Compositions with the Last Summand Odd

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Abstract

For $n \geq 1$, we let a_n count the number of compositions of the positive integer n, where the last summand is odd. We find that $a_n = \left(\frac{1}{3}\right)(-1)^{n-1} + \left(\frac{2}{3}\right)2^{n-1}$. Since J_n, the n-th *Jacobsthal number* is given as $\left(\frac{1}{3}\right)(-1)^n + \left(\frac{2}{3}\right)2^n$, for $n \geq 0$, it follows that $a_n = J_{n-1}$, for $n \geq 1$. For this reason, these compositions are often referred to as the *Jacobsthal compositions*.

In our investigation we determine results for the a_n compositions of n such as: (i) $a_{n,k}$, the number of times the positive integer k appears as a summand among these a_n compositions of n; (ii) the numbers of plus signs, summands, even summands, and odd summands that occur for these compositions of n; (iii) the sum of the even summands and the sum of the odd summands for the a_n compositions of n; (iv) the numbers of levels, rises, and descents for the a_n compositions; and (v) the number of runs that occur among these a_n compositions.

Keywords : Compositions, Jacobsthal Numbers, Levels, Rises, Descents, Runs

1. Determining the Number of Jacobsthal Compositions

For $n \geq 1$, we let a_n count the number of compositions of n where the last summand is odd. Then $a_1 = 1$ for the composition 1, $a_2 = 1$ for the composition $1+1$, and $a_3 = 3$ for the compositions $2+1$, 3, and $1+1+1$. To determine a_n, we consider the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2}.$$

Here a_{n-1} accounts for the compositions of n obtained by placing '1+' in front of a composition of $n-1$ (that has an odd last summand). The summand $2a_{n-2}$ arises from (i) placing '2+' in front of each composition of $n-2$ (that has an odd last summand); and (ii) adding 2 to the first summand of each such composition of $n-2$. This recurrence relation leads to the characteristic equation $r^2 - r - 2 = 0$, for which the characteristic roots are -1 and 2. Consequently, the solution for this recurrence relation has the form $a_n = c_1(-1)^n + c_2(2^n)$. [For this and more on the solution of recurrence relations see Chapter 7 of the text by R. Brualdi[3] or Chapter 10 of the text by R. Grimaldi[5].] With $a_1 = 1$ and $a_2 = 1$, it follows that $c_1 = -1/3$ and $c_2 = 1/3$, so

$$a_n = \left(\frac{1}{3}\right)(-1)^{n-1} + \left(\frac{2}{3}\right)2^{n-1} = J_{n-1},\ \ n \geq 1,$$

where J_n denotes the n-th Jacobsthal number. As a result of this solution we refer to these compositions of n as the Jacobsthal compositions of n. [The Jacobsthal numbers are given recursively by $J_0 = 1$, $J_1 = 1$, $J_n = J_{n-1} + 2J_{n-2}$, $n \geq 2$. These numbers arise in a variety of applications. For instance, they count (i) the number of ways to tile a $2 \times n$ chessboard using 2×2 tiles and 2×1 tiles (that can also be used as 1×2 tiles); (ii) the number of ways to place non-taking kings on a $2 \times n$ chessboard; (iii) the number of perfect matchings for the ladder graph where each cycle of length 4 is completed to a K_4; and (iv) the number of independent subsets of vertices (including the empty set) for the graph in (iii). These numbers also arise in the papers by R. Brigham, P. Chinn, and R. Grimaldi [2], D. Frey and J. Sellers [4], R. Grimaldi [6], and S. Heubach [7].]]

2. Frequency of Occurrence of Summands for the a_n Compositions

For $n \geq k \geq 1$, let $a_{n,k}$ count the number of times the summand k appears among the a_n compositions of n. The following table provides these results for $1 \leq n \leq 7$.

$n\backslash k$	1	2	3	4	5	6	7
1	1						
2	2						
3	4	1	1				
4	10	2	2				
5	22	6	4	1	1		
6	50	14	10	2	2		
7	110	34	22	6	4	1	1

Table 1

Here, for instance, we find that $a_{5,2} = 6$, since the summand 2 appears among the 11 compositions of 5 six times.

As suggested by the results in Table 1 we have the following:

Lemma 1 : For $n, k \geq 1$, $a_{n,k} = a_{n+2,k+2}$.

Proof: This follows because each individual occurrence of the summand k, among the a_n compositions of n, can be replaced by the summand $k+2$, resulting in a corresponding occurrence (of $k+2$) among the a_{n+2} compositions of $n+2$, and this correspondence is reversible.

From Lemma 1, we learn that

$$\begin{aligned} a_{n,k} &= a_{n-2m,1} = a_{n-k+1,1}, \quad \text{for } k = 2m+1 \\ a_{n,k} &= a_{n-(2m-2),2} = a_{n-k+2,2}, \quad \text{for } k = 2m. \end{aligned}$$

Consequently, we need to determine formulas for

$$\begin{aligned} a_{n,1}, \quad &\text{for } n \geq 1 \\ a_{n,2}, \quad &\text{for } n \geq 3. \end{aligned}$$

Before we do so, we need to determine $s_{n,k}$, the number of (the a_n) compositions of n that start with the summand k. We find that

$$s_{n,k} = \begin{cases} 0, \text{ if } n < k \\ 0, \text{ if } n = k, n \text{ even} \\ 1, \text{ if } n = k, n \text{ odd} \\ J_{n-k-1}, \text{ if } n > k. \end{cases}$$

To determine $a_{n,1},$ we consider the recurrence relation

$$a_{n,1} = (a_{n-1,1} + a_{n-1}) + a_{n-2,1} + (a_{n-2,1} - s_{n-2,1}),$$

where (i) $(a_{n-1,1} + a_{n-1})$ accounts for the $1's$ we get when we place '1+' at the start of each of the a_{n-1} compositions of $n-1$, for in addition to the $1's$ that already appear among these compositions of $n-1$, we must also consider each new 1 that has been appended at the start of each of these a_{n-1} compositions; (ii) $a_{n-2,1}$ accounts for the $1's$ that occur among the a_{n-2} compositions of $n-2$, where we have appended '2+'; and (iii) $(a_{n-2,1} - s_{n-2,1})$ arises when we add 2 to the first summand of each of the a_{n-2} compositions of $n-2$, for here we lost each 1 that appeared as the first summand of a composition of $n-2$. This leads to the recurrence relation

$$\begin{aligned}
a_{n,1} &= a_{n-1,1} + 2a_{n-2,1} + a_{n-1} - s_{n-2,1} \\
&= a_{n-1,1} + 2a_{n-2,1} + J_{n-2} - J_{n-4}, \; n \geq 4 \\
&= a_{n-1,1} + 2a_{n-2,1} + \left[\left(\frac{1}{3}\right)(-1)^{n-2} + \left(\frac{2}{3}\right)2^{n-2}\right] \\
&\quad - \left[\left(\frac{1}{3}\right)(-1)^{n-4} + \left(\frac{2}{3}\right)2^{n-4}\right] \\
&= a_{n-1,1} + 2a_{n-2,1} + 2^{n-3}, \; n \geq 4.
\end{aligned}$$

The roots for the characteristic equation $x^2 - x - 2 = 0$ are 2 and -1, so the particular part of the solution has the form $An\ 2^n$, while the form for the homogeneous part is $c_1(2^n) + c_2(-1)^n$. Substituting $An\ 2^n$ into the relation we learn that $A = \frac{1}{12}$, so $a_{n,1} = c_1(2^n) + c_2(-1)^n + \left(\frac{1}{12}\right) n\ 2^n$. With $a_{2,1} = 2$ and $a_{3,1} = 4$ it follows that $c_1 = \frac{5}{18}$ and $c_2 = \frac{2}{9}$. So we find that

$$\begin{aligned}
a_{1,1} &= 1 \\
a_{n,1} &= \left(\frac{5}{18}\right)2^n + \left(\frac{2}{9}\right)(-1)^n + \left(\frac{1}{12}\right) n\ 2^n \\
&= \left[\frac{5}{18} + \frac{n}{12}\right]2^n + \left(\frac{2}{9}\right)(-1)^n, \; n \geq 2.
\end{aligned}$$

To express this result in terms of the Jacobsthal numbers, we recall that $J_n = \left(\frac{1}{3}\right)(-1)^n + \left(\frac{2}{3}\right)2^n$ and observe that

$$\begin{aligned}
&(1) & J_{n-1} + J_n &= 2^n \\
&(2) & 2\,J_n - J_{n+1} &= (-1)^n \\
&(3) & J_{n+1} - J_n &= 2\,J_{n-1} \\
&(4) & J_n &= \textstyle\sum_{i=1}^{n/2} 2^{2i-1} + 1, \; n \text{ even} \\
&(5) & J_n &= \textstyle\sum_{i=0}^{(n-1)/2} 2^{2i}, \; n \text{ odd.}
\end{aligned}$$

We now find that

$$\begin{aligned}
a_{1,1} &= 1 \\
a_{n,1} &= \left(\frac{5}{18}\right)(J_{n-1} + J_n) + \left(\frac{2}{9}\right)(2J_n - J_{n+1}) + \left(\frac{1}{12}\right) n\ (J_{n-1} + J_n) \\
&= \left(\frac{-2}{9}\right) J_{n+1} + \left[\left(\frac{n}{12}\right) + \left(\frac{13}{18}\right)\right] J_n + \left[\left(\frac{n}{12}\right) + \left(\frac{5}{18}\right)\right] J_{n-1}.
\end{aligned}$$

Turning now to $a_{n,2}$, the number of $2's$ that appear among the a_n compositions of n, we consider the comparable recurrence relation

$$\begin{aligned} a_{n,2} &= a_{n-1,2} + (a_{n-2,2} + a_{n-2}) + (a_{n-2,2} - s_{n-2,2}) \\ &= a_{n-1,2} + 2\,a_{n-2,2} + J_{n-3} - J_{n-5} \\ &= a_{n-1,2} + 2\,a_{n-2,2} + 2^{n-4},\ n \geq 5. \end{aligned}$$

The form of the homogeneous part of the solution is $c_1(2^n) + c_2(-1)^n$, while the particular solution has the form $An\ 2^n$. Upon substituting the particular solution into the recurrence relation we find that $A = \frac{1}{24}$, so $a_{n,2} = c_1(2^n) + c_2(-1)^n + (\frac{1}{24})\,n\ 2^n$. Since $a_{3,2} = 1$ and $a_{4,2} = 2$, it follows that $c_1 = (\frac{-1}{36})$ and $c_2 = (\frac{-2}{9})$, so

$$\begin{aligned} a_{n,2} &= \left(\frac{-1}{36}\right)(2^n) + \left(\frac{-2}{9}\right)(-1)^n + \left(\frac{1}{24}\right)\ n\ 2^n \\ &= \left(\frac{3n-2}{72}\right) J_{n-1} + \left(\frac{3n-34}{72}\right) J_n + \left(\frac{2}{9}\right) J_{n+1},\ \ n \geq 3. \end{aligned}$$

We summarize the results of this section in the following:

Theorem 2:

$$\begin{aligned}
\text{For } k \text{ odd, } a_{k,k} &= 1 \text{ and, for } n > k, \\
a_{n,k} &= a_{n-k+1,1} = \left[\frac{5}{18} + \frac{n-k+1}{12}\right] 2^{n-k+1} \\
&\quad + \left(\frac{2}{9}\right)(-1)^{n-k+1} \\
&= \left(\frac{-2}{9}\right) J_{n-k+2} + \left[\frac{n-k+1}{12} + \frac{13}{18}\right] J_{n-k+1} \\
&\quad + \left[\frac{n-k+1}{12} + \frac{5}{18}\right] J_{n-k}. \\
\text{For } k \text{ even, } a_{k+1,k} &= 1 \text{ and, for } n > k+1, \\
a_{n,k} &= a_{n-k+2,2} = \left(\frac{-1}{36}\right) 2^{n-k+2} + \left(\frac{-2}{9}\right)(-1)^{n-k+2} \\
&\quad + \left(\frac{1}{24}\right)(n-k+2)\, 2^{n-k+2} \\
&= \left(\frac{3(n-k)+4}{72}\right) J_{n-k+1} \\
&\quad + \left(\frac{3(n-k)-28}{72}\right) J_{n-k+2} + \left(\frac{2}{9}\right) J_{n-k+3}.
\end{aligned}$$

3. Numbers of Plus Signs, Summands, Even Summands, and Odd Summands; The Sum of Even Summands; The Sum of Odd Summands

For $n \geq 1$, let pl_n count the number of plus signs that appear among the a_n compositions of n. Then

$$pl_n = (pl_{n-1} + a_{n-1}) + (pl_{n-2} + a_{n-2}) + pl_{n-2},$$

where (i) $(pl_{n-1} + a_{n-1})$ arises for when we append $'1+'$ to the front of each composition of $n-1$; (ii) $(pl_{n-2} + a_{n-2})$ arises for when we append $'2+'$ to the front of each composition of $n-2$; and (iii) pl_{n-2} accounts for the case where we add 2 to the first summand of each composition of $n-2$. This recurrence relation can then be rewritten as

$$\begin{aligned}
pl_n &= pl_{n-1} + 2\, pl_{n-2} + J_{n-2} + J_{n-3} \\
&= pl_{n-1} + 2\, pl_{n-2} + 2^{n-2},\ n \geq 3.
\end{aligned}$$

Here the homogeneous part of the solution has the form $c_1(-1)^n + c_2(2^n)$, while the particular part has the form $An\ 2^n$. Upon substituting $An\ 2^n$ into the given recurrence relation we find that $A = \frac{1}{6}$, so $pl_n = c_1(-1)^n + c_2(2^n) + \left(\frac{1}{6}\right) n\ 2^n$. With $pl_1 = 0$ and $pl_2 = 1$, it follows that $c_1 = \frac{1}{9}$ and $c_2 = \frac{-1}{9}$, so

$$
\begin{aligned}
pl_n &= \left(\frac{1}{9}\right)(-1)^n + \left(\frac{-1}{9}\right)2^n + \frac{1}{6}\ n\ 2^n \\
&= \left(\frac{1}{9}\right)(-1)^n + \left(\frac{3n-2}{18}\right)2^n \\
&= \left(\frac{3n-2}{18}\right)J_{n-1} + \left(\frac{3n+2}{18}\right)J_n - \frac{1}{9}J_{n+1},\ \ n \geq 1.
\end{aligned}
$$

Now we wish to determine $nsum_n$, the number of summands that occur among the a_n compositions of n. For each composition the number of summands is one more than the number of plus signs, so for $n \geq 1$,

$$
\begin{aligned}
nsum_n &= pl_n + a_n \\
&= \left(\frac{1}{9}\right)(-1)^n + \left(\frac{3n-2}{18}\right)2^n + \left(\frac{1}{3}\right)(-1)^{n-1} + \left(\frac{2}{3}\right)2^{n-1} \\
&= \left(\frac{3n+4}{18}\right)2^n - \left(\frac{2}{9}\right)(-1)^n \\
&= \left(\frac{3n+4}{18}\right)J_{n-1} + \left(\frac{3n-4}{18}\right)J_n + \left(\frac{2}{9}\right)J_{n+1},\ \ n \geq 1.
\end{aligned}
$$

To determine the number of even summands among the a_n compositions - denoted $nevensum_n$ - we consider the recurrence relation:

$$
\begin{aligned}
nevensum_n &= nevensum_{n-1} + (nevensum_{n-2} + a_{n-2}) + nevensum_{n-2} \\
&= nevensum_{n-1} + 2\ nevensum_{n-2} + \left(\frac{1}{3}\right)(-1)^{n-3} \\
&\quad + \left(\frac{2}{3}\right)2^{n-3}.
\end{aligned}
$$

The summand $nevensum_{n-1}$ accounts for all the even summands among the a_{n-1} compositions of $n-1$ and here these even summands do not change when we append $'1+'$. The summand $(nevensum_{n-2} + a_{n-2})$ accounts for all the even summands that appear among the a_{n-2} compositions of $n-2$ together with the additional $2's$ that are appended at the start. The summand $nevensum_{n-2}$ accounts for all the even summands among the a_{n-2} compositions of $n-2$ and here the even summands at the start stay even, but are increased by 2. For this recurrence relation the form of the solution is $c_1(-1)^n + c_2\, 2^n + An\,(-1)^n + Bn\,2^n$. Substituting $An\,(-1)^n$ into the relation, we learn that $A = \frac{-1}{9}$. A similar calculation reveals that $B = \frac{1}{18}$. So $nevensum_n = c_1(-1)^n + c_2\,2^n - \frac{1}{9}n\,(-1)^n + \frac{1}{18}n\,2^n$. With $nevensum_1 = nvensum_2 = 0$, it follows that $c_1 = \frac{2}{27}$ and $c_2 = \frac{-2}{27}$, so for $n \geq 1$,

$$\begin{aligned} nevensum_n &= \left(\frac{2}{27}\right)(-1)^n + \left(\frac{-2}{27}\right)2^n - \frac{1}{9}\,n\,(-1)^n + \frac{1}{18}\,n\,2^n \\ &= \left(\frac{2-3n}{27}\right)(-1)^n + \left(\frac{3n-4}{54}\right)2^n \\ &= \left(\frac{3n-4}{54}\right)J_{n-1} + \left(\frac{4-9n}{54}\right)J_n + \left(\frac{3n-2}{27}\right)J_{n+1}. \end{aligned}$$

If we let $noddsum_n$ count the number of odd summands among the a_n compositions, then as $nsum_n = nevensum_n + noddsum_n$, we find that

$$\begin{aligned} noddsum_n &= \left(\frac{3n+4}{18}\right)2^n - \frac{2}{9}(-1)^n - \left(\frac{2-3n}{27}\right)(-1)^n \\ &\quad - \left(\frac{3n-4}{54}\right)2^n \\ &= \left(\frac{3n+8}{27}\right)2^n + \left(\frac{3n-8}{27}\right)(-1)^n \\ &= \left(\frac{3n+8}{27}\right)J_{n-1} + \left(\frac{9n-8}{27}\right)J_n - \left(\frac{3n-8}{27}\right)J_{n+1},\ n \geq 1. \end{aligned}$$

We now wish to determine the sum of all the even summands among the a_n compositions of n. This we shall denote by $evensum_n$. To do so we need to determine $startev_n$, the number of compositions of n that start with an even summand. This leads us to the recurrence relation

$$startev_{n+2} = startev_n + a_n,$$

where (i) the summand $startev_n$ accounts for the compositions of $n+2$ that arise from the compositions of n when we add 2 to the first summand; and (ii) the summand a_n accounts for the compositions that arise when we place '2+' at the start of each of the a_n compositions of n. So now the above relation can be rewritten as

$$startev_{n+2} = startev_n + \left(\frac{1}{3}\right)(-1)^{n-1} + \left(\frac{2}{3}\right)2^{n-1}.$$

The characteristic equation for this relation is $x^2 - 1 = 0$, so the characteristic roots are 1 and -1. Consequently, the solution for this relation takes the form $c_1+c_2(-1)^n+A(2^n)+Bn\,(-1)^n$. When we substitute $A(2^n)$ into the relation $startev_{n+2} = startev_n + \left(\frac{2}{3}\right)2^{n-1}$ we learn that $A=\frac{1}{9}$. Substituting $Bn\,(-1)^n$ into the relation $startev_{n+2} = startev_n + \left(\frac{1}{3}\right)(-1)^{n-1}$ reveals that $B=\frac{-1}{6}$. Consequently, the solution has the form $startev_n = c_1 + c_2(-1)^n + \left(\frac{1}{9}\right)(2^n) - \left(\frac{1}{6}\right)n\,(-1)^n$. As $0 = startev_1 = c_1 - c_2 + \frac{7}{18}$ and $0 = startev_2 = c_1 + c_2 + \frac{1}{9}$, it follows that $c_1 = \frac{-1}{4}$ and $c_2 = \frac{5}{36}$. Therefore, for $n \geq 1$,

$$\begin{aligned} startev_n &= \left(\frac{-1}{4}\right) + \left(\frac{1}{9}\right)2^n + \left(\frac{5-6n}{36}\right)(-1)^n \\ &= \left(\frac{-1}{4}\right) + \left(\frac{1}{9}\right)J_{n-1} + \left(\frac{7-6n}{18}\right)J_n + \left(\frac{6n-5}{36}\right)J_{n+1}. \end{aligned}$$

If we let $startodd_n$ count the number of the a_n compositions that start with an odd summand, then as $startodd_n + startev_n = a_n$, it follows that

$$\begin{aligned} startodd_n &= a_n - startev_n \\ &= \frac{1}{4} + \left(\frac{2}{9}\right)2^n + \left(\frac{6n-17}{36}\right)(-1)^n \\ &= \frac{1}{4} + \left(\frac{2}{9}\right)J_{n-1} + \left(\frac{6n-13}{18}\right)J_n - \left(\frac{6n-17}{36}\right)J_{n+1},\ n \geq 1. \end{aligned}$$

And now we return to our original problem of trying to determine $evensum_n$, the sum of all the even summands that occur among the a_n compositions of n. We find that

$$\begin{aligned} evensum_{n+2} &= evensum_{n+1} + (evensum_n + 2\,a_n) \\ &\quad +(evensum_n + 2\,startev_n) \\ &= evensum_{n+1} + 2\,evensum_n \\ &\quad +2\left[\left(\frac{1}{3}\right)(-1)^{n-1} + \left(\frac{2}{3}\right)2^{n-1}\right] \\ &\quad +2\left[\left(\frac{1}{9}\right)2^n + \left(\frac{5}{36}\right)(-1)^n + \left(\frac{-1}{6}\right)n\,(-1)^n - \frac{1}{4}\right] \\ &= evensum_{n+1} + 2\,evensum_n + \left(\frac{8}{9}\right)2^n + \left(\frac{-7}{18}\right)(-1)^n \\ &\quad + \left(\frac{-1}{3}\right)n\,(-1)^n - \frac{1}{2}. \end{aligned}$$

Here the 2 in front of a_n accounts for when we append '2+' at the start of each of the a_n compositions of n. The 2 in front of $startev_n$ is for the cases where the first summand of a composition of n is even and then increased by 2. The characteristic equation for the homogeneous part of the recurrence relation is $x^2 - x - 2 = 0$, so the characteristic roots are -1 and 2. Consequently, the solution for the recurrence relation has the form

$$evensum_n = c_1(2^n) + c_2(-1)^n + An\ (-1)^n + Bn^2\ (-1)^n + C + Dn\ 2^n.$$

To determine C we substitute $evensum_n = C$ into the recurrence relation $evensum_{n+2} = evensum_{n+1} + 2\, evensum_n - \frac{1}{2}$ and find that $C = C + C - \frac{1}{2}$. Hence $C = \frac{1}{4}$. For the evaluation of D we substitute $evensum_n = Dn\ 2^n$ into the recurrence relation $evensum_{n+2} = evensum_{n+1} + 2\, evensum_n + \left(\frac{8}{9}\right) 2^n$ and learn that $4D(n+2) = 2D(n+1) + 2Dn + \left(\frac{8}{9}\right)$. From this we see that $D = \frac{4}{27}$. Finally, we substitute $An\ (-1)^n + Bn^2\ (-1)^n$ for $evensum_n$ in the recurrence relation $evensum_{n+2} = evensum_{n+1} + 2\, evensum_n + \left(\frac{-7}{18}\right)(-1)^n + \left(\frac{-1}{3}\right) n\ (-1)^n$. This takes us to $A = \frac{-1}{27}$ and $B = \frac{-1}{18}$. Consequently,

$$\begin{aligned} evensum_n &= c_1(2^n) + c_2(-1)^n + \left(\frac{-1}{27}\right) n\ (-1)^n + \left(\frac{-1}{18}\right) n^2\ (-1)^n \\ &\quad + \frac{1}{4} + \left(\frac{4}{27}\right) n\ 2^n. \end{aligned}$$

From $0 = evensum_1 = 2\,c_1 - c_2 + \frac{23}{36}$ and $0 = evensum_2 = 4\,c_1 + c_2 + \frac{123}{108}$ it follows that $c_1 = \frac{-8}{27}$ and $c_2 = \frac{5}{108}$. So for $n \geq 1$,

$$\begin{aligned} evensum_n &= \frac{1}{4} + \left(\frac{4n-8}{27}\right) 2^n + \left(\frac{5-4n-6n^2}{108}\right)(-1)^n \\ &= \frac{1}{4} + \left(\frac{4n-8}{27}\right) J_{n-1} + \left(\frac{-11+4n-6n^2}{54}\right) J_n \\ &\quad - \left(\frac{5-4n-6n^2}{108}\right) J_{n+1}. \end{aligned}$$

If we assign $oddsum_n$ to the sum of all the odd summands that occur among the a_n compositions of n, as $oddsum_n = n\ a_n - evensum_n$, we learn that for $n \geq 1$

$$\begin{aligned} oddsum_n &= \left(\frac{-1}{4}\right) + \left(\frac{6n^2-32n-5}{108}\right)(-1)^n + \left(\frac{5n+8}{27}\right) 2^n \\ &= \left(\frac{-1}{4}\right) + \left(\frac{5n+8}{27}\right) J_{n-1} + \left(\frac{6n^2-22n+11}{54}\right) J_n \\ &\quad - \left(\frac{6n^2-32n-5}{108}\right) J_{n+1}. \end{aligned}$$

4. Levels, Rises, and Descents

Early on, the concepts of levels, rises, and descents were studied for compositions of $1's$ and $2's$ in the article by Alladi and Hogatt [1].

Here we say that a *level* occurs in a composition of n when two consecutive summands in the composition are equal. If we let lev_n count the number of levels that occur among the a_n compositions of n, we find that

$$lev_1 = 0, \quad lev_2 = 1, \quad lev_3 = 2, \quad lev_4 = 5.$$

In order to develop a recurrence relation for lev_n, let us examine the levels that occur for $n = 5$ and $n = 6$. We find that

$$lev_5 = (lev_4 + s_{4,1}) + (lev_3 + s_{3,2}) - a_1,$$

where (i) the summand $(lev_4 + s_{4,1})$ takes into account the levels that exist for $n = 4$ together with the new levels that arise when we append $'1+'$ to each of the $s_{4,1}$ compositions of 4 that start with 1; (ii) the summand $(lev_3 + s_{3,2})$ accounts for the levels that exist for $n = 3$ along with the new levels we get by appending $'2+'$ to each of the $s_{3,2}$ compositions of 3 that start with 2; and (iii) a_1 takes into account the compositions of 3 that start with $'1+1'$, for by increasing the first summand from 1 to 3, we lost a level. [Recall from Section 2 that $s_{n,k} = J_{n-k-1} = a_{n-k}$.] For the case of $n = 6$ it follows that

$$lev_6 = lev_5 + s_{5,1} + lev_4 + s_{4,2} + lev_4 - a_2 + a_0,$$

where a_2 accounts for the compositions of 4 that start with $'1+1'$, and $a_0(=1)$ deals with the compositions of 4 that start with $'1+3'$. These equations for lev_5 and lev_6 can also be expressed as

$$\begin{aligned} lev_5 &= lev_4 + 2\,lev_3 + J_2 + J_0 - a_1 = lev_4 + 2\,lev_3 + J_2 \\ lev_6 &= lev_5 + 2\,lev_4 + J_3 + J_1 + a_0 - a_2 = lev_5 + 2\,lev_4 + J_3 + a_0. \end{aligned}$$

In general we find that for $n = 4k+1$, $k \geq 1$,

$$\begin{aligned} lev_n = \; & lev_{n-1} + s_{n-1,1} + lev_{n-2} + s_{n-2,2} + \\ & lev_{n-2} - a_{n-4} - a_{n-6} - \dots - \; a_3 - a_1 + a_{n-6} + a_{n-8} + \dots + a_1, \end{aligned}$$

where, for example, $-a_{n-4}$ accounts for the loss of the initial level in the compositions of $n-2$ that start with $'1+1'$, $-a_{n-6}$ for the loss of the initial level in the compositions of $n-2$ that start with $'2+2'$, ..., and

$-a_1$ for the loss of the one level in the composition of $n-2$ that starts with $'(2k-1)+(2k-1)'$. Meanwhile, the summand $'+a_{n-6}'$ counts the new initial levels of n that arise from the compositions of $n-2$ that start with $'1+3'$, the summand $'+a_{n-8}'$ counts the new initial levels of n that arise from the compositions of $n-2$ that start with $'2+4'$, ..., and the summand $'+a_1'$ counts the new initial levels of n that arise from the compositions of $n-2$ that start with $'(2k-2)+2k'$. This recurrence relation can then be simplified as

$$\begin{aligned} lev_n &= lev_{n-1} + s_{n-1,1} + lev_{n-2} + s_{n-2,2} + lev_{n-2} - a_{n-4} \\ &= lev_{n-1} + 2\ lev_{n-2} + J_{n-3} + J_{n-5} - a_{n-4} \\ &= lev_{n-1} + 2\ lev_{n-2} + J_{n-3}. \end{aligned}$$

Similar calculations provide the following, where $k \geq 1$,

$$\begin{aligned} n &= 4k+2 : lev_n = lev_{n-1} + 2\ lev_{n-2} + J_{n-3} + a_0 \\ n &= 4k+3 : lev_n = lev_{n-1} + 2\ lev_{n-2} + J_{n-3} \\ n &= 4k+4 : lev_n = lev_{n-1} + 2\ lev_{n-2} + J_{n-3} - a_0. \end{aligned}$$

The following recurrence relation deals with this cycle of four interrelated relations:

$$\begin{aligned} lev_n &= lev_{n-1} + 2\ lev_{n-2} + J_{n-3} - a_0 \left(\frac{1}{2}\right) [1+(-1)^n] (-1)^{n/2} \\ &= lev_{n-1} + 2\ lev_{n-2} + J_{n-3} - \left(\frac{1}{2}\right) [i^n + (-i)^n],\ n \geq 3. \end{aligned}$$

Since $i = \cos\frac{\pi}{2} + i\ \sin\frac{\pi}{2}$, it follows from DeMoivre's Theorem that $i^n = \cos\frac{n\pi}{2} + i\ \sin\frac{n\pi}{2}$. Likewise, $(-i)^n = (\cos(-\frac{\pi}{2}) + i\ \sin(-\frac{\pi}{2}))^n = \cos(-\frac{n\pi}{2}) + i\ \sin(-\frac{n\pi}{2}) = \cos\frac{n\pi}{2} - i\ \sin\frac{n\pi}{2}$. So $i^n + (-i)^n = 2\ \cos\frac{n\pi}{2}$ and we now have

$$\begin{aligned} lev_n &= lev_{n-1} + 2\ lev_{n-2} + J_{n-3} - \cos\frac{n\pi}{2} \\ &= lev_{n-1} + 2\ lev_{n-2} + \left[\left(\frac{1}{3}\right)(-1)^{n-3} + \left(\frac{2}{3}\right) 2^{n-3}\right] - \cos\frac{n\pi}{2}. \end{aligned}$$

The characteristic roots for the homogeneous part of the solution are -1 and 2, so the form for the particular solution here is

$$An\ (-1)^n + Bn\ 2^n + C\ \sin\frac{n\pi}{2} + D\ \cos\frac{n\pi}{2}.$$

To determine A we substitute $lev_n = An\ (-1)^n$ into the recurrence relation $lev_n = lev_{n-1} + 2\ lev_{n-2} + \left(\frac{1}{3}\right)(-1)^{n-3}$ to find that $-An =$

$A(n-1) - 2A(n-2) + (\frac{1}{3})$, so $A = \frac{-1}{9}$. A similar calculation shows us that $B = \frac{1}{18}$. Finally, we substitute $lev_n = C\ \sin\frac{n\pi}{2} + D\ \cos\frac{n\pi}{2}$ into the recurrence relation $lev_n = lev_{n-1} + 2\ lev_{n-2} - \cos\ \frac{n\pi}{2}$ to obtain

$$\begin{aligned}
C\sin\frac{n\pi}{2} + D\cos\frac{n\pi}{2} &= C\sin\frac{(n-1)\pi}{2} + D\cos\frac{(n-1)\pi}{2} \\
&\quad +2C\sin\frac{(n-2)\pi}{2} + 2D\cos\frac{(n-2)\pi}{2} - \cos\frac{n\pi}{2} \\
&= -C\cos\frac{n\pi}{2} + D\sin\frac{n\pi}{2} - 2C\sin\frac{n\pi}{2} \\
&\quad -2D\cos\frac{n\pi}{2} - \cos\frac{n\pi}{2}.
\end{aligned}$$

From this it follows that $C = D - 2C$ and $D = -C - 2D - 1$, so $C = \frac{-1}{10}$ and $D = \frac{-3}{10}$. Consequently,

$$\begin{aligned}
lev_n &= c_1(-1)^n + c_2\ 2^n + \left(\frac{-1}{9}\right) n\ (-1)^n + \left(\frac{1}{18}\right) n\ 2^n \\
&\quad + \left(\frac{-1}{10}\right)\sin\frac{n\pi}{2} + \left(\frac{-3}{10}\right)\cos\frac{n\pi}{2}.
\end{aligned}$$

From $0 = lev_1 = -c_1 + 2\,c_2 + \frac{1}{9} + \frac{1}{9} - \frac{1}{10}$ and $1 = lev_2 = c_1 + 4\,c_2 - \frac{2}{9} + \frac{8}{18} + \frac{3}{10}$ we learn that $c_1 = \frac{13}{54}$ and $c_2 = \frac{8}{135}$. Consequently, for $n \geq 1$,

$$\begin{aligned}
lev_n &= \frac{13}{54}(-1)^n + \frac{8}{135}\ 2^n - \frac{1}{9}n\ (-1)^n + \frac{1}{18}n\ 2^n \\
&\quad -\frac{1}{10}\sin\frac{n\pi}{2} - \frac{3}{10}\cos\frac{n\pi}{2} \\
&= \left(\frac{13-6n}{54}\right)(-1)^n + \left(\frac{16+15n}{270}\right)2^n - \frac{1}{10}\sin\frac{n\pi}{2} - \frac{3}{10}\cos\frac{n\pi}{2} \\
&= \left(\frac{6n-13}{54}\right)J_{n+1} + \left(\frac{146-45n}{270}\right)J_n + \left(\frac{16+15n}{270}\right)J_{n-1} \\
&\quad -\frac{1}{10}\sin\frac{n\pi}{2} - \frac{3}{10}\cos\frac{n\pi}{2}.
\end{aligned}$$

Now that we have settled the issue of the levels we shall turn our attention to the number of rises that occur among the a_n compositions of n. Given a composition of n, when two consecutive summands occur as $'x+y'$ and $x < y$, then we say that a *rise* has occurred. For example, the composition $'2+3+1+6+7'$ for 19 contains the three rises: $'2+3'$, $'1+6'$, and $'6+7'$. Now for $n \geq 1$, we shall let $rise_n$ count the number of rises that occur among the a_n compositions of n. We find that

$$rise_1 = rise_2 = rise_3 = 0,\ rise_4 = 2,\ rise_5 = 5,\ \text{and}\ rise_6 = 14.$$

To determine a general formula for $rise_n$, we consider the following summation formulas. For $n \geq 0$,

$$\begin{aligned} & J_0 + J_1 + J_2 + \ldots + J_n \\ = & \sum_{i=0}^{n} \left(\frac{1}{3}\right)(-1)^i + \left(\frac{2}{3}\right) 2^i \\ = & \begin{cases} \left(\frac{2}{3}\right)\left[2^{n+1}-1\right], & n \text{ odd} \\ \left(\frac{2}{3}\right)\left[2^{n+1}-1\right]+\frac{1}{3}, & n \text{ even} \end{cases}, \end{aligned}$$

so for $n \geq 1$,

$$J_1 + J_2 + \ldots + J_n = \begin{cases} \left(\frac{2}{3}\right)\left[2^{n+1}-1\right]-1, & n \text{ odd} \\ \left(\frac{2}{3}\right)\left[2^{n+1}-1\right]-\frac{2}{3}, & n \text{ even} \end{cases}.$$

As we did with the levels earlier in this section, we'll start by examining how to determine $rise_5$ in terms of $rise_4$ and $rise_3$. Initially we find that

$$rise_5 = rise_4 + (a_4 - s_{4,1}) + rise_3 + (a_3 - s_{3,1} - s_{3,2}) + rise_3,$$

where (i) the term $(a_4 - s_{4,1})$ accounts for the new rises that are created when '1+' is appended to any composition of 4 except those that start with 1; and (ii) the term $(a_3 - s_{3,1} - s_{3,2})$ deals with the new rises that are created when '2+' is appended to any composition of 3 except for those that start with 1 or 2. This equation can then be rewritten as

$$\begin{aligned} rise_5 &= rise_4 + (J_3 - J_2) + rise_3 + (J_2 - J_1 - J_0) + rise_3 \\ &= rise_4 + 2\ rise_3 + J_3 - (J_1 + J_0) \\ &= rise_4 + 2\ rise_3 + \left[\left(\frac{1}{3}\right)(-1)^3 + \left(\frac{2}{3}\right) 2^3\right] - \left(\frac{2}{3}\right)\left[2^2 - 1\right] \\ &= rise_4 + 2\ rise_3 + \frac{2}{3}\left(2^2\right) + \frac{1}{3}(-1)^2. \end{aligned}$$

Meanwhile for $n = 6$ we see that

$$rise_6 = rise_5 + (a_5 - s_{5,1}) + rise_4 + (a_4 - s_{4,1} - s_{4,2}) + rise_4 - a_1 - a_0.$$

For this case: (i) The term $-a_1$ accounts for the number of compositions of 4 that start with '$1+2$'. Adding 2 to 1 results in '$3+2$'. (ii) The term $-a_0$ deals with the number of compositions of 4 that start with '$1+3$'. Adding 2 to 1 now gives us '$3+3$'. In both (i) and (ii) we lose rises. We can rewrite this equation as

$$\begin{aligned} rise_6 &= rise_5 + (J_4 - J_3) + rise_4 + (J_3 - J_2 - J_1) + rise_4 - J_0 - 1 \\ &= rise_5 + 2\, rise_4 + J_4 - (J_2 + J_1 + J_0) - 1 \\ &= rise_5 + 2\, rise_4 + \frac{2}{3}\,(2^3) + \frac{1}{3}(-1)^3 \\ &= rise_5 + 2\, rise_4 + \frac{2}{3}\,(2^3) - \frac{2}{3}(-1)^3 + (-1)^3. \end{aligned}$$

However, for $n = 7$, we arrive at

$$\begin{aligned} rise_7 &= rise_6 + (a_6 - s_{6,1}) + rise_5 + (a_5 - s_{5,1} - s_{5,2}) \\ &\quad + rise_5 - a_2 - a_1 - a_0 \\ &= rise_6 + 2\, rise_5 + J_5 - (J_3 + J_2 + J_1 + J_0) - 1 \\ &= rise_6 + 2\, rise_5 + \left(\frac{2}{3}\right) 2^4 - \left(\frac{2}{3}\right)(-1)^4. \end{aligned}$$

For the general case consider $n = 4k + 1,\ k \geq 1$. Then

$$\begin{aligned} rise_n &= rise_{n-1} + (a_{n-1} - s_{n-1,1}) + rise_{n-2} \\ &\quad + (a_{n-2} - s_{n-2,1} - s_{n-2,2}) \\ &\quad + \begin{cases} rise_{n-2}, & k = 1 \\ rise_{n-2} - a_{4(k-1)} - a_{4(k-1)-1} - \dots - a_2 - a_1, & k > 1 \end{cases} \end{aligned}$$

where, for example, $-a_{4(k-1)}$ accounts for the loss of the rises that result from when we add 2 to the first summand of the compositions of $n-2$ that start with $'1+2'$; $-a_{4(k-1)-1}$ for the loss when we add 2 to the compositions of $n-2$ that start with $'1+3'$; $-a_{4(k-1)-2}$ for the loss that results when we add 2 to those compositions of $n-2$ that start with $'2+3'$; ...; $-a_1$ for the loss that results from the compositions of $n-2$ that start with $'(2k-2)+2k'$. For $n = 4k+1$ with $k \geq 1$, this recurrence relation can be rewritten as

$$\begin{aligned} rise_n &= rise_{n-1} + 2\, rise_{n-2} + J_{n-2} - (J_{n-4} + J_{n-5} + \dots + J_1 + J_0) \\ &= rise_{n-1} + 2\, rise_{n-2} + \left(\frac{2}{3}\right) 2^{n-3} - \left(\frac{2}{3}\right)(-1)^{n-3} + (-1)^{n-3}. \end{aligned}$$

Similar calculations provide the following, where $k \geq 1$:

$$\begin{aligned} n &= 4k+2 : rise_n = rise_{n-1} + 2\, rise_{n-2} + \left(\frac{2}{3}\right) 2^{n-3} - \left(\frac{2}{3}\right)(-1)^{n-3} \\ &\quad +(-1)^{n-3} \\ n &= 4k+3 : rise_n = rise_{n-1} + 2\, rise_{n-2} + \left(\frac{2}{3}\right) 2^{n-3} - \left(\frac{2}{3}\right)(-1)^{n-3} \\ n &= 4k+4 : rise_n = rise_{n-1} + 2\, rise_{n-2} + \left(\frac{2}{3}\right) 2^{n-3} - \left(\frac{2}{3}\right)(-1)^{n-3}. \end{aligned}$$

Consequently, we see that

$$\begin{aligned} rise_n &= rise_{n-1} + 2\, rise_{n-2} + \left(\frac{2}{3}\right) 2^{n-3} - \left(\frac{2}{3}\right)(-1)^{n-3} \\ &\quad + \begin{cases} 0, & n \equiv 0 \pmod 4 \\ 1, & n \equiv 1 \pmod 4 \\ -1, & n \equiv 2 \pmod 4 \\ 0, & n \equiv 3 \pmod 4 \end{cases}. \end{aligned}$$

Note that this recurrence relation is true for all $n \geq 3$. But in order to solve this recurrence relation we need to combine the four relations into one and to do this we need a formula for the sequence

$$0,\ 1,\ -1,\ 0,\ 0,\ 1,\ -1,\ 0,\ 0,\ 1,\ -1,\ 0,\ \ldots,$$

where the leading 0 is the 0-th term of the sequence. In Sloane's *Encyclopedia of Integer Sequences* [8], sequence A102560:

$$1,\ 0,\ 0,\ -1,\ 1,\ 0,\ 0,\ -1,\ 1,\ 0,\ \ldots$$

(provided by Paul Barry) can be generated by the formula $\frac{1}{2}\cos(\frac{n\pi}{2}) + \frac{1}{2}\sin(\frac{n\pi}{2}) + \frac{1}{2}\cos n\pi$. Using this as the starting point, we find that the sequence we need to determine the recurrence relation for $rise_n$ can be given by the formula

$$\frac{1}{2}\cos\left(\frac{n\pi}{2}\right) + \frac{1}{2}\sin\left(\frac{n\pi}{2}\right) - \frac{1}{2}\cos n\pi.$$

Consequently, for $n \geq 3$, we now have

$$\begin{aligned} rise_n &= rise_{n-1} + 2\, rise_{n-2} + \left(\frac{2}{3}\right) 2^{n-3} - \left(\frac{2}{3}\right)(-1)^{n-3} + \\ &\quad \frac{1}{2}\cos\left(\frac{n\pi}{2}\right) + \frac{1}{2}\sin\left(\frac{n\pi}{2}\right) - \frac{1}{2}\cos n\pi \\ &= rise_{n-1} + 2\, rise_{n-2} + \left(\frac{2}{3}\right) 2^{n-3} - \left(\frac{1}{6}\right)(-1)^{n-3} + \\ &\quad \frac{1}{2}\cos\left(\frac{n\pi}{2}\right) + \frac{1}{2}\sin\left(\frac{n\pi}{2}\right). \end{aligned}$$

Here the particular part of the solution has the form

$$An\, 2^n + Bn\, (-1)^n + C \cos\left(\frac{n\pi}{2}\right) + D \sin\left(\frac{n\pi}{2}\right).$$

Upon substituting $An\, 2^n$ for $rise_n$ in $rise_n = rise_{n-1} + 2\, rise_{n-2} + (\frac{2}{3})\, 2^{n-3}$ we find that $8An = 4A(n-1) + 4A(n-2) + \frac{2}{3}$, from which it follows that $A = \frac{1}{18}$. A comparable calculation shows us that $B = \frac{1}{18}$. To determine C and D, we substitute $C \cos\left(\frac{n\pi}{2}\right) + D \sin\left(\frac{n\pi}{2}\right)$ for $rise_n$ in $rise_n = rise_{n-1} + 2\, rise_{n-2} + \frac{1}{2}\cos\frac{n\pi}{2} + \frac{1}{2}\sin\frac{n\pi}{2}$ and learn that

$$\begin{aligned} & C\cos\left(\frac{n\pi}{2}\right) + D\sin\left(\frac{n\pi}{2}\right) \\ = \; & C\sin\left(\frac{n\pi}{2}\right) - D\cos\left(\frac{n\pi}{2}\right) - 2C\cos\left(\frac{n\pi}{2}\right) - 2D\sin\left(\frac{n\pi}{2}\right) \\ & + \frac{1}{2}\cos\left(\frac{n\pi}{2}\right) + \frac{1}{2}\sin\left(\frac{n\pi}{2}\right), \end{aligned}$$

from which it follows that $C = -D - 2C + \frac{1}{2}$ and $D = C - 2D + \frac{1}{2}$, so $C = \frac{1}{10}$ and $D = \frac{1}{5}$. Consequently,

$$rise_n = c_1(-1)^n + c_2\, 2^n + \frac{1}{18} n\, 2^n + \frac{1}{18} n\, (-1)^n + \frac{1}{10}\cos\left(\frac{n\pi}{2}\right) + \frac{1}{5}\sin\left(\frac{n\pi}{2}\right).$$

From $rise_1 = rise_2 = 0$ it then follows that $c_1 = \frac{1}{54}$ and $c_2 = \frac{-16}{135}$. So, for $n \geq 1$,

$$\begin{aligned} rise_n &= \frac{1}{54}(-1)^n - \frac{16}{135}\, 2^n + \frac{1}{18} n\, 2^n + \frac{1}{18} n\, (-1)^n \\ &\quad + \frac{1}{10}\cos\left(\frac{n\pi}{2}\right) + \frac{1}{5}\sin\left(\frac{n\pi}{2}\right) \\ &= \left(\frac{3n+1}{54}\right)(-1)^n + \left(\frac{15n-32}{270}\right) 2^n \\ &\quad + \frac{1}{10}\cos\left(\frac{n\pi}{2}\right) + \frac{1}{5}\sin\left(\frac{n\pi}{2}\right) \\ &= \left(\frac{15n-32}{270}\right) J_{n-1} + \left(\frac{45n-22}{270}\right) J_n - \left(\frac{3n+1}{54}\right) J_{n+1} \\ &\quad + \frac{1}{10}\cos\left(\frac{n\pi}{2}\right) + \frac{1}{5}\sin\left(\frac{n\pi}{2}\right). \end{aligned}$$

Finally, with the levels and rises now determined, it is time to close this section with a formula for the number of descents that occur among the a_n compositions of n. Given a composition of n, when two consecutive summands occur as $'x+y'$ with $x > y$, then we have the occurrence of a *descent*. For example, the composition $'1+2+1+8+7'$ for 19 contains the two descents: $'2+1'$ and $'8+7'$. For $n \geq 1$, we let $desc_n$ count the number of descents that occur among the a_n compositions of n. For example,

$$desc_1 = 0,\; desc_2 = 0,\; desc_3 = 1,\; desc_4 = 3,\; desc_5 = 7.$$

Since

$$desc_n + lev_n + rise_n = pl_n,$$

it follows that for $n \geq 1$

$$\begin{aligned} desc_n &= \left(\frac{3n-8}{54}\right)(-1)^n + \left(\frac{15n-14}{270}\right)2^n \\ &\quad -\frac{1}{10}\sin\left(\frac{n\pi}{2}\right) + \frac{1}{5}\cos\left(\frac{n\pi}{2}\right) \\ &= \left(\frac{15n-14}{270}\right)J_{n-1} + \left(\frac{45n-94}{270}\right)J_n - \left(\frac{3n-8}{54}\right)J_{n+1} \\ &\quad -\frac{1}{10}\sin\left(\frac{n\pi}{2}\right) + \frac{1}{5}\cos\left(\frac{n\pi}{2}\right). \end{aligned}$$

5. Runs

In this final section we shall determine the number of runs that occur among the a_n compositions of n. A *run* in a composition is a consecutive summation of one, or more, summands that are equal - where the summation is preceded by nothing or a different summand and is followed by nothing or a different summand. For example, the composition $2+2+3+1+1+1+3+3+3$ of 19 contains four runs: (i) $2+2$; (ii) 3; (iii) $1+1+1$; and (iv) $3+3+3$. If we let run_n count the number of runs that occur among the a_n compositions of n, we find the following initial values:

$$run_1 = 1,\ run_2 = 1,\ run_3 = 4,\ run_4 = 10,\ run_5 = 23.$$

To develop a recurrence relation for run_n, we consider the cases where $n = 5$ and $n = 6$. When $n = 5$ we find that

$$run_5 = run_4 + (a_4 - s_{4,1}) + run_3 + (a_3 - s_{3,2}) + run_3 + a_1,$$

where (i) the term $(a_4 - s_{4,1})$ accounts for the new runs that arise from when we append '1+' to the a_4 compositions of 4; (ii) the term $(a_3 - s_{3,2})$ accounts for the new runs that arise from when we append '2+' to the a_3 compositions of 3; and (iii) a_1 accounts for the new runs that arise from when we add 2 to the first summand - namely, '1' - of the a_1 compositions of 3 that start with '1 + 1'. This result can also be expressed as

$$\begin{aligned} run_5 &= run_4 + (J_3 - J_2) + run_3 + (J_2 - J_0) + run_3 + J_0 \\ &= run_4 + 2\,run_3 + J_3. \end{aligned}$$

In the case of $n = 6$ we learn that

$$run_6 = run_5 + (a_5 - s_{5,1}) + run_4 + (a_4 - s_{4,2}) + run_4 - a_0 + a_2.$$

Here the term $-a_0$ accounts for the runs we lose when we add 2 to the a_0 compositions of 4 that start with $'1+3'$. The term a_2 deals with the new runs that arise from adding 2 to the compositions of 4 that start with $'1+1'$. This equation can be rewritten as

$$\begin{aligned} run_6 &= run_5 + (J_4 - J_3) + run_4 + (J_3 - J_1) + run_4 - 1 + J_1 \\ &= run_5 + 2\, run_4 + J_4 - 1. \end{aligned}$$

For the general situation consider $n = 4k+1$, for $k \geq 1$. We find that

$$\begin{aligned} run_n &= run_{n-1} + (a_{n-1} - s_{n-1,1}) + run_{n-2} + (a_{n-2} - s_{n-2,2}) \\ &\quad + run_{n-2} \\ &\quad \begin{cases} + a_1, & k = 1 \\ - a_{n-6} - a_{n-8} - \ldots - a_1 + a_{n-4} + a_{n-6} + \ldots + a_1, & k \geq 2 \end{cases}, \end{aligned}$$

where, for example, $-a_{n-6}$ accounts for the runs lost when we add 2 to the first summand of the a_{n-2} compositions of $n-2$ that start with $'1+3'$; $-a_{n-6}$ accounts for the runs lost when we add 2 to the first summand of the a_{n-2} compositions of $n-2$ that start with $'2+4'$; ...; $-a_1$ the loss when we add 2 to the first summand of the a_{n-2} compositions of $n-2$ that start with $'(2k-2)+2k'$. Meanwhile, a_{n-4} accounts for the new runs that result from adding 2 to the first summand of the a_{n-2} compositions of $n-2$ that start with $'1+1'$; a_{n-6} accounts for the new runs that result from adding 2 to the first summand of the a_{n-2} compositions of $n-2$ that start with $'2+2'$; ...; a_1 accounts for the new runs that result from adding 2 to the first summand of the a_{n-2} compositions of $n-2$ that start with $'(2k-1)+(2k-1)'$. For $n = 4k+1$ with $k \geq 1$, this recurrence relation can be rewritten as

$$run_n = run_{n-1} + 2\, run_{n-2} + a_{n-1} = run_{n-1} + 2\, run_{n-2} + J_{n-2}.$$

Similar calculations provide the following for $k \geq 0$, if we define $run_0 = 0$:

$$\begin{aligned} n &= 4k+2 : run_n = run_{n-1} + 2\, run_{n-2} + J_{n-2} - 1 \\ n &= 4k+3 : run_n = run_{n-1} + 2\, run_{n-2} + J_{n-2} \\ n &= 4k+4 : run_n = run_{n-1} + 2\, run_{n-2} + J_{n-2} + 1. \end{aligned}$$

As with the levels in Section 4, for $n \geq 2$, we combine the four cases into the recurrence relation

$$\begin{aligned} run_n &= run_{n-1} + 2\,run_{n-2} + J_{n-2} + \left(\frac{1}{2}\right)[1+(-1)^n](-1)^{n/2} \\ &= run_{n-1} + 2\,run_{n-2} + J_{n-2} + \left(\frac{1}{2}\right)[i^n + (-i)^n] \\ &= run_{n-1} + 2\,run_{n-2} + J_{n-2} + \cos\left(\frac{n\pi}{2}\right) \\ &= run_{n-1} + 2\,run_{n-2} + \left(\frac{1}{3}\right)(-1)^{n-2} + \left(\frac{2}{3}\right)2^{n-2} + \cos\left(\frac{n\pi}{2}\right). \end{aligned}$$

Here the solution has the form

$$run_n = c_1(-1)^n + c_2\,2^n + An\,(-1)^n + Bn\,2^n + C\cos\left(\frac{n\pi}{2}\right) + D\sin\left(\frac{n\pi}{2}\right).$$

As we did in Section 4 we find that the particular solution is

$$\frac{1}{9}n\,(-1)^n + \frac{1}{9}n\,2^n + \frac{3}{10}\cos\left(\frac{n\pi}{2}\right) + \frac{1}{10}\sin\left(\frac{n\pi}{2}\right).$$

From the initial conditions we find that $0 = run_0 = c_1 + c_2 + \frac{3}{10}$ and $1 = run_1 = -c_1 + 2\,c_2 - \frac{1}{9} + \frac{2}{9} + \frac{1}{10} = -c_1 + 2\,c_2 + \frac{19}{90}$. This leads us to $c_1 = \frac{-25}{54}$ and $c_2 = \frac{22}{135}$, and so, for $n \geq 0$,

$$\begin{aligned} run_n &= \left(\frac{-25}{54}\right)(-1)^n + \left(\frac{22}{135}\right)2^n + \frac{1}{9}n\,(-1)^n + \frac{1}{9}n\,2^n \\ &\quad + \frac{3}{10}\cos\left(\frac{n\pi}{2}\right) + \frac{1}{10}\sin\left(\frac{n\pi}{2}\right) \\ &= \left(\frac{6n-25}{54}\right)(-1)^n + \left(\frac{15n+22}{135}\right)2^n \\ &\quad + \frac{3}{10}\cos\left(\frac{n\pi}{2}\right) + \frac{1}{10}\sin\left(\frac{n\pi}{2}\right) \\ &= \left(\frac{15n+22}{135}\right)J_{n-1} + \left(\frac{45n-103}{135}\right)J_n + \left(\frac{25-6n}{54}\right)J_{n+1} \\ &\quad + \frac{3}{10}\cos\left(\frac{n\pi}{2}\right) + \frac{1}{10}\sin\left(\frac{n\pi}{2}\right). \end{aligned}$$

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