Degree conditions for fractional (k, m)-deleted graphs *

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Abstract A graph G is called a fractional (k, m)-deleted graph if after deleting any m edges of G the resulting graph admits a fractional k-factor. In this paper, we prove that for $k \geq 2$ and $m \geq 0$, G is a fractional (k, m)-deleted graph if one of the following conditions holds: 1) $n \geq 4k + 4m - 3$, $\delta(G) \geq k + m$, and $\max\{d_G(u), d_G(v)\} \geq \frac{n}{2}$ for each pair of non-adjacent vertices u and v of G; 2) $\delta(G) \geq k + m$, $\sigma_2(G) \geq n$, $n \geq 4k + 4m - 5$ if $(k, m) \neq (3, 0)$ and $n \geq 8$ if $(k, m) = (3, 0)$. The results are best possible in some sense.

Key words: graph, fractional factor, fractional (k, m)-deleted graph, degree condition

1 Introduction

All graphs considered in this paper are finite, loopless, and without multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $x \in V(G)$, the degree and the neighborhood of x in G are denoted by $d_G(x)$ and $N_G(x)$, respectively. For $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S, and $G - S = G[V(G) \setminus S]$. For two vertex-disjoint subsets S and T of G, we use $e_G(S, T)$ to denote the number of edges with one end in S and the other end in T. We denote the minimum degree and the maximum degree of G by $\delta(G)$ and $\Delta(G)$, respectively. The *distance* $d_G(x, y)$ between two vertices x and y is defined to be the length of a shortest path connecting them. Other notation and terminology used but undefined in this paper can be found in [1].

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Let $k \geq 1$ be an integer. A spanning subgraph F of G is called a *k-factor* if $d_F(x) = k$ for each $x \in V(G)$. Let $h : E(G) \rightarrow [0,1]$ be a function. If $\sum_{x \in e} h(e) = k$ for any $x \in V(G)$, then we call $G[F_h]$ a *fractional k-factor* of G with indicator function h where $F_h = \{e \in E(G) : h(e) > 0\}$. Zhou [5] introduced the definition of a fractional (k,m)-deleted graph, that is, a graph G is called a *fractional (k,m)-deleted graph* if removing any m edges from G, the resulting graph has a fractional k-factor. A fractional (k,m)-deleted graph is simply called a fractional k-deleted graph if $m = 1$.

In what follows, we always assume that n is order of G, i.e., $n = |V(G)|$, and G is not complete. Niessen [3] proved that if G is connected graph with $\delta(G) \geq k$, kn is even, $n \geq 8k^2 + 12k + 6$, and $\max\{d_G(x), d_G(y)\} \geq \frac{n}{2}$ for any vertices x and y of G with $d_G(x,y) = 2$, then G has a k-factor. For fractional (k,m)-deleted graphs, we have the following three known results.

Theorem 1 (Zhou and Liu [7]). *Let $k \geq 2$ and $m \geq 0$ be two integers, and let G be a graph of order n with $n \geq 4k^2 + 2k - 6 + \frac{(4k^2+6k-2)m-2}{k-1}$. If $\delta(G) \geq k + m + \frac{m}{k+1}$ and*

$$\max\{d_G(u), d_G(v)\} \geq \frac{n}{2}$$

for any vertices x and y of G with $d_G(x,y) = 2$, then G is a fractional (k,m)-deleted graph.

Theorem 2 (Zhou [5]). *Let $k \geq 2$ and $m \geq 0$ be two integers. Let G be a connected graph of order n with $n \geq 9k - 1 - \sqrt{2(k-1)^2 + 2} + 2(2k+1)m$, $\delta(G) \geq k + m + \frac{(m+1)^2 - 1}{4k}$. If*

$$|N_G(x) \cup N_G(y)| \geq \frac{1}{2}(n + k - 2)$$

for each pair of non-adjacent vertices x, y of G, then G is a fractional (k,m)-deleted graph.

Theorem 3 (Zhou [4]). *Let $k \geq 1$ and $m \geq 1$ be two integers. Let G be a graph of order n with $n \geq 4k - 5 + 2(2k+1)m$. If $\delta(G) \geq \frac{n}{2}$, then G is a fractional (k,m)-deleted graph.*

More results on the topic with fractional (k,m)-deleted graphs can refer to [6]. In this paper, we give the following result:

Theorem 4 *Let $k \geq 2$ and $m \geq 0$ be two integers, and let G be a graph of order n with $n \geq 4k + 4m - 3$, $\delta(G) \geq k + m$. If*

$$\max\{d_G(u), d_G(v)\} \geq \frac{n}{2}$$

for each pair of non-adjacent vertices u and v of G, then G is a fractional (k,m)-deleted graph.

Theorem 4 shows that if we strengthen the condition $d_G(u,v) = 2$ in Theorem 1 to any non-adjacent vertices u and v, then the lower bound of n can be reduced to $O(k)$. Moreover, we will show that this bound is sharp whenever $(k,m) \neq (2,0)$.

Let $\sigma_2(G) = \min\{d_G(u) + d_G(v)\}$ for each pair of non-adjacent vertices u and v of G. Iida and Nishimura [2] studied the existence of factor by virtue of $\sigma_2(G)$, and proved that if $n \geq 4k-5$, kn is even, $\delta(G) \geq k$, and $\sigma_2(G) \geq n$, then G has a k-factor. In this paper, we use $\sigma_2(G)$ to replace the neighborhood condition in Theorem 2 and give the following result:

Theorem 5 *Let $k \geq 2$ and $m \geq 0$ be two integers, and let G be a graph of order n with $n \geq 4k+4m-5$ if $(k,m) \neq (3,0)$ and $n \geq 8$ if $(k,m) = (3,0)$. If $\delta(G) \geq k+m$ and $\sigma_2(G) \geq n$, then G is a fractional (k,m)-deleted graph.*

Also, we will show that Theorem 5 is sharp.

Corollary 6 *Let $k \geq 2$ and $m \geq 0$ be two integers. Let G be a graph of order n with $n \geq 4k+4m-5$. If $\delta(G) \geq \frac{n}{2}$, then G is a fractional (k,m)-deleted graph.*

This corollary is stronger than Theorem 3 if $k \geq 2$ and can be obtained immediately from Theorem 5 unless $(k,m) = (3,0)$. By the Remark 3 in the following section, the result holds for $(k,m) = (3,0)$ and Corollary 6 is sharp.

In order to prove our main results, we need the following lemma which is Lemma 2.2 in [4] and Lemma 2.2 in [5].

Lemma 7 (Zhou [4, 5]). *Let $k \geq 1$ and $m \geq 0$ be two integers, and let G be a graph and H a subgraph of G with m edges. Then G is a fractional (k,m)-deleted graph if and only if*

$$\delta_G(S,T) = k|S| + \sum_{x\in T} d_{G-S}(x) - k|T| \geq \sum_{x \in T} d_H(x) - e_H(S,T)$$

for all disjoint subsets S and T of $V(G)$.

Corollary 8 *Let $k \geq 1$ and $m \geq 0$ be two integers, and let G be a graph and H a subgraph of G with m edges. Then G is a fractional (k,m)-deleted graph if*

$$\delta_G(S,T) = k|S| + \sum_{x\in T} d_{G-S}(x) - k|T| \geq 2m$$

for all disjoint subsets S and T of $V(G)$.

2 Proof of Theorem 4

Suppose that G satisfies the conditions of Theorem 4, but is not a fractional (k, m)-deleted graph. From Lemma 7 and $\sum_{x\in T} d_H(x) - e_H(S,T) \leq 2m$, there exist disjoint subsets S and T of $V(G)$ such that

$$\delta_G(S,T) = k|S| + \sum_{x\in T} d_{G-S}(x) - k|T| \leq 2m - 1. \tag{1}$$

We choose subsets S and T such that $|T|$ is minimum. Obviously, $T \neq \emptyset$.

Claim 1 *$d_{G-S}(x) \leq k - 1$ for any $x \in T$.*

Proof. If $d_{G-S}(x) \geq k$ for some $x \in T$, then the subsets S and $T \setminus \{x\}$ satisfy (1). This contradicts the choice of S and T. □

Let $d_1 = \min\{d_{G-S}(x) : x \in T\}$ and choose $x_1 \in T$ such that $d_{G-S}(x_1) = d_1$. If $T - N_T[x_1] \neq \emptyset$, let $d_2 = \min\{d_{G-S}(x) : x \in T - N_T[x_1]\}$ and choose $x_2 \in T - N_T[x_1]$ such that $d_{G-S}(x_2) = d_2$. So, $d_1 \leq d_2$. Let $|S| = s$, $|T| = t$, $|N_T[x_1]| = p$. Then $p \leq d_1 + 1$, $d_{G-S}(T) \geq d_1 p + d_2(t - p)$ and $ks - kt + d_1 p + d_2(t-p) - 2m \leq k|S| - k|T| + d_{G-S}(T) - 2m < 0$. We have

$$|S| \leq \frac{k|T| - d_{G-S}(T) + (\sum_{x\in T} d_H(x) - e_H(S,T) - 1)}{k} \leq \frac{k|T| + (2m-1)}{k}.$$

Thus, $|S| \leq |T| + \frac{2m-1}{k}$. If $|S| \leq m$, then $|T| = 0$ by Claim 1 and $\delta(G) \geq k + m$, which contradicts $T \neq \emptyset$. So, $m + 1 \leq s \leq t + \frac{2m-1}{k}$. We consider following two cases:

Case 1. $T = N_T[x_1]$. In this case, $t = p \leq d_1 + 1$ and $d_2 = 0$. If $d_1 = k-1$, then $t \leq k$, $k|S| - k|T| + d_{G-S}(T) - 2m \geq ks - kt + d_1 p - 2m = ks - kt + (k-1)t - 2m \geq ks - k - 2m \geq k(m+1) - k - 2m \geq 0$, which contradicts (1). If $0 \leq d_1 \leq k-2$, then $t \leq d_1 + 1 \leq k - 1$. By $\delta(G) \geq k + m$ and $d_G(x_1) \leq s + d_1$, we have $s \geq k + m - d_1$. Thus, $k|S| - k|T| + d_{G-S}(T) - 2m \geq ks - kt + d_1 p - 2m \geq k(k + m - d_1) + (d_1 - k)t - 2m = (k - d_1)(k - t) + km - 2m > 0$, which contradicts (1).

Case 2. $T - N_T[x_1] \neq \emptyset$. We consider following three subcases.

Case 2.1. $d_1 = d_2 = k - 1$. In this subcase, $k|S| - k|T| + d_{G-S}(T) - 2m \geq ks - kt + d_1 p + d_2(t-p) - 2m = ks - kt + (k-1)p + (k-1)(t-p) - 2m = ks - t - 2m \geq 0$, which contradicts (1). In fact, if $ks \leq t + 2m - 1$, then $s \leq \frac{t+2m-1}{k}$ and $s + ks - 2m + 1 \leq s + t \leq n$. Note that x_1, x_2 are not adjacent in G. Thus, $s + k - 1 \geq \max\{d_G(x_1), d_G(x_2)\} \geq \frac{n}{2} \geq \frac{s+sk-2m+1}{2}$. So, $2k \geq (k-1)s + 2(1-m) + 1 \geq (k-1)s + 2(2-s) + 1 = (k-3)s + 5$, i.e., $k \neq 2, 3$ and $s \leq \frac{2k-5}{k-3}$. Then $\frac{2k-5}{k-3} + k - 1 \geq s + k - 1 \geq \max\{d_G(x_1), d_G(x_2)\} \geq \frac{n}{2}$, i.e., $\frac{2k-5}{k-3} + k - 1 \geq 2k + 2m - 1$ since $s + k - 1$ is an integer. Thus, $\frac{2k-5}{k-3} \geq k + 2m \geq k$, which contradicts $k \geq 4$.

Case 2.2. $0 \le d_1 \le k-2$ and $d_2 = k-1$. In this subcase, $p \le d_1+1 \le k-1$ and $s+k-1 \ge \max\{d_G(x_1), d_G(x_2)\} \ge \frac{n}{2} \ge 2k+2m-1$ since $s+k-1$ is an integer, i.e., $n \le 2s+2k-2$ and $s \ge k+2m$. Thus,

$$\begin{aligned}
& k|S| - k|T| + d_{G-S}(T) - 2m \\
\ge\ & ks - kt + d_1p + d_2(t-p) - 2m \\
\ge\ & ks - k(n-s) + (d_1 - k + 1)(d_1+1) + (k-1)(n-s) - 2m \\
=\ & (k+1)s - n - 2m - k + 1 + d_1^2 + (2-k)d_1 \\
\ge\ & (k+1)s - (2k+2s-2) - 2m - k + 1 + (\frac{k-2}{2})^2 + (2-k)\frac{k-2}{2} \\
=\ & (k-1)s - 3k - 2m + 3 + (\frac{k-2}{2})^2 + (2-k)\frac{k-2}{2} \\
\ge\ & (k-1)(k+2m) - 3k - 2m + 3 + (\frac{k-2}{2})^2 + (2-k)\frac{k-2}{2} \\
=\ & \frac{3}{4}k^2 + (2k-4)m - 3k + 2 \\
\ge\ & \frac{3}{4}k^2 - 3k + 2.
\end{aligned}$$

If $k \ge 4$, then $\frac{3}{4}k^2 - 3k + 2 > 0$, which contradicts (1).

If $k = 3$ and $m \ge 1$. We get $\frac{3}{4}k^2 + (2k-4)m - 3k + 2 \ge \frac{3}{4}k^2 - k - 2 > 0$, which contradicts (1). If $k = 3$ and $m = 0$, then $d_2 = 2$, $d_1 = 0$ or 1. If $n \ge 4k-2$, then $s \ge \frac{n}{2} - 2$ and $t \le \frac{n}{2} + 2$. So, we get $k|S| - k|T| + d_{G-S}(T) \ge k(\frac{n}{2} - 2) - k(\frac{n}{2} + 2) + 2(\frac{n}{2} + 2 - 1) \ge 0$, which contradicts (1). If $n = 4k-3$, then $s \ge 2k-3$ and $t \le n-s \le 2k$ since s is an integer. Thus, $k|S| - k|T| + d_{G-S}(T) \ge 0$, which contradicts (1).

In particular, if $k = 2$, then $n \ge 4k - 3 + 4m = 5 + 4m$, $d_1 = 0$, $d_2 = 1$ and $p \le d_1 + 1 = 1$. We need to consider three subcases.

- If $n \ge 7 + 4m$, then $s + 1 = s + k - 1 \ge \max\{d_G(x_1), d_G(x_2)\} \ge \frac{n}{2} \ge 2m + 4$ since $s + k - 1$ is an integer, i.e., $s \ge 2m + 3$. Thus, $k|S| - k|T| + d_{G-S}(T) - 2m \ge (k-1)s - 3k - 2m + 3 + (\frac{k-2}{2})^2 + (2-k)\frac{k-2}{2} \ge 0$, which contradicts (1).
- If $n = 6+4m$, then $s \ge 2+2m$ and $t \le n-s \le 4+2m$. If $t \le 3+2m$, then $k|S| - k|T| + d_{G-S}(T) - 2m \ge ks - kt + d_1p + d_2(t-p) - 2m \ge -2 + (3 + 2m - 1) - 2m = 0$, which contradicts (1). If $t = 4 + 2m$, then at least one vertex in T has degree at least two since t is even and $G = S \cup T$. Thus, $k|S| - k|T| + d_{G-S}(T) - 2m \ge ks - kt + d_1p + d_2(t-p) + 1 - 2m \ge -4 + (4 + 2m) - 2m = 0$, which contradicts (1).
- If $n = 5 + 4m = 4k - 3 + 4m$, then $s \ge 2 + 2m$ and $t \le n - s \le 3 + 2m$. Thus, $k|S| - k|T| + d_{G-S}(T) - 2m \ge ks - kt + d_1p + d_2(t-p) - 2m \ge k(2+2m) - k(3+2m) + (3 + 2m - 1) - 2m = 0$, which contradicts (1).

Case 2.3. $0 \le d_1 \le d_2 \le k-2$. In this subcase, $k - 1 - d_2 \ge 1$ and $n - s - t \ge 0$. So, $(k-1-d_2)(n-s-t) > ks - kt + d_1p + d_2(t-p) - 2m$.

Thus, $(k-d_2)(n-s)-ks > (d_1-d_2)p+(n-s-t)-2m \geq (d_1-d_2)(d_1+1)+(n-s-t)-2m \geq (d_1-d_2)(d_1+1)-2m$, i.e.,

$$(k-d_2)(n-s)-ks \geq (d_1-d_2)(d_1+1)-2m+1. \tag{2}$$

Since $n \geq 4k+4m-3$, we obtain

$$d_2\frac{n}{2} \geq d_2(2k+2m-\frac{3}{2}). \tag{3}$$

By $s+d_2 \geq \frac{n}{2}$, we have

$$(s-\frac{n}{2})(2k-d_2) \geq -d_2(2k-d_2). \tag{4}$$

Add (2), (3) and (4), we get

$$0 \geq d_1^2+d_2^2-d_1d_2+d_1-\frac{5}{2}d_2-2m+1+2md_2.$$

So,

$$(d_1-\frac{d_2-1}{2})^2+(\frac{\sqrt{3}}{2}d_2-\frac{2-2m}{3}\sqrt{3})^2-\frac{4m^2}{3}+\frac{2}{3}m-\frac{7}{12} \leq 0.$$

We have

$$-\sqrt{\frac{16m^2}{9}-\frac{8}{9}m+\frac{7}{9}}+\frac{4-4m}{3} \leq d_2 \leq \sqrt{\frac{16m^2}{9}-\frac{8}{9}m+\frac{7}{9}}+\frac{4-4m}{3}$$

by $(\frac{\sqrt{3}}{2}d_2-\frac{2-2m}{3}\sqrt{3})^2-\frac{4m^2}{3}+\frac{2}{3}m-\frac{7}{12} \leq 0$. Let

$$f(m)=\sqrt{\frac{16m^2}{9}-\frac{8}{9}m+\frac{7}{9}}+\frac{4-4m}{3}.$$

Then

$$f'(m)=\frac{\frac{16m}{9}-\frac{4}{9}}{\sqrt{\frac{16m^2}{9}-\frac{8}{9}m+\frac{7}{9}}}-\frac{4}{3}<0.$$

That is to say, $f(m)$ is a monotonically decreasing function, $d_2 \leq f(0)=\frac{\sqrt{7}+4}{3}=2,\cdots$. Therefore, $0 \leq d_1 \leq d_2 \leq 2$.

If $d_1=d_2=2$. In this case, $s \geq \frac{n}{2}-2$ and $t \leq n-s \leq \frac{n}{2}+2$. Thus, $k|S|-k|T|+d_{G-S}(T)-2m \geq k(\frac{n}{2}-2)-k(\frac{n}{2}+2)+2(\frac{n}{2}+2)-2m>0$, which contradicts (1).

If $d_1=1$ and $d_2=2$. In this case, if $n \geq 4k+4m-2$, then $s \geq \frac{n}{2}-2$ and $t \leq n-s \leq \frac{n}{2}+2$. Thus, $k|S|-k|T|+d_{G-S}(T)-2m \geq k(\frac{n}{2}-2)-k(\frac{n}{2}+2)+2+2\times\frac{n}{2}-2m \geq 0$, which contradicts (1). If $n=4k+4m-3$, then $s \ge 2k+2m-3$ and $t \le n-s \le 2k+2m$ since s is an integer. Thus, $k|S|-k|T|+d_{G-S}(T)-2m \ge k(2k+2m-3)-k(2k+2m)+2+2(2k+2m-2)-2m = k+2m-2 \ge 0$, which contradicts (1).

If $d_1=0$ and $d_2=2$. In this case, if $n \ge 4k+4m-2$, then $s \ge \frac{n}{2}-2$ and $t \le n-s \le \frac{n}{2}+2$. Thus, $k|S|-k|T|+d_{G-S}(T)-2m \ge k(\frac{n}{2}-2)-k(\frac{n}{2}+2)+2(\frac{n}{2}+2-1)-2m \ge 0$, which contradicts (1). If $n=4k+4m-3$, then $s \ge 2k+2m-3$ and $t \le n-s \le 2k+2m$ since s is an integer. Thus, $k|S|-k|T|+d_{G-S}(T)-2m \ge k(2k+2m-3)-k(2k+2m)+2(2k+2m-1)-2m = k+2m-2 \ge 0$, which contradicts (1).

If $d_1=d_2=1$. In this case, $s \ge \frac{n}{2}-d_2 = \frac{n}{2}-1$ and $t \le n-s = \frac{n}{2}+1$. If $n \ge 4k+4m-2$, then $k|S|-k|T|+d_{G-S}(T)-2m \ge k(\frac{n}{2}-1)-k(\frac{n}{2}+1)+\frac{n}{2}+1-2m \ge 0$, which contradicts (1). If $n=4k+4m-3$, then $s \ge \frac{n}{2}-1$ implies $s \ge 2k+2m-2$ and $t \le 2k+2m-1$. Thus, $k|S|-k|T|+d_{G-S}(T)-2m \ge k(2k+2m-2)-(k-1)(2k+2m-1)-2m = k-1 > 0$, which contradicts (1).

If $d_1=0$ and $d_2=1$. In this case, $s \ge \frac{n}{2}-d_2 = \frac{n}{2}-1$, $t \le n-s = \frac{n}{2}+1$ and $p \le d_1+1 = 1$. If $n \ge 4k+4m$, then $k|S|-k|T|+d_{G-S}(T)-2m \ge 0$, which contradicts (1). If $n=4k+4m-1$, then $s \ge 2k+2m-1$ and $t \le 2k+2m$. Thus, $k|S|-k|T|+d_{G-S}(T)-2m \ge k(2k+2m-1)-k(2k+2m)+(2k+2m-1)-2m = k-1 > 0$, which contradicts (1). If $n=4k+4m-2$, then $s \ge \frac{n}{2}-1 = 2k+2m-2$, $t \le 2k+2m$. If $t \le 2k+2m-1$, then $k|S|-k|T|+d_{G-S}(T)-2m \ge k(2k+2m-2)-k(2k+2m-1)+(2k+2m-1-1)-2m = k-2 \ge 0$, which contradicts (1). If $t=2k+2m$. Then at least one vertex in T has degree at least 2 since t is even and $G=S \cup T$. So, $k|S|-k|T|+d_{G-S}(T)-2m \ge k(2k+2m-2)-k(2k+2m)+(2k+2m-1)+1-2m = 0$, which contradicts (1). If $n=4k+4m-3$, then $s \ge \frac{n}{2}-1$ implies $s \ge 2k+2m-2$ and $t \le 2k+2m-1$. Thus, $k|S|-k|T|+d_{G-S}(T)-2m \ge k(2k+2m-2)-k(2k+2m-1)+(2k+2m-1-1)-2m = k-2 \ge 0$, which contradicts (1).

If $d_1=d_2=0$. In this case, $s \ge \frac{n}{2}-d_2 = \frac{n}{2}$ and $t \le \frac{n}{2}$. Thus, $k|S|-k|T|+d_{G-S}(T)-(\sum_{x\in T} d_H(x)-e_H(S,T)) \ge 0$, which contradicts Lemma 7.

This completes the proof of Theorem 4. □

Remark 1. We construct some graphs to show that the bounds in the Theorem 4 are best possible.

First, the condition that $\delta(G) \ge k+m$ cannot be replaced by $k+m-1$. Otherwise, choose a vertex v such that $d(v)=k+m-1$. Delete m edges incident to v, then the resulting graph has $\delta(G)=k-1$, which has no fractional k-factor by the definition.

Let $G = K_{2k+2m-2} \vee (2k+2m-1)K_1$. Then $n=4k+4m-3$, $\delta(G) = 2k+2m-2 \ge k+m$, but $\max\{d_G(u), d_G(v)\} = 2k+2m-2 < \frac{n}{2}$ for each non-adjacent vertices u and v in $(2k+2m-1)K_1$. Let $S=K_{2k+2m-2}$ and $T=(2k+2m-1)K_1$. Then $d_{G-S}(T)=0$ and $\sum_{x\in T} d_H(x)-e_H(S,T) =$

0. We have $k|S| + \sum_{x\in T} d_{G-S}(x) - k|T| - (\sum_{x\in T} d_H(x) - e_H(S,T)) = -k < 0$. So, G is not a fractional (k,m)-deleted graph, the condition that $\max\{d_G(u), d_G(v)\} \geq \frac{n}{2}$ is best possible.

For $k \geq 3$ or $m \geq 1$. Let $G = K_{2k+2m-3} \vee ((k+m-1)K_2 \cup K_1)$. Then $n = 4k+4m-4$, $\delta(G) = 2k+2m-3 \geq k+m$ and $\max\{d_G(u), d_G(v)\} = 2k+2m-2 = \frac{n}{2}$. Let $S = K_{2k+2m-3}$ and $T = (k+m-1)K_2 + K_1$. Let H be the set of m edges such that $H \subseteq (k+m-1)K_2$, then $\sum_{x\in T} d_H(x) - e_H(S,T) = 2m$ and $\sum_{x\in T} d_{G-S}(x) = 2k+2m-2$. We get $k|S| + \sum_{x\in T} d_{G-S}(x) - k|T| - (\sum_{x\in T} d_H(x) - e_H(S,T)) = -2k + (2k+2m-2) - 2m = -2 < 0$. Thus, G is not a fractional (k,m)-deleted graph. Therefore, the condition that $n \geq 4k+4m-3$ is best possible unless $(k,m) = (2,0)$. We will discuss $(k,m) = (2,0)$ in Remark 4.

3 Proof of Theorem 5

Suppose that G satisfies the conditions of Theorem 5, but is not a fractional (k,m)-deleted graph. From Lemma 7 and $\sum_{x\in T} d_H(x) - e_H(S,T) \leq 2m$, there exist disjoint subsets S and T of $V(G)$ such that

$$\delta_G(S,T) = k|S| + \sum_{x\in T} d_{G-S}(x) - k|T| \leq 2m-1. \tag{5}$$

We choose subsets S and T such that $|T|$ is minimal. Obviously, $T \neq \emptyset$. Let d_1, d_2, x_1, x_2, s, t, p as defined before. We have $d_1 \leq d_2$, $p \leq d_1 + 1$, $d_{G-S}(T) \geq d_1 p + d_2(t-p)$ and $ks - kt + d_1 p + d_2(t-p) - 2m \leq k|S| - k|T| + d_{G-S}(T) - 2m < 0$.

Claim 2 $d_{G-S}(x) \leq k-1$ *for any* $x \in T$.

Proof. If $d_{G-S}(x) \geq k$ for some $x \in T$, then the subsets S and $T \setminus \{x\}$ satisfy (5). This contradicts the choice of S and T. □

From what we have discussed in Section 2, we have $m+1 \leq s \leq t + \frac{2m-1}{k}$. We consider following two cases:

Case 1. $T = N_T[x_1]$. The proof can be given similarly to the discussion in Section 2.

Case 2. $T - N_T[x_1] \neq \emptyset$. We consider following three subcases.

Case 2.1. $d_1 = d_2 = k-1$. In this subcase, $k|S| - k|T| + d_{G-S}(T) - 2m \geq ks - kt + d_1 p + d_2(t-p) - 2m = ks - kt + (k-1)p + (k-1)(t-p) - 2m = ks - t - 2m > 0$, which contradicts (5). In fact, if $ks \leq t + 2m$, then $s \leq \frac{t+2m}{k}$ and $s + ks - 2m \leq s + t \leq n$. Note that x_1, x_2 are not adjacent in G. Thus, $2(s+k-1) \geq \sigma_2(G) \geq n \geq s + sk - 2m$. Then $2k \geq (k-1)s + 2(1-m) \geq (k-1)s + 2(2-s) = (k-3)s + 4$, i.e., $k \neq 2, 3$ and $s \leq \frac{2k-4}{k-3}$. Thus, $2(\frac{2k-4}{k-3} + k - 1) \geq 2(s+k-1) \geq \sigma_2(G) \geq n \geq 4k-5+4m$, i.e., $\frac{4}{k-3} \geq 2k - 7 + 4m$. We have $4 \geq (2k - 7 + 4m)(k - 3)$. If $k \geq 5$, then $4 \geq (2k-7+4m)(k-3) \geq 6$, which is a contradiction. If $k = 4$, then $s \leq 2 + \frac{2}{3}m$ by $2(s + k - 1) \geq s + sk - 2m$. Thus, $2(2 + \frac{2}{3}m + k - 1) \geq 2(s+k-1) \geq \sigma_2(G) \geq n \geq 4k - 5 + 4m$. It follows that $2k + \frac{8}{3}m - 7 \leq 0$, contradicting the fact that $k = 4$.

Case 2.2. $0 \leq d_1 \leq k-2$ and $d_2 = k-1$. In this subcase, $p \leq d_1 + 1 \leq k-1$. Since x_1 and x_2 are not adjacent in G, we have $(s+k-1)+(s+d_1) \geq \sigma_2(G) \geq n \geq 4k + 4m - 5$, i.e., $n \leq 2s + k - 1 + d_1$ and $s \geq \frac{3k+4m-d_1-4}{2}$. Thus,

$$\begin{aligned}
& k|S| - k|T| + d_{G-S}(T) - 2m \\
\geq\ & ks - kt + d_1 p + d_2(t-p) - 2m \\
\geq\ & ks - k(n-s) + (d_1 - k + 1)(d_1 + 1) + (k-1)(n-s) - 2m \\
=\ & (k+1)s - n - 2m - k + 1 + d_1^2 + (2-k)d_1 \\
\geq\ & (k+1)s - (2s + k - 1 + d_1) - 2m - k + 1 + d_1^2 + (2-k)d_1 \\
=\ & (k-1)s - 2k + 2 - 2m + d_1^2 + (1-k)d_1 \\
\geq\ & (k-1)\frac{3k+4m-d_1-4}{2} - 2k + 2 - 2m + d_1^2 + (1-k)d_1 \\
=\ & d_1^2 + \frac{3}{2}(1-k)d_1 + (k-1)\frac{3k+4m-4}{2} - 2k + 2 - 2m.
\end{aligned}$$

If $k \geq 5$, then $\frac{3}{4}(k-1) \leq k-2$. So, d_1 may take $\frac{3}{4}(k-1)$. We get

$$\begin{aligned}
& d_1^2 + \frac{3}{2}(1-k)d_1 + (k-1)\frac{3k+4m-4}{2} - 2k + 2 - 2m \\
\geq\ & \frac{9}{16}(k-1)^2 - \frac{9}{8}(k-1)^2 + (k-1)\frac{3k+4m-4}{2} - 2k + 2 - 2m \\
=\ & \frac{15}{16}k^2 - \frac{35}{8}k + 2m(k-2) + \frac{55}{16} \\
\geq\ & \frac{15}{16}k^2 - \frac{35}{8}k + \frac{55}{16} > 0,
\end{aligned}$$

which contradicts (5).

If $k = 2, 3, 4$, then

$$\begin{aligned}
& d_1^2 + \frac{3}{2}(1-k)d_1 + (k-1)\frac{3k+4m-4}{2} - 2k + 2 - 2m \\
\geq\ & (k-2)^2 + \frac{3}{2}(1-k)(k-2) + (k-1)\frac{3k+4m-4}{2} - 2k + 2 - 2m \\
=\ & k^2 - 5k + 2m(k-2) + 5.
\end{aligned}$$

If $k = 4$ or $k = 3$ and $m \geq 1$ or $k = 2$ and $m \geq 2$, then $k^2 - 5k + 2m(k-2) + 5 \geq 0$, which contradicts (5).

In particular, if $k = 3$ and $m = 0$, then $d_2 = 2$, $d_1 = 0$ or 1. If $d_1 = 0$, then $s \geq \frac{n}{2} - 1$ and $t \leq \frac{n}{2} + 1$. Thus, $k|S| - k|T| + d_{G-S}(T) \geq k(\frac{n}{2} - 1) - k(\frac{n}{2} + 1) + 2(\frac{n}{2} + 1 - 1) \geq 2k - 5 > 0$, which contradicts (5). If $d_1 = 1$. Use $n \geq 8 = 4k + 4m - 4$. We get $s \geq \frac{n-3}{2}$ and $t \leq \frac{n+3}{2}$. Thus, $k|S| - k|T| + d_{G-S}(T) - 2m \geq k\frac{n-3}{2} - k\frac{n+3}{2} + 2 + 2(\frac{n+3}{2} - 2) \geq n + 1 - 3k \geq 8 + 1 - 9 = 0$, which contradicts (5).

Assume $k = 2$. Then $d_1 = 0$ and $d_2 = 1$. If $G - S - T \neq \emptyset$, then $t \leq n - s - 1$ and $k|S| - k|T| + d_{G-S}(T) - 2m \geq 2s - 2(n - s - 1) + (n - s - p - 1) - 2m \geq 3s - n - p - 2m + 1 \geq 3s - n - 2m \geq 3s - (2s + 1) - 2m = s - 1 - 2m \geq 0$, which contradicts (5). Suppose $G - S - T = \emptyset$. If $n \geq 4k + 4m - 3 = 5 + 4m$, then $s \geq \frac{n-1}{2}$ and $t \leq \frac{n+1}{2}$. Thus, $k|S| - k|T| + d_{G-S}(T) - 2m \geq k\frac{n-1}{2} - k\frac{n+1}{2} + (\frac{n+1}{2} - 1) - 2m \geq 0$, which contradicts (5). If $n = 4k + 4m - 4 = 4 + 4m$, then $s \geq 2 + 2m$ and $t \leq 2 + 2m$ as s is an integer. Thus, $k|S| - k|T| + d_{G-S}(T) - 2m \geq k(2 + 2m) - k(2 + 2m) + (2 + 2m - 1) - 2m > 0$, which contradicts (5). If $n = 3 + 4m = 4k + 4m - 5$, then $s \geq 2m + 1$ and $t \leq 2m + 2$. If $t \leq 2m + 1$, then $s \geq 2m + 2$ and we have $k|S| - k|T| + d_{G-S}(T) - 2m \geq 0$, which contradicts (5). The last case is $s = 2m+1$ and $t = 2m+2$. Then at least one vertex in T is of degree at least 2 in $G-S$. Thus, $k|S| - k|T| + d_{G-S}(T) - 2m \geq ks - kt + d_1 p + d_2(t-p) - 2m \geq k(2m+1) - k(2m+2) + (2m+2-1) + 1 - 2m = m - 1 = 0$, which contradicts (5).

Case 2.3. $0 \leq d_1 \leq d_2 \leq k - 2$. In this subcase, $k - 1 - d_2 \geq 1$ and $n - s - t \geq 0$. So, $(k - 1 - d_2)(n - s - t) > ks - kt + d_1 p + d_2(t - p) - 2m$. Thus, $(k - d_2)(n - s) - ks > (d_1 - d_2)p + (n - s - t) - 2m \geq (d_1 - d_2)(d_1 + 1) + (n - s - t) - 2m$, i.e.,

$$(k - d_2)(n - s) - ks \geq (d_1 - d_2)(d_1 + 1) - 2m + 1. \tag{6}$$

Since $n \geq 4k + 4m - 5$, we obtain

$$d_2\frac{n}{2} \geq d_2(2k + 2m - \frac{5}{2}). \tag{7}$$

By $s \geq \frac{n-d_1-d_2}{2}$, we have

$$(s - \frac{n}{2})(2k - d_2) \geq -\frac{d_1 + d_2}{2}(2k - d_2). \tag{8}$$

Adding (6), (7) and (8), we get

$$\begin{aligned} 0 &\geq d_1^2 + \frac{d_2^2}{2} - \frac{d_1 d_2}{2} + d_1 - \frac{7}{2}d_2 - 2m + 1 + 2md_2 + (d_2 - d_1)k \\ &\geq d_1^2 + \frac{d_2^2}{2} - \frac{d_1 d_2}{2} + d_1 - \frac{7}{2}d_2 - 2m + 1 + 2md_2 + (d_2 - d_1)(d_2 + 2) \\ &= d_1^2 + \frac{3}{2}d_2^2 - \frac{3}{2}d_1 d_2 - \frac{3}{2}d_2 - d_1 - 2m + 1 + 2md_2. \end{aligned}$$

So,

$$(d_1 - (\frac{3}{4}d_2 + \frac{1}{2}))^2 + (\frac{\sqrt{15}}{4}d_2 + \frac{4m - \frac{9}{2}}{\sqrt{15}})^2 - \frac{16}{15}m^2 + \frac{2}{5}m - \frac{3}{5} \le 0.$$

We have

$$d_2 \le \sqrt{\frac{256}{225}m^2 - \frac{32}{75}m + \frac{16}{25}} + \frac{18 - 16m}{15}$$

by $(\frac{\sqrt{15}}{4}d_2 + \frac{4m-\frac{9}{2}}{\sqrt{15}})^2 - \frac{16}{15}m^2 + \frac{2}{5}m - \frac{3}{5} \le 0$. Let

$$f(m) = \sqrt{\frac{256}{225}m^2 - \frac{32}{75}m + \frac{16}{25}} + \frac{18 - 16m}{15}.$$

Then

$$f'(m) = \frac{\frac{256m}{225} - \frac{16}{75}}{\sqrt{\frac{256}{225}m^2 - \frac{32}{75}m + \frac{16}{25}}} - \frac{16}{15} < 0.$$

Thus, $f(m)$ is monotonically decreasing function, $\max f(m) = f(0) = 2$. So, $0 \le d_1 \le d_2 \le 2$.

If $d_1 = d_2 = 2$. In this case, if $n \ge 4k + 4m - 4$, then $s \ge \frac{n}{2} - 2$ and $t \le n - s \le \frac{n}{2} + 2$. Thus, $k|S| - k|T| + d_{G-S}(T) - 2m \ge k(\frac{n}{2} - 2) - k(\frac{n}{2} + 2) + 2(\frac{n}{2} + 2) - 2m \ge 0$, which contradicts (5). If $n = 4k + 4m - 5$, then $s \ge 2k + 2m - 4$ and $t \le n - s \le 2k + 2m - 1$ since s is an integer. Thus, $k|S| - k|T| + d_{G-S}(T) - 2m \ge k(2k + 2m - 4) - k(2k + 2m - 1) + 2(2k + 2m - 1) - 2m = k + 2m - 2 \ge 0$, which contradicts (5).

If $d_1 = 1$ and $d_2 = 2$. In this case, if $n \ge 4k + 4m - 3$, then $s \ge \frac{n-3}{2}$ and $t \le n - s \le \frac{n+3}{2}$. Thus, $k|S| - k|T| + d_{G-S}(T) - 2m \ge k\frac{n-3}{2} - k\frac{n+3}{2} + 2 + 2\frac{n-1}{2} - 2m \ge k + 2m - 2 \ge 0$, which contradicts (5). If $n = 4k + 4m - 4$, then $s \ge 2k + 2m - 3$ and $t \le n - s \le 2k + 2m - 1$ since s is an integer. Thus, $k|S| - k|T| + d_{G-S}(T) - 2m \ge k(2k + 2m - 3) - k(2k + 2m - 1) + 2 + 2(2k + 2m - 1 - 2) - 2m = 2k + 2m - 4 \ge 0$, which contradicts (5). If $n = 4k + 4m - 5$, then $s \ge 2k + 2m - 4$ and $t \le n - s \le 2k + 2m - 1$. Thus, $k|S| - k|T| + d_{G-S}(T) - 2m \ge k(2k+2m-4) - k(2k+2m-1) + 2 + 2(2k + 2m - 3) - 2m = k + 2m - 4 \ge 0$ if $k \ge 4$ or $m \ge 1$, which contradicts (5). If $k = 2$ and $m = 0$, then $n = 4k + 4m - 5 = 3$. In this case, $\delta(G) \ge k + m = 2$ implies $G = K_3$. Clearly, it has a fractional 2-factor. In particular, $k = 3$ and $m = 0$, then $n \ge 8 = 4k - 4 + 4m$, which contradicts $n = 4k + 4m - 5$.

If $d_1 = 0$ and $d_2 = 2$. In this case, if $n \ge 4k + 4m - 4$, then $s \ge \frac{n}{2} - 1$ and $t \le n - s \le \frac{n}{2} + 1$. Thus, $k|S| - k|T| + d_{G-S}(T) - 2m \ge k(\frac{n}{2} - 1) - k(\frac{n}{2} + 1) + 2(\frac{n}{2} + 1 - 1) - 2m = n - 2k - 2m \ge 2k + 2m - 4 \ge 0$, which contradicts (5). If $n = 4k + 4m - 5$, then $s \ge 2k + 2m - 3$ and $t \le n - s \le 2k + 2m - 2$ since s is an integer. Thus, $k|S| - k|T| + d_{G-S}(T) - 2m \ge k(2k + 2m -$

$3) - k(2k+2m-2) + 2(2k+2m-2-1) - 2m = 3k+2m-6 \geq 0$, which contradicts (5).

If $d_1 = d_2 = 1$. In this case, $s \geq \frac{n}{2} - 1$ and $t \leq n - s \leq \frac{n}{2} + 1$. If $n \geq 4k+4m-2$, then $k|S| - k|T| + d_{G-S}(T) - 2m \geq k(\frac{n}{2}-1) - k(\frac{n}{2}+1) + (\frac{n}{2}+1) - 2m \geq 0$, which contradicts (5). If $n = 4k+4m-3$, then $s \geq 2k+2m-2$ and $t \leq n - s \leq 2k+2m-1$. Thus, $k|S| - k|T| + d_{G-S}(T) - 2m \geq k(2k+2m-2) - k(2k+2m-1) + (2k+2m-1) - 2m = k-1 > 0$, which contradicts (5). If $n = 4k+4m-4$, then $s \geq 2k+2m-3$ and $t \leq 2k+2m-1$. If $s \geq 2k+2m-2$ or $t \leq 2k+2m-2$, then we have $k|S| - k|T| + d_{G-S}(T) - 2m \geq 0$. If $s = 2k+2m-3$ and $t = 2k+2m-1$, then at least one vertex in T is of degree at least 2 in T since t is odd. Thus, $k|S| - k|T| + d_{G-S}(T) - 2m \geq k(2k+2m-3) - k(2k+2m-1) + (2k+2m-1) + 1 - 2m = 0$, which contradicts (5). If $n = 4k+4m-5$, then $s \geq 2k+2m-3$ and $t \leq 2k+2m-2$ since s is an integer. Thus, $k|S| - k|T| + d_{G-S}(T) - 2m \geq k(2k+2m-3) - k(2k+2m-2) + (2k+2m-2) - 2m = k-2 \geq 0$, which contradicts (5).

If $d_1 = 0$ and $d_2 = 1$. In this case, $s \geq \frac{n-1}{2}$, $t \leq n - s = \frac{n+1}{2}$ and $p \leq d_1 + 1 = 1$. Thus, $k|S| - k|T| + d_{G-S}(T) - 2m \geq k(\frac{n-1}{2}) - k(\frac{n+1}{2}) + (\frac{n+1}{2} - 1) - 2m \geq k - 3 \geq 0$ if $k \geq 3$, which contradicts (5). If $k = 2$ and $n \geq 5 + 4m = 4k+4m-3$, then $k|S| - k|T| + d_{G-S}(T) - 2m \geq k\frac{n-1}{2} - k\frac{n+1}{2} + (\frac{n+1}{2} - 1) - 2m \geq k - 2 = 0$, which contradicts (5). If $n = 4 + 4m = 4k+4m-4$, then $s \geq 2+2m$ and $t \leq 2+2m$. Thus, $k|S| - k|T| + d_{G-S}(T) - 2m \geq k(2+2m) - k(2+2m) + (2+2m-1) - 2m > 0$, which contradicts (5). The last case is $k = 2$ and $n = 3+4m = 4k+4m-5$. Then $s \geq 2m+1$ and $t \leq 2m+2$. If $s \geq 2m+2$ or $t \leq 2m+1$, then we get $k|S| - k|T| + d_{G-S}(T) - 2m \geq 0$, which contradicts (5). Otherwise, $s = 2m+1$ and $t = 2m+2$. Then at least one vertex in T has degree at least 2 in T since t is even and $d_1 = 0$. Thus, $k|S| - k|T| + d_{G-S}(T) - 2m \geq 2(2m+1) - 2(2m+2) + (2m+1) + 1 - 2m = 0$, which contradicts (5).

If $d_1 = d_2 = 0$. In this case, $s \geq \frac{n}{2}$ and $t \leq \frac{n}{2}$. Thus, $k|S| - k|T| + d_{G-S}(T) - (\sum_{x \in T} d_H(x) - e_H(S,T)) \geq 0$, which contradicts Lemma 7.

Thus, we complete the proof of Theorem 5. □

Remark 2. We construct some graphs to show that the bounds in the Theorem 5 are best possible. First, the condition that $\delta(G) \geq k+m$ cannot be replaced by $k+m-1$ as discussed in Remark 1.

For $k \geq 3$ or $k = 2$ and $m \geq 1$, let $G = K_{2k+2m-4} \vee (k+m-1)K_2$. Then $n = 4k+4m-6$, $\delta(G) = 2k+2m-3 \geq k+m$ and $\sigma_2(G) = 4k+4m-6 = n$. Let $S = K_{2k+2m-4}$ and $H \subseteq (k+m-1)K_2 = T$. Then $k|S| - k|T| + d_{G-S}(T) - (\sum_{x \in T} d_H(x) - e_H(S,T)) = -2 < 0$. So, G is not a fractional (k,m)-deleted graph. For $(k,m) = (2,0)$ and $|G| = 2 = 4k-6$, then G has no fractional 2-factor. In particular, if $(k,m) = (3,0)$ we consider $G = K_2 \vee (K_2 \cup K_3)$. We can see that $n = 7 = 4k-5$, $\delta(G) = 3$ and $\sigma_2(G) = 7 = n$. But let $S = K_2$ (the first one) and $T = K_2 \cup K_3$, we can check that G has no fractional 3-factor. Therefore, the bound of n is sharp.

If $k \geq 3$ or $m \geq 1$. Let $G = K_{2k+2m-3} \vee (2k+2m-2)K_1$. Then $n = 4k+4m-5$, $\delta(G) = 2k+2m-3 \geq k+m$, but $\sigma_2(G) = 2k+2m-6 = n-1$. Let $S = K_{2k+2m-3}$ and $T = (2k+2m-2)K_1$. Then $d_{G-S}(T) = 0$ and $\sum_{x\in T} d_H(x) - e_H(S,T) = 0$. We have $k|S| + \sum_{x\in T} d_{G-S}(x) - k|T| - (\sum_{x\in T} d_H(x) - e_H(S,T)) = -k < 0$. So, G is not fractional (k,m)-deleted graph. The condition that $\sigma_2(G) \geq n$ is best possible unless $(k,m) = (2,0)$.

Remark 3. Now we consider Corollary 6. Note that the condition $\delta(G) \geq \frac{n}{2}$ implies $\delta(G) \geq k+m$ for $k \geq 2$ and $m \geq 0$. In particular, we consider $(k,m) = (3,0)$ and $n = 7$. From the proof of Theorem 5, we can see that if $n = 7$, then it is easy to derive that $d_1 = 1$ and $d_2 = 2$. However, since $\delta(G) \geq \frac{n}{2}$, we have $\delta(G) \geq 4$ and $s \geq 3$. Then, $\delta_G(S,T) = k|S| + \sum_{x\in T} d_{G-S}(x) - k|T| \geq 3k - 4k + 2 + 4 > 0$. Therefore, $\delta(G) \geq \frac{n}{2}$ is sufficient for G having a fractional 3-factor. By examples in Remark 2, Corollary 6 is sharp.

Remark 4. When $(k,m) = (2,0)$, the condition that $n \geq 4k+4m-3$ in Theorem 4 can be reduced to $n \geq 3$ since $\delta(G) \geq k+m = 2$ implies $n \geq 3$ and when $n = 3$, G is a complete graph. When $n = 4$, $\delta(G) \geq \frac{n}{2} = 2$ and the condition is Corollary 6 holds.

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