B in $m \times s$ matrix A is $m \times \left[\max\{0, \min(w-r+1, s-r+1)\} + \sum_{j=s-r+2}^{s} \chi(w,j)\right]$ and entries in the $1 \times r$ submatrix B can be selected in $(q-1)^2 q^{r-2}$ ways as the first and last components of the single rowed cyclic submatrix B can be chosen in $(q-1)^2$ ways and intermediate $(r-2)$ components can be chosen in q^{r-2} ways. Therefore, number of cyclic bursts of order $1 \times r$ having row weight w or less in $\mathrm{Mat}_{m\times s}(F_q)$ is given by

$$\begin{aligned} E^{1\times r}_{m\times s}(F_q, w) &= m \times \left[\max\{0, \min(w-r+1, s-r+1)\} + \sum_{j=s-r+2}^{s} \chi(w,j)\right] \\ &\quad \times (q-1)^2 q^{r-2}. \end{aligned}$$

Case 3. When $p \geq 2,\ r = 1$.

In this case, the $p \times 1$ nonzero column vector B can have (i, j) as its starting positions in $m \times s$ matrix A where i can vary from 1 to $(m-p+1)$ and j can vary from 1 to $\min([w/2], s)$. With (i, j) as the starting position of $p \times 1$ nonzero column matrix B, entries in B can be filled in

$$\sum_{\substack{\eta=0: \\ \eta j \leq w-2j}}^{p-2} (q-1)^2 \binom{p-2}{\eta} (q-1)^{\eta}$$

ways as first and last components of the column matrix B can be chosen in $(q-1)^2$ ways and intermediate $(p-2)$ components can be chosen in $\sum_{\eta=0}^{p-2}\binom{p-2}{\eta}(q-1)^{\eta}$ ways subject to constraint $\eta j \leq w-2j$ as $2j$ row weight has already been taken from the first and last components. Therefore, number of cyclic bursts of order $p \times 1$ having row weight w or less in $Mat_{m\times s}(F_q)$ is given by

$$E^{p\times 1}_{m\times s}(F_q, w) = (m-p+1)\sum_{j=1}^{\min(\lfloor \frac{w}{2}\rfloor, s)}\sum_{\substack{\eta=0:\\ \eta j\leq w-2j}}^{p-2}(q-1)^2\binom{p-2}{\eta}(q-1)^{\eta}.$$

Case 4. When $p \geq 2,\ r \geq 2$.

In this case, let the $p \times r$ nonzero cyclic submatrix B starts at the $(i,j)^{th}$ position in matrix A. Let $Q^{p}_{j,j+r-1}$ (for $1 \leq j \leq s-r+1$) denote the number of cyclic submatrices of order $p \times r$ starting at the $(i,j)^{th}$ position in matrix A and having row weight $j, j+1, \cdots, j+r-1$ with $k_j, k_{j+1}, \cdots, k_{j+r-1}$ occurences respectively such that total row weight of the submatrix is less than or equal to w. Also, $Q^{p}_{j,j+r-1(mods)}$ (for $s-r+1 < j \leq s$) denote the number of cyclic submatrices of order $p \times r$ starting at the $(i,j)^{th}$ position in matrix A and having row weight $1, 2, \cdots, (r-1)-(s-j), j, j+1, \cdots, s$ with $k_1, k_2, \cdots, k_{(r-1)-(s-j)}, k_j, k_{j+1}, \cdots, k_s$ occurences respectively with total row weight less than or equal to w. $Q^{p-1}_{j,j+r-1}$ counts the submatrices with the same property as above but with either first row or last row as zero. $Q^{p-2}_{j,j+r-1}$ counts the submatrices with the same property as above but with both first row and last rows as zero. Then the number of ways in which p rows of submatrix B can be selected is given by

$$\begin{aligned}&(Q^{p}_{j,j+r-1} - 2Q^{p-1}_{j,j+r-1} + Q^{p-2}_{j,j+r-1}) - (Q^{p}_{j,j+r-2} - 2Q^{p-1}_{j,j+r-2} + Q^{p-2}_{j,j+r-2})\\ &\qquad -(Q^{p}_{j+1,j+r-1} - 2Q^{p-1}_{j+1,j+r-1} + Q^{p-2}_{j+1,j+r-1})\end{aligned} \tag{12}$$

where $Q^{p}_{j,j+r-1}$, for $1 \leq j \leq s-r+1$, is given by

$$Q^{p}_{j,j+r-1} = \sum_{k_j,k_{j+1},\cdots,k_{j+r-1}} \frac{p!}{\prod_{l=0}^{r-1} k_{j+l}!\left(p-\sum_{l=0}^{r-1} k_{j+l}\right)!}\times$$

$$\times \left(\frac{q-1}{q}\right)^{\sum_{l=0}^{r-1} k_{j+l}} (q)^{\sum_{l=0}^{r-1}(l+1)k_{j+l}},$$

satisfying

$$k_j, k_{j+1}, \cdots, k_{j+r-1} \geq 0,$$

$$\sum_{l=0}^{r-1} k_{j+l} \leq p,$$

$$\sum_{l=0}^{r-1} (j+l) k_{j+l} \leq w,$$

and for $s-r+1 < j \leq s$, $Q^p_{j,j+r-1}$ is given by

$$Q^p_{j,j+r-1} = \sum_{k_u, k_v} \frac{p!}{\prod_{u=1}^{(r-1)-(s-j)} k_u! \prod_{v=j}^{s} k_v! \left(p - \sum_{u=1}^{(r-1)-(s-j)} k_u + \sum_{v=j}^{s} k_v\right)!} \times$$

$$\times \left(\frac{q-1}{q}\right)^{\sum_{u=1}^{(r-1)-(s-j)} k_u} \times (q)^{\sum_{u=1}^{(r-1)-(s-j)} u k_u} \times$$

$$\times \left(q-1\right)^{\sum_{v=j}^{s} k_v} \times (q)^{\sum_{v=j}^{s}[r-(s-v+1)]k_v}$$

satisfying

$$k_u, k_v \geq 0 \quad \forall \; 1 \leq u \leq (r-1)-(s-j) \;\; and \;\; j \leq v \leq s,$$

$$\sum_{u=1}^{(r-1)-(s-j)} k_u + \sum_{v=j}^{s} k_v \leq p,$$

$$\sum_{u=1}^{(r-1)-(s-j)} u k_u + \sum_{v=j}^{s} v k_v \leq w.$$

Now in the starting position (i, j) of the cyclic submatrix B, i can vary from 1 to $(m-p+1)$ and j can vary from 1 to s, therefore, summing (12) over

i and j, we get number of cyclic bursts of order pr $(2 \leq p \leq m, 2 \leq r \leq s)$ having row weight w or less $(2 \leq w \leq ms)$ and is given by

$$\begin{aligned} E^{p\times r}_{m\times s}(F_q, w) &= \sum_{i=1}^{(m-p+1)} \sum_{j=1}^{s} \Big[(Q^p_{j,j+r-1} - 2Q^{p-1}_{j,j+r-1} + Q^{p-2}_{j,j+r-1}) \\ &\quad -(Q^p_{j,j+r-2} - 2Q^{p-1}_{j,j+r-2} + Q^{p-2}_{j,j+r-2}) \\ &\quad -(Q^p_{j+1,j+r-1} - 2Q^{p-1}_{j+1,j+r-1} + Q^{p-2}_{j+1,j+r-1}) \Big] \\ &= (m-p+1)\sum_{j=1}^{s} \Big[(Q^p_{j,j+r-1} - 2Q^{p-1}_{j,j+r-1} + Q^{p-2}_{j,j+r-1}) \\ &\quad -(Q^p_{j,j+r-2} - 2Q^{p-1}_{j,j+r-2} + Q^{p-2}_{j,j+r-1}) \\ &\quad -(Q^p_{j+1,j+r-1} - 2Q^{p-1}_{j+1,j+r-1} + Q^{p-2}_{j+1,j+r-1}) \Big], \end{aligned}$$

where $Q^p_{j,j+r-1}$ (for $1 \leq j \leq s-r+1$) is given by (8) satisfying (9) and for $s-r+1 < j \leq s, Q^p_{j,j+r-1}$ is given by (10) satisfying (11). □

Remark 4.1. If $s^* < r \leq s$, then also Lemma 4.1. holds good subject to that the burst are counted with their multiplicities.

Example 4.1. Take $m = p = 2$, $r = 2$, $s = 3$, $q = 2$ and $w = 6$. Then number of cyclic bursts of order 2×2 having row weight 6 or less in $\mathrm{Mat}_{2\times3}(F_2)$ is given by (using Lemma 4.1):

$$\begin{aligned} E^{2\times2}_{2\times3}(F_2, 6) &= \sum_{j=1}^{3} \Big[(Q^2_{j,j+1} - 2Q^1_{j,j+1} + Q^0_{j,j+1}) - (Q^2_{j,j} - 2Q^1_{j,j} + Q^0_{j,j}) \\ &\quad -(Q^2_{j+1,j+1} - 2Q^1_{j+1,j+1} + Q^0_{j+1,j+1}) \Big] \\ &= \Big[(Q^2_{1,2} - 2Q^1_{1,2} + Q^0_{1,2}) - (Q^2_{1,1} - 2Q^1_{1,1} + Q^0_{1,1}) \\ &\quad -(Q^2_{2,2} - 2Q^1_{2,2} + Q^0_{2,2}) \Big] \\ &\quad + \Big[(Q^2_{2,3} - 2Q^1_{2,3} + Q^0_{2,3}) - (Q^2_{2,2} - 2Q^1_{2,2} + Q^0_{2,2}) \\ &\quad -(Q^2_{3,3} - 2Q^1_{3,3} + Q^0_{3,3}) \Big] \\ &\quad + \Big[(Q^2_{3,1} - 2Q^1_{3,1} + Q^0_{3,1}) - (Q^2_{3,3} - 2Q^1_{3,3} + Q^0_{3,3}) \end{aligned}$$

$$
\begin{aligned}
&\quad -(Q^2_{1,1} - 2Q^1_{1,1} + Q^0_{1,1})\Big] \\
&= L + M + N
\end{aligned}
$$

where

$$
\begin{aligned}
L &= (Q^2_{1,2} - 2Q^1_{1,2} + Q^0_{1,2}) - (Q^2_{1,1} - 2Q^1_{1,1} + Q^0_{1,1}) \\
&\quad -(Q^2_{2,2} - 2Q^1_{2,2} + Q^0_{2,2}), \\
M &= (Q^2_{2,3} - 2Q^1_{2,3} + Q^0_{2,3}) - (Q^2_{2,2} - 2Q^1_{2,2} + Q^0_{2,2}) \\
&\quad -(Q^2_{3,3} - 2Q^1_{3,3} + Q^0_{3,3}),
\end{aligned}
$$

and

$$
\begin{aligned}
N &= (Q^2_{3,1} - 2Q^1_{3,1} + Q^0_{3,1}) - (Q^2_{3,3} - 2Q^1_{3,3} + Q^0_{3,3}) \\
&\quad -(Q^2_{1,1} - 2Q^1_{1,1} + Q^0_{1,1}).
\end{aligned}
$$

Now,

$$
Q^2_{1,2} = \sum_{k_1,k_2} \frac{2!}{k_1!k_2!(2-(k_1+k_2))!} \left(\frac{1}{2}\right)^{k_1+k_2} 2^{k_1+2k_2}
$$

subject to

$$
\begin{aligned}
k_1, k_2 &\geq 0, \\
k_1 + k_2 &\leq 2, \qquad\qquad (13) \\
k_1 + 2k_2 &\leq 6.
\end{aligned}
$$

The feasible solutions for (k_1, k_2) satisfying (13) are given by

$$
(k_1, k_2) = (0,0), (1,0), (1,1), (2,0), (0,1), (0,2).
$$

Therefore,

$$
\begin{aligned}
Q^2_{1,2} &= 1 + \frac{2!}{1!0!1!}\left(\frac{1}{2}\right)^1 2^1 + \frac{2!}{1!1!0!}\left(\frac{1}{2}\right)^2 2^3 \\
&\quad + \frac{2!}{2!0!0!}\left(\frac{1}{2}\right)^2 2^2 + \frac{2!}{0!1!1!}\left(\frac{1}{2}\right)^1 2^2 \\
&\quad + \frac{2!}{0!2!0!}\left(\frac{1}{2}\right)^2 2^4 \\
&= 1 + 2 + 4 + 1 + 4 + 4 = 16.
\end{aligned}
$$

Again,

$$Q^1_{1,2} = \sum_{k_1,k_2} \frac{1!}{k_1!k_2!(1-(k_1+k_2)!} \left(\frac{1}{2}\right)^{k_1+k_2} 2^{k_1+2k_2}$$

subject to

$$\begin{aligned} k_1, k_2 &\geq 0, \\ k_1 + k_2 &\leq 1, \\ k_1 + 2k_2 &\leq 6. \end{aligned} \tag{14}$$

The feasible solutions for (k_1, k_2) satisfying (14) are given by

$$(k_1, k_2) = (0,0), (1,0), (0,1).$$

Therefore,

$$\begin{aligned} Q^1_{1,2} &= 1 + \frac{1!}{1!0!0!}\left(\frac{1}{2}\right)^1 2^1 + \frac{1!}{0!1!0!}\left(\frac{1}{2}\right)^1 2^2 \\ &= 1 + 1 + 2 = 4. \end{aligned}$$

A similar computation gives $Q^0_{1,2} = 1, Q^2_{1,1} = 4, Q^1_{1,1} = 2, Q^0_{1,1} = 1$, $Q^2_{2,2} = 4, Q^1_{2,2} = 2, Q^0_{2,2} = 1$.

Therefore,

$$Q^2_{1,2} - 2Q^1_{1,2} + Q^0_{1,2} = 16 - (2 \times 4) + 1 = 9, \tag{15}$$

$$Q^2_{1,1} - 2Q^1_{1,1} + Q^0_{1,1} = 4 - 4 + 1 = 1, \tag{16}$$

$$Q^2_{2,2} - 2Q^1_{2,2} + Q^0_{2,2} = 4 - 4 + 1 = 1 \tag{17}$$

Using (15), (16) and (17) in the expression for L, we get

$$L = 9 - 1 - 1 = 7.$$

For the computation of M, we compute

$$Q^2_{2,3} = 16, Q^1_{2,3} = 4, Q^0_{2,3} = 1, Q^2_{3,3} = 4, Q^1_{3,3} = 2, Q^0_{3,3} = 1.$$

Therefore, M is given by

$$\begin{aligned} M &= (Q^2_{2,3} - 2Q^1_{2,3} + Q^0_{2,3}) - (Q^2_{2,2} - 2Q^1_{2,2} + Q^0_{2,2}) \\ &\quad -(Q^2_{3,3} - 2Q^1_{3,3} + Q^0_{3,3}) \\ &= (16 - (2 \times 4) + 1) - (4 - (2 \times 2) + 1) - (4 - (2 \times 2) + 1) \\ &= 9 - 1 - 1 = 7. \end{aligned}$$

Now in the expression for N, we compute $Q^2_{3,1}, Q^1_{3,1}$ and $Q^0_{3,1}$ using (10) and are computed as follows:

$$\begin{aligned} Q^2_{3,1} &= \sum_{k_1,k_3} \frac{p!}{k_1!k_3!(p-(k_1+k_3)!} \left(\frac{1}{2}\right)^{k_1} \times 2^{k_1} \times 2^{k_3} \\ &= \sum_{k_1,k_3} \frac{p!}{k_1!k_3!(p-(k_1+k_3)!} 2^{k_3} \end{aligned}$$

subject to

$$\begin{aligned} k_1, k_3 &\geq 0, \\ k_1 + k_3 &\leq 2, \\ k_1 + 3k_3 &\leq 6. \end{aligned} \tag{18}$$

The feasible solutions for (k_1, k_3) satisfying (18) are given by

$$(k_1, k_3) = (0,0), (0,1), (1,0), (2,0), (0,2), (1,1).$$

Therefore,

$$\begin{aligned} Q^2_{3,1} &= 1 + \frac{2!}{0!1!1!} 2^1 + \frac{2!}{1!0!1!} 2^0 + \frac{2!}{2!0!0!} 2^0 + \frac{2!}{0!2!0!} 2^2 + \frac{2!}{1!1!1!} 2^1 \\ &= 1+4+2+1+4+4 = 16. \end{aligned}$$

A similar computation gives $Q^1_{3,1} = 4, Q^0_{3,1} = 1$.

Therefore,

$$Q^2_{3,1} - 2Q^1_{3,1} + Q^0_{3,1} = 16 - (2 \times 4) + 1 = 9.$$

Thus

$$\begin{aligned} N &= (Q^2_{3,1} - 2Q^1_{3,1} + Q^0_{3,1}) - (Q^2_{3,3} - 2Q^1_{3,3} + Q^0_{3,3}) \\ &\quad -(Q^2_{1,1} - 2Q^1_{1,1} + Q^0_{1,1}) \\ &= 9 - 1 - 1 = 7. \end{aligned}$$

Hence

$$E^{2\times 2}_{2\times 3}(F_2, 6) = L + M + N = 7 + 7 + 7 = 21.$$

These 21 cyclic bursts of order 2×2 in $\mathrm{Mat}_{2\times 3}(F_2)$ having row weight 6 or less are listed below:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Example 4.2. Take $m = 4, s = 2, p = 4, r = 2, q = 2$ and $w = 3$ in Lemma 4.1. Note that in this case $r > \lceil \frac{s}{2} \rceil$, therefore, in view of Remark 4.1, cyclic burst are counted with their multiplicities. The number of cyclic bursts of order 4×2 having row weight 3 or less (counted with their multiplicities) in $\mathrm{Mat}_{4\times 2}(F_2)$ is given by:

$$\begin{aligned}
E_{4\times 2}^{4\times 2}(F_2, 3) &= \sum_{j=1}^{2} \Big[(Q_{j,j+1}^4 - 2Q_{j,j+1}^3 + Q_{j,j+1}^2) - (Q_{j,j}^4 - 2Q_{j,j}^3 + Q_{j,j}^2) \\
&\quad -(Q_{j+1,j+1}^4 - 2Q_{j+1,j+1}^3 + Q_{j+1,j+1}^2) \Big] \\
&= \Big[(Q_{1,2}^4 - 2Q_{1,2}^3 + Q_{1,2}^2) - (Q_{1,1}^4 - 2Q_{1,1}^3 + Q_{1,1}^2) \\
&\quad -(Q_{2,2}^4 - 2Q_{2,2}^3 + Q_{2,2}^2) \Big] \\
&\quad + \Big[(Q_{2,1}^4 - 2Q_{2,1}^3 + Q_{2,1}^2) - (Q_{2,2}^4 - 2Q_{2,2}^3 + Q_{2,2}^2) \\
&\quad -(Q_{1,1}^4 - 2Q_{1,1}^3 + Q_{1,1}^2) \Big] \\
&= L + M
\end{aligned}$$

where

$$L = (Q_{1,2}^4 - 2Q_{1,2}^3 + Q_{1,2}^2) - (Q_{1,1}^4 - 2Q_{1,1}^3 + Q_{1,1}^2)$$

$$
\begin{aligned}
& -(Q^4_{2,2} - 2Q^3_{2,2} + Q^2_{2,2}) \\
M &= (Q^4_{2,1} - 2Q^3_{2,1} + Q^2_{2,1}) - (Q^4_{2,2} - 2Q^3_{2,2} + Q^2_{2,2}) \\
& -(Q^4_{1,1} - 2Q^3_{1,1} + Q^2_{1,1}).
\end{aligned}
$$

Now,

$$Q^4_{1,2} = \sum_{k_1,k_2} \frac{4!}{k_1!k_2!(4-(k_1+k_2)!} \left(\frac{1}{2}\right)^{k_1+k_2} 2^{k_1+2k_2},$$

subject to

$$
\begin{aligned}
k_1, k_2 &\geq 0, \\
k_1 + k_2 &\leq 4, \\
k_1 + 2k_2 &\leq 3.
\end{aligned} \tag{19}
$$

The feasible solutions for (k_1, k_2) satisfying (19) are given by

$$(k_1, k_2) = (0,0), (1,0), (2,0), (3,0), (0,1), (1,1).$$

Therefore,

$$
\begin{aligned}
Q^4_{1,2} &= 1 + \frac{4!}{1!0!3!}\left(\frac{1}{2}\right)^1 2^1 + \frac{4!}{2!0!2!}\left(\frac{1}{2}\right)^2 2^2 \\
&+ \frac{4!}{3!0!1!}\left(\frac{1}{2}\right)^3 2^3 + \frac{4!}{0!1!3!}\left(\frac{1}{2}\right)^1 2^2 \\
&+ \frac{4!}{1!1!2!}\left(\frac{1}{2}\right)^2 2^3 \\
&= 1 + 4 + 6 + 4 + 8 + 24 = 47.
\end{aligned}
$$

Again,

$$Q^3_{1,2} = \sum_{k_1,k_2} \frac{3!}{k_1!k_2!(3-(k_1+k_2)!} \left(\frac{1}{2}\right)^{k_1+k_2} 2^{k_1+2k_2},$$

subject to

$$
\begin{aligned}
k_1, k_2 &\geq 0, \\
k_1 + k_2 &\leq 3, \\
k_1 + 2k_2 &\leq 3.
\end{aligned} \tag{20}
$$

The feasible solutions for (k_1, k_2) satisfying (20) are given by

$$(k_1, k_2) = (0,0), (1,0), (2,0), (3,0), (0,1), (1,1).$$

Therefore,

$$\begin{aligned}
Q_{1,2}^3 &= 1 + \frac{3!}{1!0!2!}\left(\frac{1}{2}\right)^1 2^1 + \frac{3!}{2!0!1!}\left(\frac{1}{2}\right)^2 2^2 \\
&\quad + \frac{3!}{3!0!0!}\left(\frac{1}{2}\right)^3 2^3 + \frac{3!}{0!1!2!}\left(\frac{1}{2}\right)^1 2^2 + \frac{3!}{1!1!1!}\left(\frac{1}{2}\right)^2 2^3 \\
&= 1+3+3+1+6+12 = 26.
\end{aligned}$$

A similar computation gives

$$Q_{1,2}^2 = 12, Q_{1,1}^4 = 15, Q_{1,1}^3 = 8, Q_{1,1}^2 = 4, Q_{2,2}^4 = 5, Q_{2,2}^3 = 4, Q_{2,2}^2 = 3.$$

Therefore

$$Q_{1,2}^4 - 2Q_{1,2}^3 + Q_{1,2}^2 = 47 - (2 \times 26) + 12 = 7, \tag{21}$$

$$Q_{1,1}^4 - 2Q_{1,1}^3 + Q_{1,1}^2 = 15 - (2 \times 8) + 4 = 3, \tag{22}$$

$$Q_{2,2}^4 - 2Q_{2,2}^3 + Q_{2,2}^2 = 5 - (4 \times 2) + 3 = 0. \tag{23}$$

Thus

$$\begin{aligned}
L &= (Q_{1,2}^4 - 2Q_{1,2}^3 + Q_{1,2}^2) - (Q_{1,1}^4 - 2Q_{1,1}^3 + Q_{1,1}^2) \\
&\quad - (Q_{2,2}^4 - 2Q_{2,2}^3 + Q_{2,2}^2) \\
&= 7 - 3 - 0 = 4. \quad \text{(using (21), (22) and (23))}
\end{aligned}$$

For the computation of M, we compute

$$Q_{2,1}^4 = 47, Q_{2,1}^3 = 26, Q_{2,1}^2 = 12. \tag{24}$$

Therefore, M is given by

$$\begin{aligned}
M &= (Q_{2,1}^4 - 2Q_{2,1}^3 + Q_{2,1}^2) - (Q_{2,2}^4 - 2Q_{2,2}^3 + Q_{2,2}^2) \\
&\quad - (Q_{1,1}^4 - 2Q_{1,1}^3 + Q_{1,1}^2) \\
&= 47 - (2 \times 26) + 12 - 0 - 3 = 7 - 0 - 3 = 4. \quad \text{(using (22), (23) and (24))}
\end{aligned}$$

Hence

$$E^{4\times 2}_{4\times 2}(F_2,3) = L + M = 4 + 4 = 8.$$

These 8 cyclic bursts (counted with their multiplicities) of order 4×2 having row weight 3 or less in $\mathrm{Mat}_{4\times 2}(F_2)$ are given by:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ of multiplicity 2,} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix} \text{ of multiplicity 2,}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ of multiplicity 2,} \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ of multiplicity 2.}$$

Now, we obtain a lower bound on the number of parity check digits for the correction of cyclic bursts of order $pr(1 \le p \le m, 1 \le r \le s)$ having row weight w or less $(1 \le w \le ms)$.

Theorem 4.1. *An (n, k) linear m-metric array code $V \subseteq Mat_{m\times s}(F_q)$ where $n = ms$ that corrects all cyclic bursts of order pr $(1 \le p \le m, 1 \le r \le s^*)$ having row weight w or less $(1 \le w \le ms)$ must satisfy*

$$q^{n-k} \ge 1 + E^{p\times r}_{m\times s}(F_q, w) \tag{25}$$

where $E^{p\times r}_{m\times s}(F_q, w)$ is given by (7) in Lemma 4.1.

Proof. The proof follows from the fact that the number of available cosets must be greater than or equal to the number of correctable error matrices including the null matrix. □

Remark 4.2. Taking $p = m \ge 2, r = s \ge 2$ (in view of Remark 4.1) in Lemma 4.1, we get

$$\begin{aligned} E^{m\times s}_{m\times s}(F_q, w) &= \sum_{j=1}^{s}(Q^{m}_{j,j+s-1} - 2Q^{m-1}_{j,j+s-1} + Q^{m-2}_{j,j+s-1}) \\ &\quad -(Q^{m}_{j,j+s-2} - 2Q^{m-1}_{j,j+s-2} + Q^{m-2}_{j,j+s-2}) \\ &\quad -(Q^{m}_{j+1,j+s-1} - 2Q^{m-1}_{j+1,j+s-1} + Q^{m-2}_{j+1,j+s-1}), \end{aligned} \tag{26}$$

where $Q^m_{j,j+s-1}$ is given by

$$Q^m_{j,j+s-1} = \sum_{k_1,k_2,\cdots,k_s} \frac{p!}{\prod_{i=1}^{s} k_i! \left(m - \sum_{i=1}^{s} k_i\right)!} \times \tag{27}$$
$$\times \left(\frac{q-1}{q}\right)^{\sum_{i=1}^{s} k_i} (q)^{\sum_{i=1}^{s} ik_i},$$

and $k_1, k_2, \cdots k_s$ are integers satisfying

$$\begin{aligned} k_1, k_2, \cdots, k_s &\geq 0, \\ \sum_{i=1}^{s} k_i &\leq m, \\ \sum_{i=1}^{s} ik_i &\leq w. \end{aligned} \tag{28}$$

Further, if we exempt the condition that first and last rows as well as first and last columns in the cyclic burst to be nonzero, then $E^{m\times s}_{m\times s}(F_q, w)$ given in (26) reduces to $s \times \text{vol}(S_w)$, where $\text{vol}(S_w)$ [13] is the volume of sphere of radius w in $\text{Mat}_{m\times s}(F_q) \cong F_q^{ms}$ equipped with row metric and the bound obtained in Theorem 4.1 reduces to

$$q^{n-k} \geq s \ \text{vol}(S_w). \tag{29}$$

Further, if we restrict the number of starting columns in the cyclic bursts of order $m \times s$ to be the first column only, then $E^{m\times s}_{m\times s}(F_q, w)$ given in (26) reduces to

$$E^{m\times s}_{m\times s}(F_q, w) = \text{vol}(S_w), \tag{30}$$

and result in Theorem 4.1 becomes

$$q^{n-k} \geq \text{vol}(S_w). \tag{31}$$

which is the Hamming bound for random error correction obtained in [13] for m-metric array codes.

Now, we obtain a bound for the correction of all cyclic bursts of order pr or less ($1 \le p \le m, 1 \le r \le s^*$) having row weight w or less.

Theorem 4.2. *An (n, k) linear m-metric array code $V \subseteq Mat_{m\times s}(F_q)$ where $n = ms$ that corrects all cyclic bursts of order pr or less ($1 \le p \le m, 1 \le r \le s^*$) having row weight w or less ($1 \le w \le ms$) must satisfy*

$$q^{n-k} \ge 1 + \sum_{c=1}^{p} \sum_{d=1}^{r} E_{m\times s}^{c\times d}(F_q, w)$$

where $E_{m\times s}^{c\times d}(F_q, w)$ is given by Lemma 4.1.

Proof. Follows directly from Theorem 4.1 and Definition 3.2. □

5. Reiger's Bound for Cyclic Bursts in m-Metric Array Codes

To prove Reiger's bound in m-metric array codes, we first prove a lemma:

Lemma 5.1. *An (n, k) linear m-metric array code $V \subseteq Mat_{m\times s}(F_q)$ where $n = ms$ that has no cyclic burst of order pr or less ($1 \le p \le m, 1 \le r \le s$) as a code array must have at least pr parity check digits.*

Proof. Let $V \subseteq \text{Mat}_{m\times s}(F_q)$ be any (n, k) linear m-metric array code ($n = ms$) over F_q. Consider ζ to be the collection of all those elements of $\text{Mat}_{m\times s}(F_q)$ which have all their nonzero components (if at all they have) confined to first p rows and first r columns. Then $\zeta \ne \emptyset$ as null matrix (or null array) belongs to it. We claim that no two arrays in ζ can belong to the same coset of the standard array. Let, if possible, $A, B \in \zeta$ such that $A, B \in$ same coset of the standard array. This gives

$$A - B \in V. \tag{32}$$

But by the nature of the elements of ζ, $A, B \in \zeta \Rightarrow A - B \in \zeta$ and, therefore, $A - B$ is a cyclic burst of order pr or less. Since no cyclic burst of order pr or less is a code array, therefore, we have

$$A - B \notin V. \tag{33}$$

(32) and (33) lead to a contradiction. Therefore, no two members in ζ can be in the same coset of the standard array. Since the number of available cosets $= q^{n-k}$ and number of elements in $\zeta = q^{pr}$. Therefore, we must have

$$q^{n-k} \geq q^{pr}$$
$$\text{or} \quad n-k \geq pr.$$

Hence the lemma. □

Now, we prove Reiger's bound for m-metric array codes.

Theorem 5.1. *(Reiger's bound.) In order to correct all cyclic bursts of order pr or less ($1 \leq p \leq m, 1 \leq r \leq s$), an (n,k) linear m-metric array code $V \subseteq Mat_{m\times s}(F_q)$ where $n = ms$ must have at least 2pr parity check digits. Further, in order to correct all cyclic bursts of order pr or less and simultaneously detect all cyclic bursts of order pl or less ($l \geq r, s \geq l+r$), the code must have at least $p(l+r)$ parity check digits.*

Proof. Consider an array in $\text{Mat}_{m\times s}(F_q)$ which is a cyclic burst of order $2pr$ or less (here $2pr$ means either $p \times 2r$ or $2p \times r$). Such an array can be expressed as a sum or difference of two arrays in $\text{Mat}_{m\times s}(F_q)$ each of which is a cyclic burst of order pr or less. Since the linear m-metric array code corrects all cyclic bursts of order pr or less, therefore, all cyclic bursts of order pr or less must belong to different cosets of the standard array i.e. the difference or sum of two cyclic bursts of order pr or less can not be a code array. This implies that the array under discussion which is a cyclic burst of order $2pr$ or less is expressible as a sum or difference of two cyclic bursts of order pr or less and can not be a code array. Therefore, a cyclic burst of order $2pr$ or less can not be a code array. Thus, by Lemma 5.1, the linear m-metric array code must have at least $2pr$ parity check digits.

Again, consider a cyclic burst of order $p \times (l+r)$ or less. Since the linear m-metric array code corrects all cyclic bursts of order pr or less and simultaneously detects all cyclic bursts of order pl or less ($l \geq r, l+r \leq s$), therefore, all correctable or detectable error arrays must belong to different cosets of the standard array unless the error array is the same. Since a cyclic burst of order $p \times (l+r)$ or less can be expressed as a sum or difference of two arrays, one of which is a cyclic burst of order pr or less and other one is a cyclic burst of order pl or less, therefore, the array which is a cyclic burst of order $p \times (l+r)$ or less can not be a code array. Accordingly the code must have at least $p(l+r)$ parity check digits. □

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References

[1] M. Blaum, P.G. Farrell and H.C.A. van Tilborg, *Array Codes*, in Handbook of Coding Theory, (Ed.: V. Pless and Huffman), Vol. II, Elsevier, North-Holland, 1998, pp.1855-1909.

[2] C.N. Campopiano, *Bounds on Burst Error Correcting Codes*, IRE. Trans., IT-8 (1962), 257-259.

[3] S.T. Dougherty and M.M. Skriganov, *MacWilliams duality and the Rosenbloom-Tsfasman metric*, Moscow Mathematical Journal, 2 (2002), 83-99.

[4] S.T. Dougherty and M.M. Skriganov, *Maximum Distance Separable Codes in the ρ-metric over Arbitrary Alphabets*, Journal of Algebraic Combinatorics, 16 (2002), 71-81.

[5] P. Fire, *A Class of Multiple-Error-Correcting Binary Codes for Non-Independent Errors*, Sylvania Reports RSL-E-2, 1959, Sylvania Reconnaissance Systems, Mountain View, California.

[6] E.M. Gabidulin and V.V. Zanin, *Matrix codes correcting array errors of size* 2×2, International Symp. on Communication Theory and Applications, Ambleside, U.K., 11-16 June, 1993.

[7] S. Jain, *Bursts in m-Metric Array Codes*, Linear Algebra and Its Applications, 418 (2006), 130-141.

[8] S. Jain, *Campopiano-Type Bounds in Non-Hamming Array Coding*, Linear Algebra and Its Applications, 420 (2007), 135-159.

[9] S. Jain, *CT Bursts-From Classical to Array Coding*, Discrete Mathematics, 308-309 (2008), 1489-1499.

[10] S. Jain, *An Algorithmic Approach to Achieve Minimum ρ-Distance at least d in Linear Array Codes*, Kyushu Journal of Mathematics, 62 (2008), 189-200.

[11] W.W. Peterson and E.J. Weldon, Jr., Error Correcting Codes, 2nd Edition, MIT Press, Cambridge, Massachusetts, 1972.

[12] S.H. Reiger, *Codes for the Correction of Clustered Errors*, IRE-Trans., IT-6 (1960), 16-21.

[13] M.Yu. Rosenbloom and M.A. Tsfasman, *Codes for m-metric*, Problems of Information Transmission, 33 (1997), 45-52.

[14] M. Schwartz and T. Etzion, *Two dimensional cluster correcting codes*, IEEE Trans. Info. Theory, 55 (2005), 2121-2132.

[15] I. Siap, *The Complete Weight Enumerator for Codes over $M_{m\times s}(F_q)$*, Lecture Notes on Computer Sciences, 2260 (2001), 20-26.

[16] I. Siap, *A MacWilliams Type Identity*, Turk. J. Maths., 26 (2002), 465-473.

[17] D.C. Voukalis, *A new series of concatenated codes subject to matrix type-B codes*, IEEE. Trans. Info. Theory, Vol. IT-28 (1982), p.522.