DURFEE SQUARE AREAS AND ASSOCIATED PARTITION IDENTITIES

AUBREY BLECHER, ARNOLD KNOPFMACHER‡, AND AUGUSTINE MUNAGI

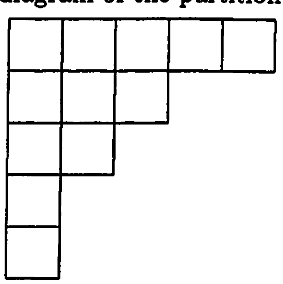
ABSTRACT. A partition of an integer n is a representation $n = a_1 + a_2 + \cdots + a_k$, with integer parts $a_1 \geq a_2 \geq \cdots \geq a_k \geq 1$. The Durfee square is the largest square of points in the graphical representation of a partition. We consider generating functions for the sum of areas of the Durfee squares for various different classes of partitions of n. As a consequence, interesting partition identities are derived. The more general case of Durfee rectangles is also treated as well as the asymptotic growth of the mean area over all partitions of n.

1. INTRODUCTION

A partition λ of a positive integer n is a non-increasing sequence of positive integers whose sum is n. If λ represents a partition of n, we will also write $\lambda \vdash n$ or $|\lambda| = n$. For example, $(20, 7, 5, 5, 4, 4, 1, 1, 1) \vdash 48$.

A partition $(\lambda_1, \ldots, \lambda_k)$ may be represented graphically by a *Young diagram*, which is a left-justified array of boxes, with λ_i boxes in the i^{th} row. This standard definition as well as the one for a Durfee square below are to be found, for example, in [1].

Example: The Young diagram of the partition $(5, 3, 2, 1, 1)$ of 12 is



Date: January 22, 2013.
1991 *Mathematics Subject Classification.* Primary: 05A17; Secondary: 11P57.
Key words and phrases. partitions, Durfee squares, identities.
‡This material is based upon work supported by the National Research Foundation under grant number 2053740.

The largest square in the Young diagram of a partition is called the Durfee square. If there are λ_i boxes in the ith row then its size k can be described as the number of parts λ_i satisfying $\lambda_i \geq i$. In our example, the Durfee square has length 2 and area 4. Durfee squares in partitions have been studied previously with respect to the length of the side in [3, 4, 6] but the area does not seem to have been considered. In [4, 6] the emphasis was on asymptotic estimates for the length, whereas the main aim of the present paper is on deriving interesting partition identities.

In particular, by counting partitions in various classes according to the size of their Durfee squares we obtain the following partition identities.

Theorem 1. *For partitions into odd parts we have*

(1)

$$\sum_{k=0}^{\infty} q^{4k^2+2k}(1 - q^{4k+2} + q^{6k+3}) \prod_{i=1}^{k+1} \frac{1}{1-q^{2i-1}} \prod_{i=1}^{2k+1} \frac{1}{1-q^{2i}} = \prod_{i=1}^{\infty} \frac{1}{1-q^{2i-1}}.$$

For partitions into even parts we have

$$(2)\quad \sum_{k=0}^{\infty} q^{4k^2}(1 - q^{4k+2} + q^{6k+2}) \prod_{i=1}^{k} \frac{1}{1-q^{2i}} \prod_{i=1}^{2k+1} \frac{1}{1-q^{2i}} = \prod_{i=1}^{\infty} \frac{1}{(1-q^{2i})}.$$

For partitions into distinct parts we have

$$(3)\quad \sum_{k=0}^{\infty} q^{\frac{3k^2-k}{2}} \frac{1+q^{2k}}{1+q^k} \prod_{i=1}^{k} \frac{1+q^i}{1-q^i} = \prod_{i=1}^{\infty}(1+q^i).$$

For partitions into distinct odd parts we have

$$(4)\quad \sum_{k=0}^{\infty} q^{8k^2} \frac{(1 - q^{2k+1} + q^{6k+1} - q^{8k+2} + q^{10k+3})}{1-q^{2k+1}} \times \prod_{i=1}^{k}(1+q^{2i-1}) \prod_{i=1}^{2k} \frac{1}{1-q^{2i}} = \prod_{i=1}^{\infty}(1+q^{2i-1}).$$

For partitions into distinct even parts we have

$$(5)\quad \sum_{k=0}^{\infty} q^{8k^2-2k}(1 - q^{2k} + q^{4k} - q^{4k+2} + q^{6k+2} - q^{8k+2} + q^{10k+2}) \prod_{i=1}^{k}(1+q^{2i}) \times \prod_{i=1}^{2k+1} \frac{1}{1-q^{2i}} = \prod_{i=1}^{\infty}(1+q^{2i}).$$

Next, let $a_P(n)$ denote the sum of the areas of the Durfee squares over all partitions of n belonging to a given class P of partitions, and let $A_P(x) = \sum_{n\geq 0} a_P(n) q^n$ be the associated generating function for the sum of the areas. For example, if P is the set of all partitions, then for $n = 4$ we find

$a_P(n) = 1+1+2+1+1 = 6$ from the partitions (4), (3, 1), (2, 2), (2, 1, 1) and (1, 1, 1, 1).

Theorem 2. *The generating functions $A_P(x)$ for partitions according to the sum of areas of their Durfee squares are as follows:*
For partitions into odd parts,

$$(6)\quad \sum_{k=0}^{\infty} q^{4k^2+2k} \prod_{i=1}^{k} \frac{1}{1-q^{2i-1}} \prod_{i=1}^{2k} \frac{1}{1-q^{2i}} \left[4k^2 + \frac{(2k+1)^2 q^{2k+1}}{(1-q^{2k+1})(1-q^{4k+2})}\right].$$

For partitions into even parts,

$$(7)\qquad \sum_{k=0}^{\infty} q^{4k^2} \prod_{i=1}^{k} \frac{1}{1-q^{2i}} \prod_{i=1}^{2k} \frac{1}{1-q^{2i}} \left[4k^2 + \frac{(2k+1)^2 q^{6k+2}}{1-q^{4k+2}}\right].$$

For partitions into distinct parts,

$$(8)\qquad \sum_{k=0}^{\infty} k^2 q^{\frac{3k^2-k}{2}} \frac{1+q^{2k}}{1+q^k} \prod_{i=1}^{k} \frac{1+q^i}{1-q^i}.$$

For partitions into distinct odd parts,

$$(9)\qquad \sum_{k=0}^{\infty} q^{8k^2} \left(4k^2 + \frac{(2k+1)^2 q^{6k+1}\left(1-q^{2k+1}+q^{4k+2}\right)}{1-q^{2k+1}}\right) \times \prod_{i=1}^{k}(1+q^{2i-1}) \prod_{i=1}^{2k} \frac{1}{1-q^{2i}}.$$

For partitions into distinct even parts,

$$(10)\qquad \sum_{k=0}^{\infty} q^{8k^2-2k} \left((2k+1)^2 q^{10k+2} + 4k^2(1-q^{2k}+q^{4k})(1-q^{4k+2})\right) \times \prod_{i=1}^{k}(1+q^{2i}) \prod_{i=1}^{2k+1} \frac{1}{1-q^{2i}}.$$

These results are established in Sections 2 and 3. In Section 4 we generalise the problem to the area of Durfee rectangles. Finally in Section 5 the asymptotic behaviour of the areas of Durfee squares is briefly discussed.

2. The area of Durfee Squares

2.1. Durfee square areas for unrestricted partitions. It is well known that the generating function for partitions with Durfee square of length k is given by

$$\frac{q^{k^2}}{\prod_{i=1}^{k}(1-q^i)^2}.$$

Thus the generating function for the sum of areas of Durfee squares for all partitions of n is given by

$$\sum_{k=1}^{\infty} \frac{k^2 q^{k^2}}{\prod_{i=1}^{k}(1-q^i)^2}. \tag{11}$$

Similarly the generating function, (2.4), for the sum of areas of Durfee squares in self-conjugate partitions of n is given by

$$\sum_{k=1}^{\infty} \frac{k^2 q^{k^2}}{\prod_{i=1}^{k}(1-q^{2i})}.$$

Durfee squares have also been studied with respect to basis partitions (see [4] p.784). If a partition λ has Durfee square of length k, then the rank vector of the partition is $(\lambda_1 - \lambda'_1, \ldots, \lambda_k - \lambda'_k)$ where λ' is the conjugate partition. Although infinitely many partitions have a given rank vector, there is a unique such partition with a minimum sum of its parts. The generating function for basis partitions with Durfee square of size k is known to be

$$\frac{q^{k^2} \prod_{i=1}^{k}(1+q^i)}{\prod_{i=1}^{k}(1-q^i)}.$$

Hence the sum of areas of Durfee squares for basis partitions of n is

$$\sum_{k=1}^{\infty} \frac{k^2 q^{k^2} \prod_{i=1}^{k}(1+q^i)}{\prod_{i=1}^{k}(1-q^i)}.$$

2.2. **Durfee squares for partitions into odd parts.** Consider partitions with odd parts with Durfee square of size $2k$. Then the partition to the right of the Durfee square can be decomposed into a column of length $2k$ followed by a partition into at most $2k$ parts, all even numbers. Beneath the Durfee square we may place a partition into odd parts with largest part at most $2k-1$. The generating function for this case is then

$$q^{4k^2} q^{2k} \prod_{i=1}^{k} \frac{1}{1-q^{2i-1}} \prod_{i=1}^{2k} \frac{1}{1-q^{2i}}.$$

Next consider partitions with odd parts with Durfee square of size $2k+1$. Here the partition to the right of the Durfee square has at most $2k+1$ parts, all even. Beneath the Durfee square we may place a partition into odd parts with largest part at most $2k+1$. The generating function for this case is then

$$q^{(2k+1)^2} \prod_{i=1}^{k+1} \frac{1}{1-q^{2i-1}} \prod_{i=1}^{2k+1} \frac{1}{1-q^{2i}}.$$

Combining the two disjoint cases and summing over k leads to the following identity for partitions into odd parts (equivalently, partitions into distinct parts)

$$\sum_{k=0}^{\infty} q^{4k^2+2k}(1 - q^{4k+2} + q^{6k+3}) \prod_{i=1}^{k+1} \frac{1}{1-q^{2i-1}} \prod_{i=1}^{2k+1} \frac{1}{1-q^{2i}} = \prod_{i=1}^{\infty} \frac{1}{1-q^{2i-1}},$$

which establishes (1) above. Similarly as per (6) above, the generating function for the total areas of the Durfee squares is

$$\sum_{k=0}^{\infty} q^{4k^2+2k} \prod_{i=1}^{k} \frac{1}{1-q^{2i-1}} \prod_{i=1}^{2k} \frac{1}{1-q^{2i}} \left[4k^2 + \frac{(2k+1)^2 q^{2k+1}}{(1-q^{2k+1})(1-q^{4k+2})}\right].$$

2.3. **Durfee squares for partitions into even parts.** Consider partitions with even parts with Durfee square of size $2k$. Then the partition to the right of the Durfee square is a partition into at most $2k$ parts, all even. Beneath the Durfee square we may place a partition into even parts with largest part at most $2k$. The generating function for this case is then

$$q^{4k^2} \prod_{i=1}^{k} \frac{1}{1-q^{2i}} \prod_{i=1}^{2k} \frac{1}{1-q^{2i}}.$$

Next consider partitions with even parts with Durfee square of size $2k+1$. Here the partition to the right of the Durfee square can be decomposed into a column or length $2k+1$ followed by a partition into at most $2k+1$ parts, all even. Beneath the Durfee square we may place a partition into even parts with largest part at most $2k$. The generating function for this case is then

$$q^{(2k+1)^2} q^{(2k+1)} \prod_{i=1}^{k} \frac{1}{1-q^{2i}} \prod_{i=1}^{2k+1} \frac{1}{1-q^{2i}}.$$

Combining the two disjoint cases and summing over k leads to the following identity for partitions into even parts

$$\sum_{k=0}^{\infty} q^{4k^2}(1 - q^{4k+2} + q^{6k+2}) \prod_{i=1}^{k} \frac{1}{1-q^{2i}} \prod_{i=1}^{2k+1} \frac{1}{1-q^{2i}} = \prod_{i=1}^{\infty} \frac{1}{(1-q^{2i})}.$$

This proves (2) above. Similarly as per (7) above, the generating function for the total areas of the Durfee squares is

$$\sum_{k=0}^{\infty} q^{4k^2} \prod_{i=1}^{k} \frac{1}{1-q^{2i}} \prod_{i=1}^{2k} \frac{1}{1-q^{2i}} \left[4k^2 + \frac{(2k+1)^2 q^{6k+2}}{1-q^{4k+2}}\right].$$

3. PARTITIONS INTO DISTINCT PARTS

We consider the same questions as above but now for partitions with distinct parts.

3.1. Durfee square areas. Consider partitions with distinct parts with Durfee square of size k. We need to consider two cases. In the first case, the partition to the right of the Durfee square has exactly k parts. We decompose this into a triangular array of $1+2+\cdots+k$ parts from bottom to top followed by a partition into at most k parts. Beneath the Durfee square we may place a partion into distinct parts with largest part at most k. The generating function for this case is then

$$q^{k^2}\prod_{i=1}^{k}(1+q^i)q^{k(k+1)/2}\prod_{i=1}^{k}\frac{1}{1-q^i}.$$

In the second case the partition to the right of the Durfee square has exactly $k-1$ parts. We decompose this into a triangular array of $1+2+\cdots+k-1$ parts from bottom to top followed by a partition into at most $k-1$ parts. Beneath the Durfee square we may place a partion into distinct parts with largest part at most $k-1$. The generating function for this case is then

$$q^{k^2}\prod_{i=1}^{k-1}(1+q^i)q^{k(k-1)/2}\prod_{i=1}^{k-1}\frac{1}{1-q^i}.$$

Combining the two disjoint cases and summing over k leads to the following identity for partitions into distinct parts

$$\sum_{k=0}^{\infty}q^{\frac{3k^2-k}{2}}\frac{1+q^{2k}}{1+q^k}\prod_{i=1}^{k}\frac{1+q^i}{1-q^i}=\prod_{i=1}^{\infty}(1+q^i),$$

which proves (3). Consequently the generating function (8), for the total areas of the Durfee squares is

$$\sum_{k=0}^{\infty}k^2q^{\frac{3k^2-k}{2}}\frac{1+q^{2k}}{1+q^k}\prod_{i=1}^{k}\frac{1+q^i}{1-q^i}.$$

3.2. Durfee squares for partitions into distinct odd parts. Consider partitions with distinct odd parts with Durfee square of size $2k$. Then the partition to the right of the Durfee square can be decomposed into a triangular array of $1+3+5+\cdots+(2(2k-1)+1)$ parts from bottom to top followed by a partition into at most $2k$ parts, all even numbers. Beneath the Durfee square we may place a partition into distinct odd parts with largest part at most $2k-1$. The generating function for this case is then

$$q^{4k^2}q^{4k^2}\prod_{i=1}^{k}(1+q^{2i-1})\prod_{i=1}^{2k}\frac{1}{1-q^{2i}}.$$

Next consider partitions with distinct odd parts with Durfee square of size $2k+1$. In the first case the partition to the right of the Durfee square has exactly $2k$ parts. We decompose this into a triangular array of $0+2+4+\cdots+2(2k)$ parts from bottom to top followed by a partition into at most $2k$ parts all even numbers. Beneath the Durfee square we may place a partion into distinct odd parts with largest part at most $2k-1$. The generating function for this case is then

$$q^{(2k+1)^2}q^{2k(2k+1)}\prod_{i=1}^{k}(1+q^{2i-1})\prod_{i=1}^{2k}\frac{1}{1-q^{2i}}.$$

In the second case the partition to the right of the Durfee square has exactly $2k+1$ parts. We decompose this into a triangular array of $2+4+\cdots+2(2k+1)$ parts from bottom to top followed by a partition into at most $2k+1$ parts all even numbers. Beneath the Durfee square we may place a partition into distinct odd parts with largest part at most $2k+1$. The generating function for this case is then

$$q^{(2k+1)^2}q^{(2k+1)(2k+2)}\prod_{i=1}^{k+1}(1+q^{2i-1})\prod_{i=1}^{2k+1}\frac{1}{1-q^{2i}}.$$

Combining the three disjoint cases and summing over k gives

$$\sum_{k=0}^{\infty}q^{8k^2}\prod_{i=1}^{k}(1+q^{2i-1})\prod_{i=1}^{2k}\frac{1}{1-q^{2i}}\left(1+q^{6k+1}+\frac{q^{10k+3}(1+q^{2k+1})}{1-q^{4k+2}}\right).$$

After simplifying this leads to the following identity, (namely (4)), for partitions into distinct odd parts

$$\sum_{k=0}^{\infty}q^{8k^2}\frac{(1-q^{2k+1}+q^{6k+1}-q^{8k+2}+q^{10k+3})}{1-q^{2k+1}}\prod_{i=1}^{k}(1+q^{2i-1})\prod_{i=1}^{2k}\frac{1}{1-q^{2i}}$$
$$=\prod_{i=1}^{\infty}(1+q^{2i-1}).$$

Similarly the generating function, (9), for the total areas of the Durfee squares is

$$\sum_{k=0}^{\infty}q^{8k^2}\left(4k^2+(2k+1)^2\left[q^{6k+1}+\frac{q^{10k+3}(1+q^{2k+1})}{1-q^{4k+2}}\right]\right)$$

$$\times \prod_{i=1}^{k}(1+q^{2i-1}) \prod_{i=1}^{2k} \frac{1}{1-q^{2i}}$$

or equivalently

$$\sum_{k=0}^{\infty} q^{8k^2} \left(4k^2 + \frac{(2k+1)^2 q^{6k+1}\left(1-q^{2k+1}+q^{4k+2}\right)}{1-q^{2k+1}}\right)$$
$$\times \prod_{i=1}^{k}(1+q^{2i-1}) \prod_{i=1}^{2k} \frac{1}{1-q^{2i}}.$$

Note that although the number of partitions with distinct odd parts equals the number of self-conjugate partitions of n, the sum of areas of the Durfee squares is different in each case.

3.3. **Durfee squares for partitions into distinct even parts.** Consider partitions with distinct even parts with Durfee square of size $2k+1$. Then the partition to the right of the Durfee square can be decomposed into a triangular array of $1+3+5+\cdots+(2(2k)+1)$ parts from bottom to top followed by a partition into at most $2k+1$ parts, all even. Beneath the Durfee square we may place a partition into distinct even parts with largest part at most $2k$. The generating function for this case is then

$$q^{2(2k+1)^2} \prod_{i=1}^{k}(1+q^{2i}) \prod_{i=1}^{2k+1} \frac{1}{1-q^{2i}}.$$

Next consider partitions with distinct even parts with Durfee square of size $2k$. In the first case the partition to the right of the Durfee square has exactly $2k-1$ parts. We decompose this into a triangular array of $0+2+4+\cdots+2(2k-1)$ parts from bottom to top followed by a partition into at most $2k-1$ parts all even. Beneath the Durfee square we may place a partition into distinct even parts with largest part at most $2k-2$. The generating function for this case is then

$$q^{8k^2-2k} \prod_{i=1}^{k-1}(1+q^{2i}) \prod_{i=1}^{2k-1} \frac{1}{1-q^{2i}}.$$

In the second case the partition to the right of the Durfee square has exactly $2k$ parts. We decompose this into a triangular array of $2+4+\cdots+2(2k)$ parts from bottom to top followed by a partition into at most $2k$ parts all even. Beneath the Durfee square we may place a partition into distinct even parts with largest part at most $2k$. The generating function for this case is then

$$q^{8k^2+2k} \prod_{i=1}^{k}(1+q^{2i}) \prod_{i=1}^{2k} \frac{1}{1-q^{2i}}.$$

Combining the three disjoint cases and summing over k leads to the following identity (i.e. (5)),for partitions into distinct even parts

$$\sum_{k=0}^{\infty} q^{8k^2-2k}(1 - q^{2k} + q^{4k} - q^{4k+2} + q^{6k+2} - q^{8k+2} + q^{10k+2}) \times \prod_{i=1}^{k}(1+q^{2i}) \prod_{i=1}^{2k+1} \frac{1}{1-q^{2i}} = \prod_{i=1}^{\infty}(1+q^{2i}).$$

The generating function for the total areas of the Durfee squares is therefore given by (10) above, namely

$$\sum_{k=0}^{\infty} q^{8k^2-2k} \left((2k+1)^2 q^{10k+2} + 4k^2(1-q^{2k}+q^{4k})(1-q^{4k+2})\right) \times \prod_{i=1}^{k}(1+q^{2i}) \prod_{i=1}^{2k+1} \frac{1}{1-q^{2i}}.$$

4. The area of Durfee Rectangles

We can extend (11) to count areas of vertical and horizontal Durfee rectangles. A vertical (horizontal) Durfee rectangle is the largest filled rectangle in the Young diagram of a partition that contains the Durfee square where the width (height) is the same as that of the Durfee square. We use the decomposition of a partition as a Durfee square of sides k with firstly, a partition with at most k parts to the right of it. Also a rectangle of $r \geq 0$ rows of width k below the Durfee square and beneath that a partition with parts of size at most $k-1$. The area of such a vertical Durfee rectangle is $k^2 + kr$. Then summing over k and r we find the generating function for the sum of vertical Durfee rectangle areas to be

$$\sum_{k=1}^{\infty} \sum_{r=0}^{\infty} \frac{(k^2+kr)q^{k^2+kr}(1-q^k)}{((q;q)_k)^2}.$$

By conjugation of partitions this is also the generating function for the sum of horizontal Durfee rectangle areas.

Now we consider the case of the general Durfee rectangle defined to be the maximum of the vertical and horizontal Durfee rectangles. We pair each partition with its conjugate and compute the rectangle area for that member of the pair for which the vertical rectangle is greater or equal to the horizontal rectangle. We refine the previous decomposition. To the right of the Durfee square we have a rectangle of height k and width l where $0 \leq l \leq r$ and then a partition with at most $k-1$ parts. First we consider the case where $l \neq r$. this gives

$$\sum_{k=1}^{\infty}\sum_{r=0}^{\infty}\sum_{l=0}^{r-1}\frac{(k^2+kr)q^{k^2+kr+kl}}{((q;q)_{k-1})^2}$$

$$=\sum_{k=1}^{\infty}\frac{kq^{k^2}}{((q;q)_{k-1})^2}\frac{q^k\left(q^{2k}+k\left(1-q^{2k}\right)+q^k+1\right)}{\left(1-q^k\right)\left(1-q^{2k}\right)^2}.$$

To count the area for both the partition and its conjugate we need to double the coefficients above. Next we consider the case where the horizontal and vertical rectangles are equal, that is, $l=r$. The area of a partition and its conjugate are both counted with this generating function:

$$\sum_{k=1}^{\infty}\sum_{r=0}^{\infty}\frac{(k^2+kr)q^{k^2+2kr}}{((q;q)_{k-1})^2}=\sum_{k=1}^{\infty}\frac{kq^{k^2}}{((q;q)_{k-1})^2}\frac{k+q^{2k}-kq^{2k}}{(1-q^{2k})^2}.$$

Thus the sum of areas of the general Durfee rectangle over all partitions of n has generating function

$$2\sum_{k=1}^{\infty}\frac{kq^{k^2}}{((q;q)_{k-1})^2}\frac{q^k\left(q^{2k}+k\left(1-q^{2k}\right)+q^k+1\right)}{\left(1-q^k\right)\left(1-q^{2k}\right)^2}$$

$$+\sum_{k=1}^{\infty}\frac{kq^{k^2}}{((q;q)_{k-1})^2}\frac{k+q^{2k}-kq^{2k}}{(1-q^{2k})^2}$$

$$=\sum_{k=1}^{\infty}\frac{kq^{k^2}}{((q;q)_{k-1})^2\left(1-q^{2k}\right)^2}\frac{\left(1+q^k\right)\left(k\left(1-q^{2k}\right)+\left(2+q^k\right)q^k\right)}{1-q^k}.$$

5. Remarks on asymptotic estimates for the area of Durfee squares

In [3, 4, 6] a local limit theorem is established for the size of the Durfee square of a random partition, namely,

$$\mathbb{P}(length=c\sqrt{n}+tn^{1/4})=\frac{n^{-1/4}}{\sqrt{2\pi\sigma^2}}e^{-t^2/(2\sigma^2)}(1+o(1)),$$

uniformly for $t=o(n^{1/12})$, with $c=\frac{\sqrt{6}\log 2}{\pi}$. More generally it is implicit in [3, 4] that a similar limit law holds for any of the special classes of partitions considered in Sections 2 and 3: Uniformly for $t=o(n^{1/12})$,

$$\mathbb{P}(length=c_F\sqrt{n}+tn^{1/4})=\frac{n^{-1/4}}{\sqrt{2\pi\sigma_F^2}}e^{-t^2/(2\sigma_F^2)}(1+o(1)),$$

for certain constants c_F and σ_F^2 which depend on the particular case considered. In particular, this immediately implies a corresponding local limit law for the area in each of these cases. Uniformly for $t = o(n^{1/12})$,

$$\mathbb{P}(area = c_F^2 n + 2c_F t n^{3/4}) = \frac{n^{-1/4}}{\sqrt{2\pi\sigma_F^2}} e^{-t^2/(2\sigma_F^2)}(1 + o(1)),$$

with constants c_F and σ_F^2 as above. We remark that one has a parity restriction in the case of self-conjugate partitions: the area of the Durfee square must be congruent to n (mod 2) (so that local limit theorem only holds for even/odd values, respectively).

A consequence of the local limit laws is that the mean area of the Durfee square tends to $c_F^2 n$ as $n \to \infty$. Or equivalently, the proportion of the area of the Young diagram that lies in the Durfee square tends to $c_F^2 n$ as $n \to \infty$.

Many of the constants c_F have been computed in [3]. Thus, for example we have, in the case of all partitions

$$c_F^2 = \frac{6 \log^2 2}{\pi^2} \approx 0.29208$$

In the case of partitions into distinct parts

$$c_F^2 = \frac{12 \log^2 (1 + \sqrt{5})/2}{\pi^2} \approx 0.28155.$$

In the case of self-conjugate partitions

$$c_F^2 = \frac{6 \log^2 2}{\pi^2} \approx 0.29208.$$

In both the case of partitions into odd parts or partitions into even parts

$$c_F^2 = \frac{12 \log^2 (1 + \sqrt{5})/2}{\pi^2} \approx 0.28155.$$

These constants give good agreement with numerical computations for $n = 1000$. For example, in the case of all partitions we find a mean area of $0.29764n$ when $n = 1000$.

Acknowledgement We thank Stephan Wagner for helpful advice concerning the application of the known asymptotic results on the size of the Durfee square to the case of the area of the Durfee square.

References

[1] G. E. Andrews, The Theory of Partitions, Addison-Wesley, Reading 1976; reprinted, Cambridge University Press, Cambridge, 1984, 1998.

[2] L. Comtet, Advanced Combinatorics. The art of finite and infinite expansions, D. Reidel Publishing Co., The Netherlands, Revised and enlarged edition, 1974.

[3] E.R. Canfield, S. Corteel and C.D. Savage, Durfee polynomials, Electron. J. Combin. 5 (1998), #R32.

[4] E.R. Canfield, From recursions to asymptotics: Durfee and dilogarithmic deductions, Advances in Applied Math. 34 (2005), 768–797.
[5] P. A. MacMahon, Combinatory Analysis, Vol. 2, Cambridge, 1917, reprinted by Chelsea, 1984.
[6] L. R. Mutafchiev, On the size of the Durfee square of a random integer partition, Journal of Computational and Applied Mathematics 142, (2002), 173–184.
[7] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, published electronically at http://www.research.att.com/ njas/sequences/.
[8] R. P. Stanley, Enumerative Combinatorics, Volume I, Wadsworth & Brooks-Cole, Monterey 1986.
[9] H. Wilf, generatingfunctionology, Academic Press Inc, 1994.

Aubrey Blecher, School of Mathematics, University of the Witwatersrand, P. O. Wits, 2050 Johannesburg, South Africa
E-mail address: Aubrey.Blecher@wits.ac.za

Arnold Knopfmacher, The John Knopfmacher Centre for Applicable Analysis and Number Theory, School of Mathematics, University of the Witwatersrand, P. O. Wits, 2050 Johannesburg, South Africa
E-mail address: Arnold.Knopfmacher@wits.ac.za

Augustine Munagi, The John Knopfmacher Centre for Applicable Analysis and Number Theory, University of the Witwatersrand, P. O. Wits, 2050 Johannesburg, South Africa
E-mail address: Augustine.Munagi@wits.ac.za