Catalan triangle, standard Young words and standard Young tableaux

Carol J. Wang

Department of Mathematics
Beijing Technology and Business University
Beijing 100048, P.R. China

wang_jian@th.btbu.edu.cn

Abstract. In this paper, we introduce a new sequence called standard Young words, which are defined as quaternary words with interesting restrictions. First, we show that the cardinality of standard Young words of length n is related to Catalan triangle sequence and we establish a bijection from the set of standard Young words to the set of pairs of non-intersection lattice paths. Then we set a one-to-one correspondence between the set of standard Young words and the set of standard Young tableaux of two rows, which results in the correspondence between the statistics of standard Young words and standard Young tableaux, such as sign and descents.
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1 Introduction

Let $X = \{a, b, c, d\}$ and let S be the set of words consisting of the letters in X. Let $S_1 \subset S$ be the set of words on X starting with letter a. Suppose that $w = w_1 \cdots w_n \in S$ be a word of length n. The *shift operator* $g^k \colon S \to S$ is defined by

$$g^k(w) = w_1 \cdots w_k, \text{ for any } 1 \le k \le n.$$

Let Y be a set of two dimensional vectors on $\mathbb{N} \cup \{0\}$. For any $k = 1, \ldots, n$, we define a map $f \colon S_1 \to Y$ as follows:

$$f(g^k(w)) = f \circ g^k(w) = \sum_{i=1}^{k} f_1(w_i), \tag{1.1}$$

where $f_1 \colon X \to Y$ such that

$$f_1(w_i) = \begin{cases} (1,0), & \text{if } i = 1, \\ (2,0), & \text{if } w_i = a \text{ and } i \neq 1, \\ (1,1), & \text{if } w_i = b \text{ or } c, \\ (0,2), & \text{if } w_i = d. \end{cases} \tag{1.2}$$

For example, $w = abcadd$ is a word of length 5 in S, then

$$f \circ g^3(w) = (1,0) + (1,1) + (1,1) = (3,2),$$

$$f \circ g^6(w) = (1,0) + (1,1) + (1,1) + (2,0) + (0,2) + (0,2) = (5,6).$$

Definition 1.1. Assume that $f \circ g^k(w) = (l_k, m_k)$. If $l_k > m_k$ holds for any $1 \leq k \leq n$, then we call the word $w = w_1 \cdots w_n$ a *standard Young word.*

Let $W(n)$ denote the set of standard Young words of length n. Let $W(n,r)$ denote the set of standard Young words w of length n such that $\#a - \#d = r$, $r \geq 1$, where $\#x$ denotes the number of letter $x \in X$ in w. Let $w(n,r)$ be the cardinality of $W(n,r)$. We show the recurrence relation of $w(n,r)$ as follows: for $1 \leq r \leq n$,

$$w(n,r) = w(n-1,r-1) + 2w(n-1,r) + w(n-1,r+1) \qquad (1.3)$$

with the initial condition $w(1,1) = 1$ and $w(n,r) = 0$ if $n < r$. The proof is shown in Property 2.1.

Shapiro [6] introduced a triangular array of numbers $B(n,r)$, which is called a Catalan triangle, A039598 and A009766 in OEIS [5]. The numbers $B(n,r)$, $n \geq 1$, are defined recursively as follows:

$$B(n,r) = \begin{cases} 1, & \text{if } r = n = 1, \\ B(n-1,r-1) + 2B(n-1,r) + B(n-1,r+1), & \text{if } 1 \leq r \leq n, \\ 0, & \text{otherwise.} \end{cases} \qquad (1.4)$$

See Table 1.1 for the triangular array. Note that the first column of Catalan triangle is made up of Catalan numbers, i.e., $B(n,1) = C_n$; the second column is the Cayley numbers, A002057 in OEIS [5].

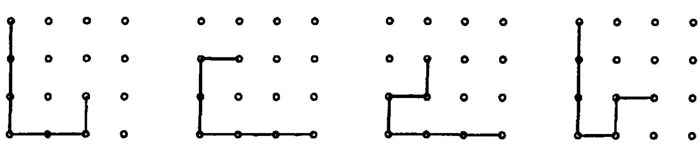
$n\backslash r$	1	2	3	4	5	6
1	1					
2	2	1				
3	5	4	1			
4	14	14	6	1		
5	42	48	27	8	1	
6	132	165	110	44	10	1
⋮		…		…		

Table 1.1: Catalan triangle $B(n,r)$.

Moreover, Koshy [3] gave the explicit formula for $B(n,r)$:

$$B(n,r) = \frac{r}{n}\binom{2n}{n-r}, \quad n \geq r.$$

He also gave a geometric interpretation for $B(n,r)$. Consider the lattice paths of length n consisting of steps $E = (1,0)$ and $N = (0,1)$ in the first quadrant. Denote such lattice path of length n by a sequence of its lattice points $v_0v_1\cdots v_n$, and denote v_i by its coordinate (a_i, b_i). Consider two paths $u_0u_1\cdots u_n$ and $v_0v_1\cdots v_n$ with the same length n, where $u_n = (a_n, b_n)$ and $v_n = (c_n, d_n)$. Then define $|a_n - c_n|$ is the distance between the two paths. Equivalently, the distance between two paths is the absolute difference of the x-coordinates of the terminal points of the two paths. The two paths intersect if $u_i = v_i$ for some i, where $0 < i \leq n$. Koshy [3] showed that $B(n,r)$ counts the number of pairs of nonintersecting paths of length n originated at $(0,0)$ with distance r, where $1 \leq r \leq n$. For example, the four pairs of lattice paths of length 3 originated at $(0,0)$ with distance 2 is illustrated in Figure 1.1.



$B(3,2) = 4$

Figure 1.1: The four pairs of paths of length 3 with distance 2.

Furthermore, Deutsch [2] give another combinatorial interpretation of Catalan triangle. He describe that the number of standard Young tableaux of shape $(n+r-1, n-r)$ is also enumerated by Catalan triangle $B(n,r)$.

Above all, it is easy to see that $w(n,r) = B(n,r)$ from (1.4) and (1.3). In this paper, we shall establish a bijection from the set of standard Young words to the set of pairs of non-intersection lattice paths. Then we set a one-to-one correspondence between the set of standard Young words and the set of standard Young tableaux of two rows, which results in some interesting correspondence between the statistics of standard Young words and standard Young tableaux, especially for the sign and the descents of standard Young words and standard Young tableaux.

2 Standard Young words

Review that $W(n,r)$ denotes the set of standard Young words w of length n such that $\#a - \#d = r$, $r \geq 1$ and $w(n,r)$ denotes its cardinality. In this section, we first give the recurrence relation of $w(n,r)$ and then we establish two bijections between the set of standard Young words and the set of pairs of non-intersecting lattice paths and the set of standard Young tableaux of two rows, respectively.

First we establish the recurrence relation of $w(n,r)$.

Proposition 2.1. *For $1 \leq r \leq n$, the number $w(n,r)$ satisfies the recurrence relation*

$$w(n,r) = w(n-1,r-1) + 2w(n-1,r) + w(n-1,r+1) \tag{2.1}$$

with the initial condition $w(1,1) = 1$ and $w(n,r) = 0$ if $n < r$.

Proof. Given a word $w = w_1 \cdots w_n \in W(n,r)$, by (1.1) and (1.2), we have that for any $1 \leq k \leq n$, $\#a > \#d$ in $g^k(w)$. When $n = r = 1$, it is obvious that $w(1,1) = 1$. For $n \geq 2$, by induction hypothesis, let us consider the following three cases.

If $w_n = a$, then there are $w(n-1,r-1)$ words $w' = w_1 \cdots w_{n-1}$ such that $\#a - \#d = r - 1$; If $w_n = b$ or c, then there are $w(n-1,r)$ words of length $n-1$ satisfying $\#a - \#d = r$; Otherwise, if $w_n = d$, then there are $w(n-1,r+1)$ words of length $n-1$ with $\#a - \#d = r+1$. Hence, (2.1) holds. ∎

By comparing the recurrence relation of Catalan triangle $B(n,r)$, we have

Corollary 2.2. *The number of standard Young words of length n with $\#a - \#d = r$ is counted by the sequence Catalan triangle $B(n,r)$, i.e., $w(n,r) = B(n,r)$.*

Especially, since $B(n,1)$ is equal to the n-th Catalan number C_n, we have

Corollary 2.3. *The number of standard Young words of length n with $\#a - \#d = 1$ is counted by the n-th Catalan number C_n.*

Now we give a bijection from the set of standard Young words to the set of pairs of non-intersecting lattice paths, which establishes a combinatorial proof for Corollary 2.3.

Theorem 2.4. *There is a bijection from the set of pairs of non-intersecting lattice paths of length n with distance r and the set of standard Young words of length n with $\#a - \#d = r$.*

Proof. Let $P(n,r)$ denote the set of pairs of non-intersecting lattice paths of length n with distance r. To establish the correspondence, we define a map φ from $P(n,r)$ to $W(n,r)$.

Given a pair of non-intersecting lattice paths (P,Q) of length n with distance r, denote the i-th steps of P and Q by P_i and Q_i, respectively. Let us decompose the pair of paths (P,Q) by their steps and then we have the following four cases.

(1) If $(P_i,Q_i) = (N,E)$, then we set $w_i = a$;

(2) If $(P_i,Q_i) = (N,N)$, then we set $w_i = b$;

(3) If $(P_i,Q_i) = (E,E)$, then we set $w_i = c$;

(4) If $(P_i,Q_i) = (E,N)$, then we set $w_i = d$.

Then for $i = 1,\ldots,n$, we let $w = w_1 \cdots w_n$, which results in a word of S. Moreover, since (P,Q) is a pair of non-intersecting lattice paths with distance $r \geq 1$, we have $w_1 = a$ and the number of pairs of steps (N,E) is greater than the number of pairs of (E,N). In other words, $\#a > \#d$ in w. By (1.1), (1.2) and Definition 1.1, we conclude that the word w is a standard Young word in $W(n)$. On the other hand, because a pair of steps (N,E) increases the distance of paths (P,Q) by one, a pair of steps (E,N) decreases the distance of paths (P,Q) by one, and the distance between P and Q is r, we have $\#a - \#d = r$. Thus w is a standard Young word of length n with $\#a - \#d = r$.

It is routine to verify that for any standard Young word of length n with $\#a - \#d = r$, one can reverse every step of the map φ to obtain a pair of non-intersecting lattice paths. Thus the map φ is a bijection. This completes the proof. ∎

For example, Figure 2.1 is an illustration of φ for $n = 3, r = 1$.

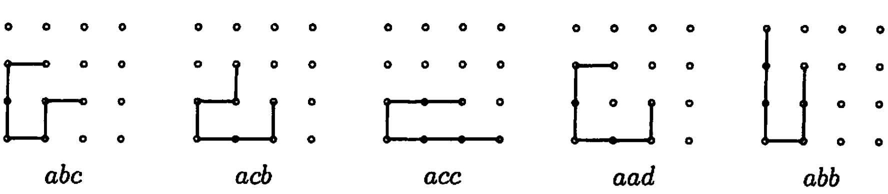


Figure 2.1: The bijection φ for $n = 3$ and $r = 1$.

Next we establish a bijection between standard Young words of length n and standard Young tableaux of two rows, which also establish a combinatorial proof for Corollary 2.3 and the result of Deutsch [2].

Theorem 2.5. *There exists a one-to-one correspondence ψ from the set of standard Young tableaux of shape $(n+r-1, n-r)$ and the set of standard Young words of length n with $\#a - \#d = r$.*

Proof. To establish the correspondence, we define a map ψ from the set of standard Young tableaux of shape $(n+r-1, n-r)$ and the set of standard Young words of length n with $\#a - \#d = r$. Let $w = w_1 \cdots w_n \in W(n, r)$ such that $w_1 = a$ and $\#a - \#d = r$. We proceed to construct a standard Young tableau $T = \psi(w)$ of shape $(n+r-1, n-r)$.

If $n = 1$, namely $w = w_1 = a$ and $r = 1$, then let T be a tableau of shape 1 filled with 1. Otherwise, if $n \geq 2$, then let $T^{(1)}$ be the tableau of shape 1 filled with 1 for $w_1 = a$. For any $i = 2, \ldots, n$, we define $T^{(i)}$ from $T^{(i-1)}$ by considering the following four cases.

(1) If $w_i = a$, then we place $2i-2$ and $2i-1$ at the end of the first row of $T^{(i-1)}$; See Figure 2.2.

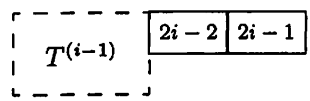
$g^i(w) = w_1 \cdots w_{i-1} a \rightarrow T^{(i)} =$ 

Figure 2.2: If $w_i = a$.

(2) If $w_i = b$, then we place $2i-2$ and $2i-1$ at the end of the first and the second row of $T^{(i-1)}$, respectively; See Figure 2.3.

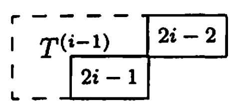
$g^i(w) = w_1 \cdots w_{i-1} b \rightarrow T^{(i)} =$ 

Figure 2.3: If $w_i = b$.

(3) If $w_i = c$, then we place $2i-1$ and $2i-2$ at the end of the first and the second row of $T^{(i-1)}$, respectively; See Figure 2.4.

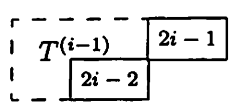
$g^i(w) = w_1 \cdots w_{i-1} c \rightarrow T^{(i)} =$ 

Figure 2.4: If $w_i = c$.

(4) If $w_i = d$, then we place $2i-2$ and $2i-1$ at the end of the second row of $T^{(i-1)}$; See Figure 2.5.

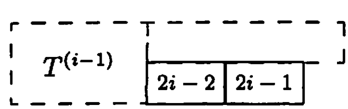
$g^i(w) = w_1 \cdots w_{i-1} d \rightarrow T^{(i)} =$ 

Figure 2.5: If $w_i = d$.

Finally, let $T = T^{(n)}$ and the above procedure implies that the sequences consisting of the digits in each row of T are increasing, respectively. Since $T^{(1)} = \psi(w_1)$ is a tableau of one element, and for $i \geq 2$, each w_i corresponds to add two digits $2i-2$ and $2i-1$ to $T^{(i-1)}$, the elements $2i-2$ and $2i-1$ can not be in the same column of $T^{(i)}$ and there are $2n-1$ elements in T. On the other hand, by Definition 1.1 and the condition $\#a - \#d = r \geq 1$, we have for any $i \geq 2$, if $w_i = d$, then the first row is longer than the second row in $T^{(i-1)}$ by at least two boxes, which implies that each column of $T^{(i)}$ is also increasing. Note that the length of the first row of T is equal to $1 + (\#a - 1) \times 2 + \#b + \#c$, and the length of the second row of T is equal to $\#b + \#c + \#d \times 2$. In other words, from $\#a - \#d = r$, we have that the difference between the length of the two rows in T is equal to $2r - 1$. Above all, $T = T^{(n)}$ is a standard Young tableau of shape $(n+r-1, n-r)$.

It is easy to see that the above construction is reversible. Hence we conclude that ψ is a desired bijection. ∎

As an example, (2.2) illustrates bijection ψ.

$$T = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 5 & 6 & 7 \\ \hline 3 & 4 & 8 & 9 \\ \cline{1-4}\end{array} \Longleftrightarrow w = abcad \in W(5,1). \tag{2.2}$$

3 The sign of standard Young words

In this section, we consider the sign of standard Young words and give the relation between the sign of such words and that of standard Young tableaux.

Definition 3.1. The *sign* of a standard Young word $w = w_1 \cdots w_n$ is defined recursively as follows. If $n = 0$, i.e., w is empty, then let $\mathrm{sign}(w) = 1$. If $n \geq 1$, then we define $\mathrm{sign}(w)$ as

$$\mathrm{sign}(w) = \mathrm{sign}(w_1 \cdots w_n) = \begin{cases} \mathrm{sign}(w_1 \cdots w_{n-1}), & \text{if } w_n = a \text{ or } d; \\ (-1)^m \mathrm{sign}(w_1 \cdots w_{n-1}), & \text{if } w_n = b; \\ (-1)^{m+1} \mathrm{sign}(w_1 \cdots w_{n-1}), & \text{if } w_n = c, \end{cases} \tag{3.1}$$

where m is the total number of b and c in $w_1 \cdots w_{n-1}$.

As an example, $\mathrm{sign}(aba) = 1 \cdot (-1)^0 \cdot 1 = 1$, and $\mathrm{sign}(acc) = (-1)^{1+1} \cdot (-1)^{0+1} \cdot 1 = -1$.

The sign of a standard Young tableau T is defined as the number of transitions of the permutation $\pi(T)$ obtained by reading T as a book, see Stanley [7]. For example, $\mathrm{sign}\left(\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}\right) = \mathrm{sign}(12435) = -1$.

In [4], a graph representation of the sign of permutations is introduced. Given a permutation $\pi = \pi_1 \cdots \pi_n$ of the set $[n]$, its diagram consists of two rows such that $1, \ldots, n$ are listed in the first row increasingly and each element i of the first row is connected by a line with π_i in the second row. The lines should go from top to bottom and at most two lines should cross at any point. For example, the diagram of the permutation $\pi = 41352$ is illustrated in Figure 3.1. Note that the diagram of a permutation π gives an equivalent definition of the sign of π represented by the number of crossings of lines in its diagram. Let $\mathrm{NC}(\pi)$ denote the number of crossings of lines in the diagram of π, then

$$\mathrm{sign}(\pi) = (-1)^{\mathrm{NC}(\pi)}. \tag{3.2}$$

As an example, since $\mathrm{NC}(41352) = 5$, $\mathrm{sign}(41352) = (-1)^5 = -1$.

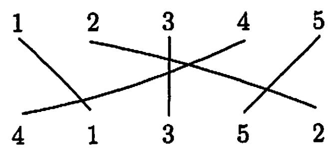


Figure 3.1: The diagram of $\pi = 41352$.

By Theorem 2.5, we have the following conclusion.

Theorem 3.2. *Let $w = w_1 \cdots w_n$ be a standard Young word in $W(n)$ and let $T = \psi(w)$. Then we have*

$$\mathrm{sign}(w) = \mathrm{sign}(T). \tag{3.3}$$

Proof. We prove this conclusion by induction. If $n = 1$, namely $w = w_1 = a$ and $T = \psi(w) = \boxed{1}$, then it is obvious that $\mathrm{sign}(w) = \mathrm{sign}(T) = 1$;

Assume that $\psi(w_1 \cdots w_k) = T^{(k)}$ and $\mathrm{sign}(w_1 \cdots w_k) = \mathrm{sign}(T^{(k)})$ holds, for $k \leq n - 1$. Now suppose that $k = n$ and let $w' = w_1 \cdots w_{n-1}$, $T' = T^{(n-1)} = \psi(w')$ and let π', π be the permutations obtained from T' and T, respectively. Now $w = w_1 \cdots w_n = w'w_n \in W(n)$ such that $w_1 = a$ and let the total number of letters b and c in $w_1 \cdots w_{n-1}$ be m. We consider the following cases.

Case 1: If $w_n = a$. From the bijection ψ, $2n - 2$ and $2n - 1$ are increasingly inserted to the right of the first row of T', which resulted in T. By comparing the diagram of π' with that of π, we obtain that the parity of $\mathrm{NC}(\pi)$ is the same as the parity of $\mathrm{NC}(\pi')$. Figure 3.2 illustrates the diagram of π. Thus $\mathrm{sign}(T) = \mathrm{sign}(T')$. On the other hand, $\mathrm{sign}(w) = \mathrm{sign}(w_1 \cdots w_{n-1})$. Then $\mathrm{sign}(w_1 \cdots w_n) = \mathrm{sign}(T)$ holds.

Case 2: If $w_n = b$. From the bijection ψ, to obtain T, $2n - 2$ and $2n - 1$ are inserted to the right of the first row and the second row of T', respectively. Let ℓ be the number of entries in the second row of T' and let

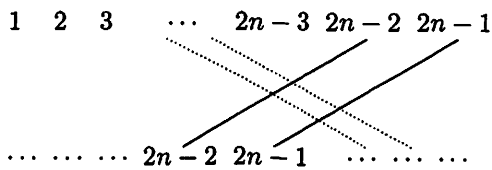


Figure 3.2: If $w_n = a$.

m be the total number of letters b and c in w'. By comparing the diagram of π' with that of π, we obtain that

$$\begin{aligned}\mathrm{NC}(\pi) &= \mathrm{NC}(\pi') \cdot (-1)^{\ell} \\ &= \mathrm{NC}(\pi') \cdot (-1)^{m+2\#d} \\ &= (-1)^m \mathrm{NC}(\pi').\end{aligned}$$

See Figure 3.3. Thus $\mathrm{sign}(T) = (-1)^m \mathrm{sign}(T')$. Moreover, $\mathrm{sign}(w) = (-1)^m \mathrm{sign}(w_1 \cdots w_{n-1})$. Then $\mathrm{sign}(w_1 \cdots w_n) = \mathrm{sign}(T)$.

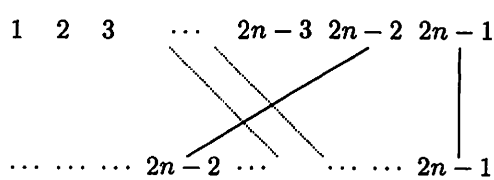


Figure 3.3: If $w_n = b$.

Case 3: If $w_n = c$. From the bijection ψ, in $T = \psi(w)$, the entries $2n-1$ and $2n-2$ are inserted to the right of the first row and the second row of T', respectively. Let ℓ and m be defined as in Case 2. By comparing the diagram of π' with that of π, we obtain that

$$\begin{aligned}\mathrm{NC}(\pi) &= \mathrm{NC}(\pi') \cdot (-1)^{\ell+1} \\ &= \mathrm{NC}(\pi') \cdot (-1)^{m+2\#d+1} \\ &= (-1)^{m+1} \mathrm{NC}(\pi').\end{aligned}$$

See Figure 3.4. Thus $\mathrm{sign}(T) = (-1)^{m+1} \mathrm{sign}(T')$. On the other hand,

$$\mathrm{sign}(w) = (-1)^{m+1} \mathrm{sign}(w_1 \cdots w_{n-1}).$$

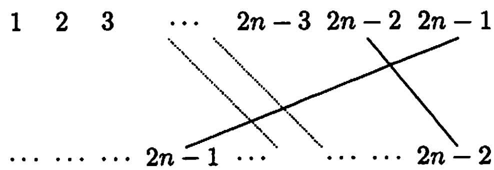


Figure 3.4: If $w_n = c$.

Hence $\text{sign}(w_1 \cdots w_n) = \text{sign}(T)$ holds.

Case 4: $w_n = d$. From the bijection ψ, T is obtained by inserting $2n-2$ and $2n-1$ to the right of the second row of T' increasingly. By comparing the diagram of π' with that of π, we obtain that $\text{NC}(\pi) = \text{NC}(\pi')$, see Figure 3.5. Thus $\text{sign}(T) = \text{sign}(T')$. On the other hand, when $w_n = d$, $\text{sign}(w) = \text{sign}(w_1 \cdots w_{n-1})$. Hence $\text{sign}(w_1 \cdots w_n) = \text{sign}(T)$ holds. Above all, the conclusion holds for any $n \geq 1$. ∎

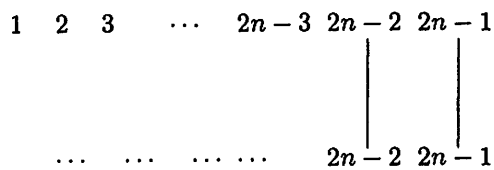


Figure 3.5: If $w_n = d$.

4 The descents of standard Young words

In this section, we introduce the descents of standard Young words and discuss its correspondence with the descents of standard Young tableaux.

Definition 4.1. Given a standard Young word $w = w_1 \cdots w_n$, if $w_i = b$, or $w_i = a, c$ which are immediately followed by c or d, then w_i is defined as a *descent* of w. The set of descents of w is denoted by $D(w)$ and its cardinality is denoted by $d(w)$.

As an example, let $w = w_1 \cdots w_6 = abbcad$. Since there are two letters b and a consecutive ad in w, $D(w) = \{w_2, w_3, w_5\}$ and $d(w) = 3$.

A standard Young tableau T has a *descent* at i if the entry $i + 1$ is strictly south and weakly to the west of i in T. The set of all descents in

T is denoted by $D(T)$ and the cardinality of $D(T)$ is denoted by $d(T)$. For example, let

$$T = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 4 & 7 & 8 & 9 \\ \hline 3 & 5 & 6 \\ \cline{1-3} \end{array},$$

then $D(T) = \{2,4\}$ and $d(T) = 2$. As a consequence of Theorem 2.5, we have

Theorem 4.2. *Let $w = w_1 \cdots w_n$ be a standard Young word in $W(n)$ and let $T = \psi(w)$. Then we have $d(w) = d(T)$, and*

(i) if $w_i = b \in D(w)$, then $2i-2 \in D(T)$;

(ii) if $w_i = a$ or c is a descent of w, then $2i-1$ is a descent of T.

Proof. From bijection ψ, for any $i = 1, \ldots, n$, $w_i = a, b, c$ or d is mapped into tableau T by inserting the entries $2i-2$ and $2i-1$. On the other hand, a descent k of a standard Young tableau T corresponds to two consecutive entries k and $k+1$ such that $k+1$ is strictly south and weakly to the west of k in T. So we need to discuss the parity of the descent k in T.

If k is even. Assume $k = 2i-2$, $i \geq 2$. Then $k+1 = 2i-1$ is odd and is strictly to the south of $k = 2i-2$, then the insertion of $2i-2$ and $2i-1$ corresponds to letter b in bijection ψ.

Otherwise, if k is odd. Assume $k = 2i-1$, $i \geq 1$. Then $k+1 = 2i$ is even and the entries $2i-2$ and $2i-1$ in T correspond to w_i, and the entries $2i$ and $2i+1$ in T is mapped into w_{i+1} such that $2i$ is strictly located to the south of $2i-1$. Then $w_i w_{i+1}$ could be ac, cc, ad or cd.

Above all, we have (i) and (ii) holds and $d(T) = d(W)$. ∎

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