

# ON THE CRITICAL NUMBER OF FINITE GROUPS (II)

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**ABSTRACT.** Let  $G$  be a finite group and  $S$  a subset of  $G \setminus \{0\}$ . We call  $S$  an *additive basis of  $G$*  if every element of  $G$  can be expressed as a sum over a nonempty subset in some order. Let  $cr(G)$  be the smallest integer  $t$  such that every subset of  $G \setminus \{0\}$  of cardinality  $t$  is an additive basis of  $G$ . In this paper, we determine  $cr(G)$  for the following cases: (i)  $G$  is a finite nilpotent group; (ii)  $G$  is a group of even order which possesses a subgroup of index 2.

## 1. INTRODUCTION AND MAIN RESULTS

Let  $G$  be a finite additively written group (not necessarily commutative). Let  $S = \{a_1, \dots, a_k\}$  be a subset of  $G \setminus \{0\}$ . Define  $\sum(S) = \{a_{i_1} + \dots + a_{i_l} \mid i_1, \dots, i_l \text{ are distinct and } 1 \leq l \leq k\}$ , and for any  $1 \leq r \leq k$ , define  $\sum_r(S) = \{a_{i_1} + \dots + a_{i_r} \mid i_1, \dots, i_r \text{ are distinct}\}$ . We call  $S$  an *additive basis of  $G$*  if  $\sum(S) = G$ . The critical number  $cr(G)$  of  $G$  is the smallest integer  $t$  such that every subset  $S$  of  $G \setminus \{0\}$  with  $|S| \geq t$  forms an additive basis of  $G$ .

Let  $Z_n$  be the cyclic group of  $n$  elements.  $cr(G)$  was first introduced and studied by Erdős and Heilbronn in 1964 for  $G = Z_p$  where  $p$  is a prime. With many mathematicians' efforts, after nearly half a century,  $cr(G)$  has been determined for all finite abelian groups recently (see [3][5][7][10][11][13]).

However, the problem to determine  $cr(G)$  for  $G$  non-abelian is widely open. So far, we only have the following results in this direction.

**Theorem 1.1.** ([8], [14]) *Let  $G$  be a finite group of order  $n$  and  $p$  be the smallest prime divisor of  $n$ . Then,  $cr(G) = n/p + p - 2$  providing one of the following conditions holds,*

- (i)  $G$  is nilpotent,  $p \geq 149$  and  $n \geq 120p^2$ ;

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(ii) There exists a subgroup of  $G$  with index  $p$  and the other prime divisor of  $n$  (if exists) is larger than  $6p$ ,  $p \geq 149$  and  $n \geq 120p^2$ ;

(iii)  $G$  be a non-abelian group of order  $pq \geq 10$  where  $q$  is a prime.

In this paper we shall determine  $cr(G)$  for all groups  $G$  as stated in the abstract by showing the following two results.

**Theorem 1.2.** *Let  $G$  be a finite nilpotent group of odd order and let  $p$  be the smallest prime dividing  $|G|$ . If  $\frac{|G|}{p}$  is a composite number then  $cr(G) = |G|/p + p - 2$ .*

**Theorem 1.3.** *Let  $G$  be a finite non-abelian group of even order  $n$  which possesses a subgroup of index 2. Then,*

(i) *if  $n = 6$  then  $cr(G) = cr(S_3) = 4$ , where  $S_3$  denotes the symmetric group of six elements;*

(ii)  *$cr(G) = n/2$ , otherwise.*

**Remark 1.4.** *The proofs of Theorem 1.2 and 1.3 will be heavily based on the ideas contained in [10] and [11] respectively.*

**Remark 1.5.** *From Theorem 1.1, Theorem 1.2, Theorem 1.3, and the fact that  $cr(G)$  has been determined for all finite abelian groups we know that, the critical number  $cr(G)$  also has been determined for all finite nilpotent groups and all finite groups of even order which possesses a subgroup of index two. However, for finite groups which contains no subgroup with index  $p$  ( $p$  is the smallest prime divisor of the order of  $G$ ), we even can't guess the exact value of  $cr(G)$ .*

## 2. NOTATION AND PRELIMINARY LEMMAS

**Lemma 2.1.** ([12]) *Let  $G$  be a finite group. Let  $A$  and  $B$  be subsets of  $G$  such that  $|A| + |B| > |G|$ . Then  $A + B = G$ , where  $A + B = \{a + b \mid a \in A, b \in B\}$ .*

M. B. Nathanson ([12], Lemma 2.1) stated the conclusion of Lemma 2.1 for abelian groups, but the method used there does work for the nonabelian groups. For convenience, we repeat the proof here.

*Proof of Lemma 2.1.* For  $g \in G$ , let  $g - B = \{g - b : b \in B\}$ . Since

$$\begin{aligned} |G| &\geq |A \cup (g - B)| \\ &= |A| + |g - B| - |A \cap (g - B)| \\ &= |A| + |B| - |A \cap (g - B)|, \end{aligned}$$

it follows that

$$|A \cap (g - B)| \geq |A| + |B| - |G| \geq 1,$$

and so there exist a element  $a \in A$  and  $b \in B$  such that  $g = a + b$ . This completes the proof of the lemma.  $\square$

**Lemma 2.2.** ([5]) *Let  $p, q$  be two primes and  $G$  be a finite abelian group of order  $pq$ . Let  $S$  be a subset of  $G$  such that  $0 \notin S$  and  $|S| = p + q - 1$ . Then  $\sum(S) = G$ .*

Let  $G$  be a finite group. Let  $B \subset G$  and  $x \in G$ . As usual, we write  $\lambda_B(x) = |(B + x) \setminus B|$ . For any  $B, x$ , Olson proved in [2]

$$(2.1) \quad \lambda_B(x) = \lambda_B(-x)$$

and

$$(2.2) \quad \lambda_B(x) = \lambda_{G \setminus B}(x).$$

We use the following property which is implicit in [2]: Let  $G$  be a finite group. Let  $S$  be a subset of  $G$  such that  $0 \notin S$ . Put  $B = \sum(S)$ . For every  $y \in S$ , we have

$$\begin{aligned} \lambda_B(y) &= |(\sum(S) + y) \setminus \sum(S)| \\ &\leq |(\sum(S) + y) \setminus (\sum(S \setminus y) + y)| \\ &= |\sum(S) \setminus \sum(S \setminus y)| \\ &= |\sum(S)| - |\sum(S \setminus y)|. \end{aligned}$$

By above analysis we get the following inequality

$$(2.3) \quad |\sum(S)| \geq |\sum(S \setminus y)| + \lambda_B(y).$$

We also use the following result of Olson.

**Lemma 2.3.** ([4]) *Let  $G$  be a finite group and let  $S$  be a generating subset of  $G$  such that  $0 \notin S$ . Let  $B$  be a subset of  $G$  such that  $|B| \leq |G|/2$ . Then there is a  $x \in S$  such that*

$$\lambda_B(x) \geq \min((|B| + 1)/2, (|S \cup -S| + 2)/4).$$

**Lemma 2.4.** *Let  $G$  be a finite group of odd order. Let  $S$  be a subset of  $G$  such that  $S \cap -S = \emptyset$  and  $|S| \geq 3$ . Then  $|\sum(S)| \geq 2|S|$ .*

*Proof.* We proceed by induction on  $|S|$ . For  $|S| = 3$ , set  $S = \{a, b, c\}$ . In order to prove  $|\sum(S)| \geq 6$ , we distinguish three cases.

**Case 1.**  $a + b = c$ . We consider the sequence  $(a, b, c, a + c, c + b, a + b + c)$ . If  $a + c = b$ , then  $2a = 0$ , a contradiction. If  $c + b = a$ , then  $2b = 0$ , a contradiction. If  $c + b = a + c$ , then  $b = a$ , a contradiction. If  $c + b = a + b + c$ , then  $b = c$ , a contradiction. If  $b = a + b + c$ , then  $a + c = 0$ , a contradiction. By above analysis we have that  $\{a, b, c, a + c, c + b, a + b + c\}$  is a set, then  $|\sum(S)| \geq 6$ .

**Case 2.**  $a + c = b$ . The proof is similar to Case 1.

**Case 3.**  $b + c = a$ . The proof is similar to Case 1.

**Case 4.**  $a + b \neq c, a + c \neq b, b + c \neq a$ . Now we have that either  $a, b, c, a + b, a + c, b + c$  or  $a, b, c, a + b, a + c, a + b + c$  are pairwise distinct. This proves the lemma for  $|S| = 3$ .

Now assume that the lemma is true for smaller  $|S|$ . Set  $B = \sum(S)$ . Applying Lemma 2.3 to  $B$  or  $G \setminus B$  and using (2.2), there exists a  $y \in S$  such that  $\lambda_B(y) \geq 2$ . By (2.3),  $|B| \geq |\sum(S \setminus y)| + 2 \geq 2|S|$ . This completes the proof.  $\square$

Let  $X$  be a subset of  $G$  with cardinality  $k$ . Let  $\{x_i, 1 \leq i \leq k\}$  be an ordering of  $X$ . For  $0 \leq i \leq k$ , set  $X_i = \{x_j | 1 \leq j \leq i\}$  and  $B_i = \sum(X_i)$ . The ordering  $\{x_1, \dots, x_k\}$  will be called a *resolving sequence* of  $X$  if for all  $i$ ,  $\lambda_{B_i}(x_i) = \max\{\lambda_{B_i}(x_j); 1 \leq j \leq i\}$ . The *critical index* of the resolving sequence is the maximal integer  $t$  such that  $X_{t-1}$  generates a proper subgroup of  $G$ .

Clearly, every nonempty subset  $S$  not containing 0 admits a resolving sequence. Moreover, the critical index is  $\geq 1$ .

We shall write  $\lambda_i = \lambda_{B_i}(x_i)$ . By induction we have using (2.3) for all  $1 \leq j \leq k$ ,

$$|\sum(X)| \geq \lambda_k + \dots + \lambda_j + |B_{j-1}|.$$

Put  $\delta(m) = 0$  if  $m$  is odd and  $= 1$  otherwise. If  $\sum(X) < n/2$ , by Lemma 2.3,  $\lambda_i \geq (i + 1 + \delta(i))/2$  for all  $i \geq t$ . In particular for all  $s \geq t$ , we have

$$(2.4) \quad |\sum(X)| \geq (k + s + 3)(k - s + 1)/4 - 1/2 + |B_{s-1}|.$$

**Lemma 2.5.** *Let  $G$  be a finite group of order 9, and let  $A, B$  be two subsets of  $G$ .*

- (i) *If  $|A| = 3$  and  $A$  is zero-sum free then  $|\sum(A)| \geq 6$ .*
- (ii) *If  $|A| = 3$  and  $0 \notin A$  then  $|\sum(A)| \geq 5$ .*
- (iii) *If  $|A| = 4$  and  $0 \notin A$  then  $|\sum(A)| \geq 7$ .*
- (iv) *If  $|A| = 4$  then  $|\sum_2(A)| \geq 5$ .*
- (v) *If  $|A| = 4$  and  $|B| \geq 2$  then  $|A + B| \geq 5$ .*

*Proof.* By the basic knowledge of group theory we know that  $G$  is abelian.

- (i) One can find a proof in [9].
- (ii) Let  $A = \{a_1, a_2, a_3\}$ . Assume to the contrary that

$$|\sum(A)| \leq 4.$$

It follows that  $|\{a_1, a_2, a_3\} \cap \{a_1 + a_2, a_1 + a_3, a_2 + a_3\}| \geq 2$ . Without loss of generality we may assume that  $a_1 = a_2 + a_3$  and  $a_2 = a_1 + a_3$ . Therefore,  $a_1 + a_2 = a_2 + a_3 + a_1 + a_3$ . Hence,  $2a_3 = 0$ . Thus,  $a_3 = 0$  for  $|G| = 9$ , a contradiction with  $A \subset G \setminus \{0\}$ .

- (iii) Let  $A = \{a_1, a_2, a_3, a_4\}$ . Assume to the contrary that

$$|\sum(A)| \leq 6.$$

If  $A$  is zero-sum free then  $a_1 + a_2 + a_3 + a_4 \notin \sum(\{a_1, a_2, a_3\})$ . By (i) we obtain that  $|\sum(A)| \geq |\{a_1 + a_2 + a_3 + a_4\} \cup \sum(\{a_1, a_2, a_3\})| = 1 + |\sum(\{a_1, a_2, a_3\})| \geq 7$ , a contradiction. Hence,

$$0 \in \sum(A).$$

Therefore,

$$|\sum(A) \setminus \{0\}| \leq 5.$$

This together with (i) implies that

(\*)  $A$  contains no zero-sum free sequence of length 3.

By rearranging if necessary we may assume that  $a_1 + a_2 + a_3 \neq 0$ . By (\*) we may assume that  $a_1 + a_2 = 0$  (by rearranging if necessary). Since  $a_1 \neq a_2$ , either  $a_1 + a_3 + a_4 \neq 0$  or  $a_2 + a_3 + a_4 \neq 0$ . Without loss of generality we assume that  $a_1 + a_3 + a_4 \neq 0$ . It follows from (\*) and  $a_1 + a_2 = 0$  that  $a_3 + a_4 = 0$ . Now we have

$$A = \{a_1, -a_1, a_3, -a_3\}.$$

Since  $\{0\} \cup A = \{0, a_1, -a_1, a_3, -a_3\} \subset \sum(A)$ , by the contrary hypothesis we infer that  $\{a_1 + a_3, -(a_1 + a_3)\} \cap A \neq \emptyset$ . By the symmetry of  $A$  we may assume that  $a_1 + a_3 \in A$ . Therefore,  $a_1 + a_3 = -a_1$  or  $a_1 + a_3 = -a_3$ . Again by the symmetry of  $A$  we may assume that  $a_1 + a_3 = -a_1$ . Thus,  $a_3 = -2a_1$ . Now we have  $A = \{a_1, -a_1, 2a_1, -2a_1\}$ . Since  $|G| = 9$  and  $a_1 \neq -2a_1$ , it is easy to see that  $0, a_1, -a_1, 2a_1, -2a_1, 3a_1, -3a_1$  are 7 distinct elements from  $\sum(A)$ , a contradiction.

(iv) Let  $A = \{a_1, a_2, a_3, a_4\}$ . Assume to the contrary that  $|\sum_2(A)| \leq 4$ . It follows that  $|\{a_1 + a_2, a_1 + a_3, a_1 + a_4\} \cap \{a_2 + a_3, a_2 + a_4, a_3 + a_4\}| \geq 2$ . By rearranging if necessary we assume that  $a_1 + a_2 = a_3 + a_4$  and  $a_1 + a_3 = a_2 + a_4$ . Thus,  $a_1 + a_2 + a_1 + a_3 = a_3 + a_4 + a_2 + a_4$ . It follows that  $a_1 = a_4$ , a contradiction.

(v) Let  $B = \{b_1, b_2\}$ . Assume to the contrary that  $|A + B| \leq 4$ . It follows that  $|A + B| = 4$ , and  $b_1 + A = b_2 + A = A + B$ . Therefore,  $\sum_{a \in A}(b_1 + a) = \sum_{a \in A}(b_2 + a)$ . Hence,  $|A|b_1 + \sum_{a \in A} a = |A|b_2 + \sum_{a \in A} a$ . Thus,  $4b_1 = 4b_2$  and  $b_1 = b_2$ , a contradiction.  $\square$

**Lemma 2.6.** *If  $|G| = 27$  then  $cr(G) = 10$ .*

*Proof.* We only need to check the case that  $G$  is non-abelian. Since  $G$  is a nilpotent group,  $G$  possesses a normal subgroup  $K$  of index 3. Suppose  $G/K = \langle 1 + K \rangle$ . Let  $x \in 1 + K$  and  $T = (K \setminus \{0\}) \cup \{x\}$ . It is easy to see that  $-1 + K \not\subset \sum(T)$ . This shows that  $cr(G) \geq 10$ . So it suffices to prove that  $cr(G) \leq 10$ . Let  $S \subset G \setminus \{0\}$  and  $|S| = 10$ . We want to show that  $\sum(S) = G$ .

From the basic knowledge on  $p$ -groups (see [6]) we know that there exist exactly four distinct maximal subgroups of  $G$  and each is a normal subgroup

of order 9, and  $G$  equals to the union of these maximal subgroups. Since  $|S| = 10 = 2 \times 4 + 2$ , there exists a maximal subgroup  $H$  of  $G$  such that

$$|S \cap H| \geq 3.$$

Now we fix  $a \in G \setminus H$ . Then,  $G = H \cup (a + H) \cup (2a + H)$ . It suffices to prove the following inclusions hold simultaneously:

$$H \subset \sum(S), \quad a + H \subset \sum(S), \quad 2a + H \subset \sum(S).$$

Let  $A = (a + H) \cap S$  and  $B = (2a + H) \cap S$ . Suppose

$$A = \{a + a_1, \dots, a + a_r\}, \quad B = \{2a + b_1, \dots, 2a + b_t\},$$

where  $r \geq t \geq 0$ ,  $r + t = 10 - |S \cap H|$ , and  $a_i, b_j \in H$ .

Since  $H$  is a normal subgroup of  $G$ , we also have that

$$a + a_i = a'_i + a, \quad 2a + b_j = b'_j + 2a,$$

where  $a'_i, b'_j \in H$ .

We distinguish three cases.

**Case 1.**  $|S \cap H| \geq 5$ . By Lemma 2.2 we get

$$H = \sum(S \cap H) \subset \sum(S).$$

Since  $|S \cap H| \leq |H \setminus \{0\}| = 8$ ,  $|S \cap (G \setminus H)| \geq 2$ . Therefore,  $(\sum(S \cap (G \setminus H))) \cap (a + H) \neq \emptyset$  and  $(\sum(S \cap (G \setminus H))) \cap (2a + H) \neq \emptyset$ . It follows from  $H = \sum(S \cap H)$  that  $a + H \subset \sum(S)$  and  $2a + H \subset \sum(S)$ .

**Case 2.**  $|S \cap H| = 4$ . Now we have

$$r + t = 6.$$

By Lemma 2.5(iii) we obtain that

$$|\sum(S \cap H)| \geq 7.$$

**Subcase 2.1.**  $r = t = 3$ . Note that  $(2a + b_i) + (a + a_i) = b'_i + 2a + a + a_i = b'_i + 3a + a_i = h + a_i$ , where  $h = b'_i + 3a \in H$ . By Lemma 2.1  $\sum(S \cap H) + \{h + a_1, h + a_2, h + a_3\} = H$ . Therefore  $H \subset \sum(S \cap H) + (2a + b_1) + A \subset \sum(S)$ . Again by Lemma 2.1 we have that  $a + H \subset A + \sum(S \cap H) \subset \sum(S)$  and  $2a + H \subset B + \sum(S \cap H) \subset \sum(S)$ .

**Subcase 2.2.**  $r \geq 4$  and  $t \geq 1$ . Similar to Subcase 2.1 we know that  $H \subset \sum(S)$  and  $a + H \subset \sum(S)$ . Note that  $\sum_2(A) \supset \{a + a_1 + a + a_2, a + a_1 + a + a_3, a + a_1 + a + a_4\}$ . Therefore,  $\sum_2(A) = 2a + C$  with  $C \subset H$  and  $|C| \geq 3$ . By Lemma 2.1, we have  $2a + H = 2a + C + \sum(S \cap H) = \sum_2(A) + \sum(S \cap H) \subset \sum(S)$ .

**Subcase 2.3.**  $r = 6$ . Similarly to Subcase 2.2 one can prove that  $a + H \subset \sum(S)$  and  $2a + H \subset \sum(S)$ . Since  $\{a + a_1 + a + a_2 + a + a_3, a + a_1 + a + a_2 + a + a_4, a + a_1 + a + a_2 + a + a_5\} \subset \sum_3(A)$ , we infer

that  $|\sum_3(A)| \geq 3$ . Note that  $\sum_3(A) \subset H$ . By Lemma 2.1, we have  $H = \sum_3(A) + \sum(S \cap H) \subset \sum(S)$ .

**Case 3.**  $|S \cap H| = 3$ . By Lemma 2.5 we get

$$|\sum(S \cap H)| \geq 5.$$

In this case we have

$$r + t = 7.$$

**Subcase 3.1.**  $r = 4$  and  $t = 3$ . Note that  $A + B = \{a'_1 + a, a'_2 + a, a'_3 + a, a'_4 + a\} + \{2a + b_1, 2a + b_2, 2a + b_3\} = \{a'_1 + 3a, a'_2 + 3a, a'_3 + 3a, a'_4 + 3a\} + \{b_1, b_2, b_3\} \subset H$ . Since  $3a \in H$ , by Lemma 2.5(v),  $|A + B| \geq 5$ . It follows from Lemma 2.1 that  $H = (A + B) + \sum(S \cap H) \subset \sum(S)$ . Note that  $a + a_i + (2a + b_1) + (a + a_j) = a + a_i + (b'_1 + 2a) + (a + a_j) = a + a_i + (b'_1 + 3a + a_j) = a + b'_1 + 3a + a_i + a_j$ . Therefore,  $a + b'_1 + 3a + \sum_2\{a_1, a_2, a_3, a_4\} \subset \sum_3(A \cup B)$ . By Lemma 2.5(iv),  $|\sum_2\{a_1, a_2, a_3, a_4\}| \geq 5$ . It follows from Lemma 2.1 that  $a + H = a + b'_1 + 3a + \sum_2(A) + \sum(S \cap H) \subset \sum_3(A \cup B) + \sum(S \cap H) \subset \sum(S)$ . Note that  $2a + b_i + (a + a_k) + (2a + b_j) = 2a + b_i + (a'_k + a) + (2a + b_j) = 2a + a'_k + 3a + b_j + b_i = 2a + 3a + a'_k + b_j + b_i$ . Therefore,  $2a + 3a + \{a'_1, a'_2, a'_3, a'_4\} + \sum_2\{b_1, b_2, b_3\} \subset \sum_3(A \cup B)$ . By Lemma 2.5,  $|\{a'_1, a'_2, a'_3, a'_4\} + \sum_2\{b_1, b_2, b_3\}| \geq 5$ . Again by Lemma 2.1 we have  $2a + H = 2a + 3a + \{a'_1 + a'_2 + a'_3 + a'_4\} + \sum_2\{b_1, b_2, b_3\} + \sum(S \cap H) \subset \sum(S)$ .

**Subcase 3.2.**  $r = 5$  and  $t = 2$ . Similarly to above one can prove that  $H \subset \sum(S)$  and  $a + H \subset \sum(S)$ . Since  $A + (2a + b_1) + (2a + b_2) \subset 2a + H$ , by Lemma 2.1 we infer that  $2a + H = A + (2a + b_1) + (2a + b_2) + \sum(S \cap H) \subset \sum(S)$ .

**Subcase 3.3.**  $r = 6$  and  $t = 1$ . Similarly to above one can prove that  $H \subset \sum(S)$  and  $a + H \subset \sum(S)$ . Note that  $\{a + a_1\} + \{a + a_2, \dots, a + a_6\} \subset \sum_2(A)$ . Therefore,  $|\sum_2(A)| \geq 5$ . Now by Lemma 2.1 and  $\sum_2(A) \subset 2a + H$  we obtain that  $2a + H = \sum_2(A) + \sum(S \cap H) \subset \sum(S)$ .

**Subcase 3.4.**  $r = 7$  and  $t = 0$ . Similarly to above one can prove  $a + H \subset \sum(S)$  and  $2a + H \subset \sum(S)$ . Note that  $(a + a_1) + (a + a_2) + \{(a + a_3), \dots, (a + a_7)\} \subset \sum_3(A) \subset H$ . Therefore,  $|\sum_3(A)| \geq 5$ . Again by Lemma 2.1,  $H = \sum_3(A) + \sum(S \cap H) \subset \sum(S)$ .  $\square$

### 3. THE PROOFS OF THE MAIN RESULTS

*Proof of Theorem 1.2.*

Set  $|G| = n$ . Since  $G$  is a nilpotent group,  $G$  possesses a normal subgroup  $K$  of index  $p$ . Suppose  $G/K = \langle 1 + K \rangle$ . Let  $B$  be any subset of  $p - 2$  elements in  $1 + K$  and  $T = (K \setminus \{0\}) \cup B$ . It is easy to see that  $-1 + K \not\subset \sum(T)$ . This shows that  $cr(G) \geq n/p + p - 2$ . So it suffices to prove that  $cr(G) \leq n/p + p - 2$ .

Let  $S$  be any subset of  $G \setminus \{0\}$  with cardinality  $|S| = n/p + p - 2$ . We need to show  $\sum(S) = G$ . We proceed by induction on the number of prime divisors of  $n$  (counted with multiplicity). By the hypothesis we know that  $n = 27$  or  $n \geq 45$ . By Lemma 2.6 we may assume that  $n \geq 45$ . Set  $k(n) = (n/p + p - 2)/2$ . We shall write sometimes  $k$  instead of  $k(n)$ . Clearly we may partition  $S = X \cup Y$  so that  $|X| = |Y| = k$ ,  $X \cap -X = Y \cap -Y = \emptyset$  and  $|\sum(X)| \leq |\sum(Y)|$ .

The result holds by Lemma 2.1 if  $|\sum(X)| > n/2$ . Suppose the contrary. Since  $n$  is odd, we have

$$(3.1) \quad |\sum(X)| \leq (n - 1)/2.$$

Let  $\{x_i; 1 \leq i \leq k\}$  be a resolving sequence for  $X$  with critical index  $t$ .

By Lemma 2.4 and note that  $n \geq 45$ , in a similar way to the proof of Theorem 3.1 in [10] we can prove that

$$(3.2) \quad t \geq n/p^2 + p.$$

Let  $H$  be the proper subgroup generate by  $X_{t-1}$ . Let  $p'$  be the smallest prime divisor of  $n/p$ . By (3.2),  $|H \cap S| \geq n/(pp') + p' - 1$ . If  $n/p$  is the product of more than two primes, then by the induction hypothesis,  $\sum(S \cap H) = H$ . If  $n/p$  is the product of two primes, then by Theorem 1.1 and Lemma 2.2,  $\sum(S \cap H) = H$ .

Since  $|H| > n/(pp')$ , we see easily that  $q = |G/H|$  is a prime. Since  $G$  is nilpotent,  $H$  is a normal subgroup of  $G$ . Clearly  $|S \setminus H| \geq q - 1$ . Let  $a_1, \dots, a_{q-1}$  be distinct elements from  $S \setminus H$ . We denote by  $\bar{a}_i$  the image of  $a_i$  in  $G/H$  under the canonical morphism.

By the Cauchy-Davenport Theorem(cf.[12]),  $\{0, \bar{a}_1\} + \dots + \{0, \bar{a}_{q-1}\} = G/H$ . It follows that  $\sum(a_1, \dots, a_{q-1}) + H = G$ . The theorem now follows since  $\sum(S \cap H) = H$ .  $\square$

### *Proof of Theorem 1.3.*

Since  $G$  possesses a subgroup of index 2, in a similar way to the proof of Theorem 1.2 we can show that  $cr(G) \geq n/2$ . So, it suffices to prove that  $cr(G) \leq n/2$ . In a similar way to the proof of Lemma 2.6 we can checked the theorem for  $n \leq 14$ (one can find the structures of nonabelian groups for the case in [6]). Now assume that  $n \geq 16$ . Let  $S$  be a subset of  $G \setminus \{0\}$  of size  $n/2$ . Let  $T = S \cup \{0\}$ .

Now fix a subgroup  $H$  of index 2. Then, for any  $g \in G$ ,  $H + 2g = H$ , so that  $2g \in H$ . Also the sets  $T$  and  $g - T$  cannot be disjoint, because of their sizes, so  $g$  has a representation as  $t_1 + t_2$  with  $t_i \in T$ . If  $g \notin H$ , since  $2g \in H$ , it means that  $t_1 \neq t_2$  in its representation  $g = t_1 + t_2$ . Tossing away 0, if it is one of  $t_i$ 's, we have express  $g$  as a subset sum in  $S$ .

So from now on, we assume  $g \in H$ , and split the proof into three cases according to  $k =: |T \cap H|$ .

Case 1.  $k \geq (n/2) - 1$ . obviously.

Case 2.  $3 \leq k \leq (n/2) - 2$ . Consider the collection of sums  $h + j$  with  $h \in T \cap H$  and  $j \in T \cap (G \setminus H)$ . These  $k(|T| - k)$  sums belong to  $G \setminus H$ , so some element  $v$  occurs in this collection with multiplicity at least

$$\lceil \frac{k(|T|-k)}{|G \setminus H|} \rceil = \lceil \frac{k(n/2+1-k)}{n/2} \rceil \geq \lceil \frac{3(n/2-2)}{n/2} \rceil = 3.$$

In other words, we can write  $v = h_i + j_i$ , for  $i = 1, 2, 3$ , such that the  $h_i$  (resp.,  $j_i$ ) are distinct elements of  $T \cap H$  (resp.,  $T \cap (G \setminus H)$ ). Since  $g - v \notin H$ , and since as above  $T$  and  $(g - v) - T$  are not disjoint, we can write  $g - v = h + j$  or  $g - v = j' + h'$  with  $h, h' \in H$  and  $j, j' \in T \cap (G \setminus H)$ . Pick  $i$  so that  $h_i \neq h$  and  $j_i \neq j$  or  $h_i \neq h'$  and  $j_i \neq j'$  (which is possible since there are three choice for  $i$ ). Then we have  $g = h + j + h_i + j_i$  or  $g = j' + h' + h_i + j_i$ , which is a sum of distinct elements of  $T$ . Omitting 0 as one of the terms, if present, gives a subset from  $S$ .

Case 3.  $k \leq 2$ . Now  $T$  contains  $G \setminus H$ , with the possible exception of a single element  $r$ . Fix  $v \in T \setminus H$ . The  $\frac{n}{2}(n/2 - 1)^2$  sums  $x_1 + x_2 + x_3$  with  $x_1, x_2 \in G \setminus (H \cup \{r\})$  and  $x_3 \in G \setminus H$ . In particular,  $g - v$  can be represented  $(n/2 - 1)^2$  ways as such a sum. Exactly  $n/2 - 1$  of these sums have  $x_1 = x_2$ ,  $n/2 - 1$  have  $x_2 = x_3$ , and  $n/2 - 1$  have  $x_1 = x_3$ . Also,  $n/2 - 1$  of these sums have  $x_1 = v$ ,  $n/2 - 1$  sums have  $x_2 = v$ , and  $n/2 - 1$  sums have  $x_3 = v$ . Similar  $n/2 - 1$  of these sums have  $x_3 = r$ . There exists a form  $g - v = v + v + x_3$ . Thus there remain at least

$$(n/2)^2 - 7(n/2 - 1) + 1 = \frac{(n-2)(n-16)}{4} + 1 > 0.$$

sums  $x_1 + x_2 + x_3$  equaling  $g - v$  with distinct  $x_i \in G \setminus H$  not equal either  $v$  or  $r$ . So there exists a subset sum representation  $g = x_1 + x_2 + x_3 + v$ . This completes the proof of the theorem.  $\square$

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