Binding number and fractional (g, f, n', m)-critical deleted graph

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Abstract: A graph G is called a fractional (g, f, n', m)-critical deleted graph if after deleting any n' vertices of G the remaining graph is a fractional (g, f, m)-deleted graph. In this paper, we give two binding number conditions for a graph to be a fractional (g, f, n', m)-critical deleted graph.

Key words: fractional critical deleted graph; binding number

1 Introduction

All graphs considered in this paper are finite, loopless, and without multiple edges. Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. Let $n = |V(G)|$. For a vertex $x \in V(G)$, the degree and the neighborhood of x in G are denoted by $d_G(x)$ and $N_G(x)$, respectively. Let $\Delta(G)$ and $\delta(G)$ denote the maximum degree and the minimum degree of G, respectively. For $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S, and let $G - S = G[V(G)\backslash S]$. For two disjoint subsets S and T of $V(G)$, we use $e_G(S,T)$ to denote the number of edges with one end in S and the other in T. The *binding number* bind(G) of a graph G is defined as follows:

$$\text{bind}(G) = \min\{\frac{|N_G(X)|}{|X|} | \emptyset \neq X \subseteq V(G), N_G(X) \neq V(G)\}.$$

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Suppose that g and f are two integer-valued functions on $V(G)$ such that $0 \le g(x) \le f(x)$ for all $x \in V(G)$. A *fractional (g,f)-factor* is a function h that assigns to each edge of a graph G a number in [0,1] so that for each vertex x we have $g(x) \le d_G^h(x) \le f(x)$, where $d_G^h(x) = \sum_{e \in E(x)} h(e)$ is called the *fractional degree* of x in G. If $g(x) = f(x)$ for all $x \in V(G)$, then a fractional (g,f)-factor is a fractional f-factor. Moreover, if $g(x) = f(x) = k$ ($k \ge 1$ is an integer) for all $x \in V(G)$, then a fractional (g,f)-factor is just a fractional k-factor.

A graph G is called a *fractional (g,f,m)-deleted graph* if for each edge subset $H \subseteq E(G)$ with $|H| = m$, there exists a fractional (g,f)-factor h such that $h(e) = 0$ for all $e \in H$. That is, after removing any m edges, the resulting graph still has a fractional (g,f)-factor. A graph G is called a *fractional (g,f,n')-critical graph* if after delated any n' vertices from G, the resulting graph still has a fractional (g,f)-factor.

The first author of this paper first introduced the concept of a fractional (g,f,n',m)-critical deleted graph [1]. A graph G is called a *fractional (g,f,n',m)-critical deleted graph* if after deleting any n' vertices from G, the resulting graph is still a fractional (g,f,m)-deleted graph. If $g(x) = f(x)$ for all $x \in V(G)$, then fractional (g,f,m)-deleted graph, fractional (g,f,n')-critical graph, and fractional (g,f,n',m)-critical deleted graph are fractional (f,m)-deleted graph, fractional (f,n')-critical graph, and fractional (f,n',m)-critical deleted graph, respectively. Furthermore, if $g(x) = f(x) = k$ ($k \ge 1$ is an integer) for all $x \in V(G)$, then fractional (g,f,m)-deleted graph, fractional (g,f,n')-critical graph, and fractional (g,f,n',m)-critical deleted graph are just fractional (k,m)-deleted graph, fractional (k,n')-critical graph, and fractional (k,n',m)-critical deleted graph, respectively. Some results on degree condition for fractional (k,m)-deleted graphs can be seen in [2].

Zhou [4] gave the binding number condition for a graph to be a fractional (k,m)-deleted graph.

Theorem 1 (Zhou [4]) *Let $k \ge 2$ and $m \ge 0$ be two integers, and let G be a graph of order n with $n \ge 4k - 6 + \frac{2m}{k-1}$. If*

$$\mathrm{bind}(G) > \frac{(2k-1)(n-1)}{k(n-2) - 2m + 2},$$

then G is a fractional (k,m)-deleted graph.

Zhou and Shen [6] obtained the binding number condition for fractional (f,n')-critical graphs.

Theorem 2 (Zhou and Shen [6]) *Let G be a graph of order n, and let a, b, and n' be non-negative integers such that $2 \le a \le b$, and let f be an* *integer-valued function defined on $V(G)$ such that $a \le f(x) \le b$ for each $x \in V(G)$. If* $\text{bind}(G) > \frac{(a+b-1)(n-1)}{(an-(a+b)-bn'+2)}$ *and* $n \ge \frac{(a+b)(a+b-3)}{a} + \frac{bn'}{a-1}$, *then G is fractional (f, n')-critical.*

Zhou, Bian and Liu [5] proved that under some specific conditions, binding number condition for fractional (f, n')-critical graphs can be improved.

Theorem 3 (Zhou, Bian and Liu [5]) *Let G be a graph of order n, and let a, b, and n' be non-negative integers such that $2 \le a \le b$, and*

$$n \ge \frac{(a+b-1)(a+b-2)-2}{a} + \frac{bn'}{a-1}.$$

Let f be an integer-valued function defined on $V(G)$ such that $a \le f(x) \le b$ for each $x \in V(G)$. If G satisfies

$$\text{bind}(G) \ge \frac{(a+b-1)(n-1)}{a(n-1)-bn'}$$

and

$$\delta(G) \ne \lfloor \frac{(b-1)n+a+b+bn'-2}{a+b-1} \rfloor,$$

then G is a fractional (f, n')-critical graph.

In this paper, we will extend Theorems 2 and 3 to fractional (g, f, n', m)-critical deleted graphs. Our two main results as follows:

Theorem 4 *Let G be a graph of order n, and let a, b, n', and m be non-negative integers such that $2 \le a \le b$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \le g(x) \le f(x) \le b$ for each $x \in V(G)$. If* $\text{bind}(G) > \frac{(a+b-1)(n-1)}{an-(a+b)-bn'-2m+2}$ *and* $n \ge \frac{(a+b)(a+b-3)}{a} + \frac{bn'+2m}{a-1}$, *then G is a fractional (g, f, n', m)-critical deleted graph.*

Theorem 5 *Let G be a graph of order n, and let a, b, n', and m be non-negative integers such that $2 \le a \le b$, $n \ge \frac{(a+b-1)(a+b-2)-2}{a} + \frac{bn'+2m}{a-1}$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \le g(x) \le f(x) \le b$ for each $x \in V(G)$. If G satisfies*

$$\text{bind}(G) \ge \frac{(a+b-1)(n-1)}{a(n-1)-bn'-2m}$$

and

$$\delta(G) \ne \lfloor \frac{(b-1)n+a+b+bn'+2m-2}{a+b-1} \rfloor,$$

then G is a fractional (g, f, n', m)-critical deleted graph.

Clearly, Theorem 5 is stronger than Theorem 4 under some conditions. We will show that Theorem 4 is best possible for certain combinations of (a, b, n', m).

Let $m = 0$ in Theorems 4 and 5, we get two results on binding number condition for fractional (g, f, n')-critical graphs, which are an extension of Theorems 2 and 3, respectively.

Corollary 1 *Let G be a graph of order n, and let a, b, and n' be non-negative integers such that $2 \le a \le b$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \le g(x) \le f(x) \le b$ for each $x \in V(G)$. If $\mathrm{bind}(G) > \frac{(a+b-1)(n-1)}{an-(a+b)-bn'+2}$ and $n \ge \frac{(a+b)(a+b-3)}{a} + \frac{bn'}{a-1}$, then G is a fractional (g, f, n')-critical graph.*

Corollary 2 *Let G be a graph of order n, and let a, b, and n' be non-negative integers such that $2 \le a \le b$, $n \ge \frac{(a+b-1)(a+b-2)-2}{a} + \frac{bn'}{a-1}$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \le g(x) \le f(x) \le b$ for each $x \in V(G)$. If G satisfies*

$$\mathrm{bind}(G) \ge \frac{(a+b-1)(n-1)}{a(n-1)-bn'}$$

and

$$\delta(G) \ne \lfloor \frac{(b-1)n+a+b+bn'-2}{a+b-1} \rfloor,$$

then G is a fractional (g, f, n')-critical graph.

If $n' = 0$ in Theorems 4 and 5, then we obtain binding number condition for fractional (g, f, m)-deleted graphs as follows:

Corollary 3 *Let G be a graph of order n, and let a, b, and m be non-negative integers such that $2 \le a \le b$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \le g(x) \le f(x) \le b$ for each $x \in V(G)$. If $\mathrm{bind}(G) > \frac{(a+b-1)(n-1)}{an-(a+b)-2m+2}$ and $n \ge \frac{(a+b)(a+b-3)}{a} + \frac{2m}{a-1}$, then G is a fractional (g, f, m)-deleted graph.*

Corollary 4 *Let G be a graph of order n, and let a, b, and m be non-negative integers such that $2 \le a \le b$, $n \ge \frac{(a+b-1)(a+b-2)-2}{a} + \frac{2m}{a-1}$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \le g(x) \le f(x) \le b$ for each $x \in V(G)$. If G satisfies*

$$\mathrm{bind}(G) \ge \frac{(a+b-1)(n-1)}{a(n-1)-2m}$$

and

$$\delta(G) \ne \lfloor \frac{(b-1)n+a+b+2m-2}{a+b-1} \rfloor,$$

then G is a fractional (g, f, m)-deleted graph.

The proof of our main results is based on the following lemma:

Lemma 1 (Gao [1]) *Let G be a graph, g, f be two integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for each $x \in V(G)$. Let n', m be two non-negative integers. Then G is fractional (g, f, n', m)-critical deleted graph if and only if*

$$\begin{aligned} & f(S) - g(T) + d_{G-S}(T) \\ \geq & \max_{U \subseteq S, |U|=n', H \subseteq E(G-U), |H|=m} \{f(U) + \sum_{x \in T} d_H(x) - e_H(T, S)\} \quad (1) \end{aligned}$$

for all disjoint subsets S, T of $V(G)$ with $|S| \geq n'$.

2 Proof of Theorem 4

Suppose that G satisfies conditions of Theorem 4 but is not a fractional (g, f, n', m)-critical deleted graph. Obviously, $T \neq \emptyset$. Otherwise, (1) holds. By Lemma 1 and the fact $\sum_{x \in T} d_H(x) - e_H(T, S) \leq 2m$, there exist disjoint subsets S and T of $V(G)$ such that

$$f(S) - g(T) + d_{G-S}(T) \leq bn' + 2m - 1, \quad (2)$$

where $|S| \geq n'$. We choose S and T such that $|T|$ is minimum. Thus, for each $x \in T$, we have $d_{G-S}(x) \leq g(x) - 1 \leq b - 1$. Otherwise, if there exists some $x \in T$ such that $d_{G-S}(x) \geq g(x)$, then S and $T \setminus \{x\}$ also satisfy (2). This contradicts the choice of S and T.

Let $d = \min\{d_{G-S}(x) : x \in T\}$. Then $0 \leq d \leq b - 1$, and

$$f(S) + d_{G-S}(T) - f(T) \geq a|S| + d|T| - b|T|.$$

Thus,

$$bn' + 2m - 1 \geq a|S| - (b - d)|T|. \quad (3)$$

We choose $x_1 \in T$ such that $d_{G-S}(x_1) = d$. We shall get some contradictions in the following two cases.

Case 1. $1 \leq d \leq b - 1$.

Let $Y = (V(G) \setminus S) \setminus N_{G-S}(x_1)$. Then $x_1 \in Y \setminus N_G(Y)$. Thus, $Y \neq \emptyset$, $N_G(Y) \neq V(G)$, and $|N_G(Y)| \geq \text{bind}(G)|Y|$. We get

$$n - 1 \geq |N_G(Y)| \geq \text{bind}(G)|Y| = \text{bind}(G)(n - d - |S|),$$

that is to say,

$$|S| \geq n - d - \frac{n-1}{\text{bind}(G)} > n - d - \frac{an - (a+b) - bn' - 2m + 2}{a + b - 1}. \quad (4)$$

Case 1.1. $3 \le d \le b-1$.

By (3) and $|T| \le n - |S|$, we obtain

$$bn' + 2m > a|S| - (b-d)(n-|S|) = (a+b-d)|S| - (b-d)n. \quad (5)$$

Multiplying (5) by $(a+b-1)$, and combining (4), we have

$$\begin{aligned} 0 &> (a+b-1)[(a+b-d)|S| - (b-d)n - bn' - 2m] \\ &> (a+b-d)[(a+b-1)(n-d) - an + (a+b) + bn' + 2m - 2] \\ &\quad -(a+b-1)[(b-d)n + bn' + 2m] \\ &= (d-1)[an - bn' - 2m - (a+b-d)(a+b-1)] - (a+b-d) \\ &> (d-2)(a+b-d) \quad (6) \end{aligned}$$

However, by $3 \le d \le b-1 < a+b$, (6) does not hold, a contradiction.

Case 1.2. $d = 2$.

Obviously, $b \ge 3$. We first prove the following claim.

Claim 1 $(b-2)(a+b-1)|T| \le a[(n-2)(a+b-1) - an + a + b + bn' + 2m - 2] - (a+b-1)(bn'+2m) + (a+b-1)$; *that is,* $|T| \le \frac{a}{b-2}(n-2-\frac{an-(a+b)-bn'-2m+2}{a+b-1}) - \frac{bn'+2m}{b-2} + \frac{1}{b-2}$.

Proof of Claim 1. If $(b-2)(a+b-1)|T| \ge a[(n-2)(a+b-1) - an + a + b + bn' + 2m - 2] - (a+b-1)(bn'+2m) + (a+b-1) + 1$, then

$$\begin{aligned} |T| &\ge \frac{a}{b-2}(n-2-\frac{an-(a+b)-bn'-2m+2}{a+b-1}) \\ &\quad -\frac{bn'+2m}{b-2} + \frac{1}{b-2} + \frac{1}{(b-2)(a+b-1)}. \end{aligned}$$

Thus, from (4) and $n \ge \frac{(a+b)(a+b-3)}{a} + \frac{bn'+2m}{a-1}$, we get

$$\begin{aligned} |S| + |T| &> n - 2 - \frac{an-(a+b)-bn'-2m+2}{a+b-1} \\ &\quad + \frac{a}{b-2}(n-2-\frac{an-(a+b)-bn'-2m+2}{a+b-1}) \\ &\quad -\frac{bn'+2m}{b-2} + \frac{1}{b-2} + \frac{1}{(b-2)(a+b-1)} \\ &= n + \frac{an-(a+b)(a+b-2)+(a+b-2)(bn'+2m)}{(b-2)(a+b-1)} \\ &\quad -\frac{bn'+2m}{b-2} + \frac{1}{b-2} + \frac{1}{(b-2)(a+b-1)} \end{aligned}$$

$$
\begin{aligned}
&\geq n+\frac{(a+b)(a+b-3)+\frac{a}{a-1}(bn'+2m)}{(b-2)(a+b-1)} \\
&\quad -\frac{(a+b)(a+b-2)-(a+b-2)(bn'+2m)}{(b-2)(a+b-1)} \\
&\quad -\frac{bn'+2m}{b-2}+\frac{1}{b-2}+\frac{1}{(b-2)(a+b-1)} \\
&\geq n+\frac{(a+b-1)(bn'+2m)-(a+b)}{(b-2)(a+b-1)}-\frac{bn'+2m}{b-2} \\
&\quad +\frac{1}{b-2}+\frac{1}{(b-2)(a+b-1)} \\
&= n.
\end{aligned}
$$

This contradicts the fact $|S|+|T|\leq n$. □

By Claim 1, (3) and (4), we derive the following contradiction:

$$
\begin{aligned}
&bn'+2m-1 \\
&\geq a|S|-(b-2)|T| \\
&> a(n-2-\frac{an-(a+b)-bn'-2m+2}{a+b-1}) \\
&\quad -(b-2)(\frac{a}{b-2}(n-2-\frac{an-(a+b)-bn'-2m+2}{a+b-1}) \\
&\quad -\frac{bn'+2m}{b-2}+\frac{1}{b-2}) \\
&= bn'+2m-1.
\end{aligned}
$$

Case 1.3. $d=1$.

Case 1.3.1. $|T|\leq \frac{a}{b-1}(n-1-\frac{an-(a+b)-bn'-2m+2}{a+b-1})-\frac{bn'+2m-1}{b-1}$.

Using (3) and (4), we get the following contradiction:

$$
\begin{aligned}
&bn'+2m-1 \\
&\geq a|S|-(b-1)|T| \\
&> a(n-1-\frac{an-(a+b)-bn'-2m+2}{a+b-1}) \\
&\quad -(b-1)(\frac{a}{b-1}(n-1-\frac{an-(a+b)-bn'-2m+2}{a+b-1}) \\
&\quad -\frac{bn'+2m-1}{b-2}) \\
&= bn'+2m-1.
\end{aligned}
$$

Case 1.3.2. $|T|> \frac{a}{b-1}(n-1-\frac{an-(a+b)-bn'-2m+2}{a+b-1})-\frac{bn'+2m-1}{b-1}$.

By (4), we have

$$
\begin{aligned}
& |S|+|T| \\
> \;& n-1-\frac{an-(a+b)-bn'-2m+2}{a+b-1} \\
& +\frac{a}{b-1}(n-1-\frac{an-(a+b)-bn'-2m+2}{a+b-1})-\frac{bn'+2m-1}{b-1} \\
= \;& \frac{a+b-1}{b-1}(n-1-\frac{an-(a+b)-bn'-2m+2}{a+b-1})-\frac{bn'+2m-1}{b-1} \\
= \;& \frac{(b-1)n+bn'+2m-1}{b-1}-\frac{bn'+2m-1}{b-1}=n,
\end{aligned}
$$

which contradicts the fact that $|S|+|T|\le n$.

Case 2. $d=0$.

In this case, we first show the following claim.

Claim 2 $\frac{an-(a+b)-bn'-2m+2}{n-1}>1$.

Proof of Claim 2. Since $n\ge \frac{(a+b)(a+b-3)}{a}+\frac{bn'+2m}{a-1}$, we have

$$
\begin{aligned}
& an-(a+b)-bn'-2m+2-(n-1) \\
= \;& (a-1)n-(a+b)-bn'-2m+3 \\
\ge \;& (a-1)(\frac{(a+b)(a+b-3)}{a}+\frac{bn'+2m}{a-1})-(a+b)-bn'-2m+3 \\
= \;& \frac{(a-1)(a+b)(a+b-3)}{a}-(a+b)+3 \\
\ge \;& 2(a+b-3)-(a+b)+3 \\
= \;& a+b-3>0.
\end{aligned}
$$

Thus, we get $\frac{an-(a+b)-bn'-2m+2}{n-1}>1$. □

Let $h=|\{x : x\in T, d_{G-S}(x)=0\}|$, and $Y=V(G)\setminus S$. By $d=0$, we have $N_G(Y)\ne V(G)$. Also, by $T\ne\emptyset$, we have $Y\ne\emptyset$. So, $|N_G(Y)|\ge \text{bind}(G)|Y|$. Therefore,

$$n-h\ge |N_G(Y)|\ge \text{bind}(G)|Y|=\text{bind}(G)(n-|S|).$$

So,

$$|S|\ge n-\frac{n-h}{\text{bind}(G)}>n-\frac{(n-h)(an-(a+b)-bn'-2m+2)}{(a+b-1)(n-1)}. \tag{7}$$

By (2), (7), Claim 2 and the fact $|T| \leq n - |S|$, we get

$$\begin{aligned}
& bn' + 2m - 1 \\
\geq\ & f(S) + d_{G-S}(T) - g(T) \\
\geq\ & a|S| + |T| - h - b|T| = a|S| - (b-1)|T| - h \\
\geq\ & a|S| - (b-1)(n - |S|) - h = (a+b-1)|S| - (b-1)n - h \\
>\ & (a+b-1)(n - \frac{(n-h)(an - (a+b) - bn' - 2m + 2)}{(a+b-1)(n-1)}) \\
& -(b-1)n - h \\
=\ & an - \frac{(n-h)(an - (a+b) - bn' - 2m + 2)}{n-1} - h \\
\geq\ & an - \frac{(n-1)(an - (a+b) - bn' - 2m + 2)}{n-1} - 1 \\
=\ & bn' + 2m + (a+b) - 3 > bn' + 2m,
\end{aligned}$$

which is a contradiction. □

Remark 1. The condition that $\mathrm{bind}(G) > \frac{(a+b-1)(n-1)}{(an-(a+b)-bn'-2m+2)}$ in Theorem 4 cannot be replaced by $\mathrm{bind}(G) \geq \frac{(a+b-1)(n-1)}{(an-(a+b)-bn'-2m+2)}$. Let $2 \leq a = b, n' \geq 0$ be integers. Let $n = ((a+b-1)(a+b-2) + (a+b-2) + (a+2b-1)n' + (a+b+1)m)/a$, $l = ((a+b+n'+m-1))/2$, and $h = n - 2l = n-(a+b+n'+m-1) = ((a+b-1)(b-2)+(a+b-2)+(2b-1)n'+(b+1)m)/a$ be integers. Let $G = K_h \vee lK_2$. $X = V(lK_2)$, and for each $x \in X$, we have $|N_H(X \setminus x)| = n - 1$. By the definition of bind(G), we have

$$\mathrm{bind}(G) = \frac{N_G(X \setminus x)}{X \setminus x} = \frac{n-1}{2l-1} = \frac{(a+b-1)(n-1)}{an - (a+b) - bn' - 2m + 2}.$$

Let $S = V(K_h)$, $T = V(lK_2)$, H be any subgraph of $G[T]$ with m edges. Then $|S| = h \geq n'$, $|T| = 2l$, and $\sum_{x \in T} d_H(x) - e_H(S,T) = 2m$. Since $a = b$, we have $g(x) = a = b = f(x)$ for all $x \in V(H)$. Thus,

$$\begin{aligned}
& f(S) + d_{G-S}(T) - g(T) \\
=\ & a|S| - (b-1)|T| \\
=\ & a\frac{(a+b-1)(b-2) + (a+b-2) + (2b-1)n' + (b+1)m}{a} \\
& -(b-1)(a+b+n'+m-1) \\
=\ & bn' + 2m - 1 < bn' + 2m \\
=\ & \max_{U \subseteq S, |U| = n', H \subseteq E(G-U), |H| = m} \{f(U) + \sum_{x \in T} d_H(x) - e_H(T,S)\}.
\end{aligned}$$

By Lemma 1, G is not a fractional (g, f, n', m)-critical deleted graph.

3 Proof of Theorem 5

To show Theorem 5, we need the following lemma, which is a neighborhood condition for a graph G to be a fractional (g, f, n', m)-critical deleted graph.

Lemma 2 *Let a, b, n', n and m are non-negative integers such that $1 \leq a \leq b$. Let G be a graph with order n such that $n \geq \frac{(a+b-1)(a+b-2)+bn'+2m-2}{a}$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) \leq b$ for each $x \in V(G)$. If*

$$|N_G(X)| > \frac{(b-1)n + |X| + bn' + 2m - 1}{a+b-1}$$

holds for each non-empty independent subset $X \subseteq V(G)$, and

$$\delta(G) > \frac{(b-1)n + a + b + bn' + 2m - 2}{a+b-1},$$

then G is a fractional (g, f, n', m)-critical deleted graph.

If $n' = 0$ in Lemma 2, we obtain the following neighborhood condition on fractional (g, f, m)-deleted graphs.

Corollary 5 *Let a, b, n and m are non-negative integers such that $1 \leq a \leq b$. Let G be a graph with order n such that $n \geq \frac{(a+b-1)(a+b-2)+2m-2}{a}$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) \leq b$ for each $x \in V(G)$. If*

$$|N_G(X)| > \frac{(b-1)n + |X| + 2m - 1}{a+b-1}$$

holds for each non-empty independent subset $X \subseteq V(G)$, and

$$\delta(G) > \frac{(b-1)n + a + b + 2m - 2}{a+b-1},$$

then G is a fractional (g, f, m)-deleted graph.

Taking $m = 0$ in Lemma 2, we have the following corollary, which is stronger than Theorem 5 in [3].

Corollary 6 *Let a, b, n' and n are non-negative integers such that $1 \leq a \leq b$. Let G be a graph with order n such that $n \geq \frac{(a+b-1)(a+b-2)+bn'-2}{a}$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) \leq b$ for each $x \in V(G)$. If*

$$|N_G(X)| > \frac{(b-1)n + |X| + bn' - 1}{a+b-1}$$

holds for each non-empty independent subset $X \subseteq V(G)$, and

$$\delta(G) > \frac{(b-1)n+a+b+bn'-2}{a+b-1},$$

then G is a fractional (g,f,n')-critical graph.

Proof of Lemma 2. Suppose that G satisfies conditions of Lemma 2, but is not a fractional (g,f,n',m)-critical deleted graph. Obviously, $T \neq \emptyset$. By Lemma 1, there exist disjoint S and T satisfying

$$f(S) - g(T) + d_{G-S}(T) \leq bn' + 2m - 1, \tag{8}$$

where $|S| \geq n'$. We choose S and T such that $|T|$ is minimum. Thus, we have $d_{G-S}(x) \leq g(x) - 1 \leq b - 1$ for each $x \in T$.

Let $d = \min\{d_{G-S}(x) | x \in T\}$, then

$$0 \leq d \leq b-1,$$

$$\delta(G) \leq d + |S|. \tag{9}$$

Now, we consider the following two cases according to the value of d.

Case 1. $1 \leq d \leq b-1$.

Using (8) and (9) and noting $|S| + |T| \leq n$ and $b - d \geq 1$, we get

$$\begin{aligned}
& bn' + 2m - 1 \\
\geq\ & f(S) + d_{G-S}(T) - g(T) \\
\geq\ & a|S| + d|T| - b|T| \\
=\ & a|S| - (b-d)|T| \\
\geq\ & a|S| - (b-d)(n - |S|) \\
=\ & (a+b-d)|S| - (b-d)n \\
\geq\ & (a+b-d)(\delta(G) - d) - (b-d)n \\
\geq\ & (a+b-d)\left(\frac{(b-1)n+a+b+bn'+2m-1}{a+b-1} - d\right) - (b-d)n.
\end{aligned}$$

That is,

$$0 \geq (a+b-d)\left(\frac{(b-1)n+a+b+bn'+2m-1}{a+b-1} - d\right) - (b-d)n - bn' - 2m + 1. \tag{10}$$

Multiplying (10) by $(a+b-1)$, we obtain

$$0 \geq (d-1)(an - (a+b-d)(a+b-1) - bn' - 2m) + (a+b-1). \tag{11}$$

Case 1.1. $d = 1$.

Due to (11), we have $0 \geq (a+b-1)$, contrary to the fact that $1 \leq a \leq b$.

Case 1.2. $d = 2$.

As $d \leq b-1$, we see $b \geq 3$. By (11) and $n \geq \frac{(a+b-1)(a+b-2)+bn'+2m-2}{a}$, we have the following obvious contradiction:

$$\begin{aligned} 0 &\geq (d-1)(an-(a+b-d)(a+b-1)-bn'-2m)+(a+b-1) \\ &= (an-(a+b-2)(a+b-1)-bn'-2m)+(a+b-1) \\ &\geq -2+(a+b-1) \geq a \geq 1. \end{aligned}$$

Case 1.3. $3 \leq d \leq b-1$.

Clearly, $b \geq 4$ since $d \leq b-1$. By (11) and $n \geq \frac{(a+b-1)(a+b-2)+bn'+2m-2}{a}$, we derive the following contradiction:

$$\begin{aligned} 0 &\geq (d-1)(an-(a+b-d)(a+b-1)-bn'-2m)+(a+b-1) \\ &\geq (d-1)(an-(a+b-3)(a+b-1)-bn'-2m)+(a+b-1) \\ &\geq (d-1)(a+b-3)+(a+b-1) \geq 2(a+b-3)+(a+b-1) \\ &= 3(a+b)-7 > 0. \end{aligned}$$

Case 2. $d = 0$.

Let $Y = \{x \in T \mid d_{G-S}(x) = 0\}$. Obviously, $Y \neq \emptyset$, and Y is an independent set. Thus, by Lemma 2, we have

$$\frac{(b-1)n+|Y|+bn'+2m-1}{a+b-1} < |N_G(Y)| \leq |S|. \tag{12}$$

In view of (12) and $|S|+|T| \leq n$, we get

$$\begin{aligned} & f(S)+d_{G-S}(T)-g(T) \\ \geq\; & a|S|+d_{G-S}(T)-b|T| \\ \geq\; & a|S|+|T|-|Y|-b|T| \\ =\; & a|S|-(b-1)|T|-|Y| \\ \geq\; & a|S|-(b-1)(n-|S|)-|Y| \\ =\; & (a+b-1)|S|-(b-1)n-|Y| \\ >\; & (a+b-1)\left(\frac{(b-1)n+|Y|+bn'+2m-1}{a+b-1}\right)-(b-1)n-|Y| \\ =\; & bn'+2m-1. \end{aligned}$$

This contradicts (8). □

Remark 2. The following example shows that the neighborhood condition $|N_G(X)| > \frac{(b-1)n+|X|+bn'+2m-1}{a+b-1}$ in Lemma 2 cannot be replaced by $|N_G(X)| \geq \frac{(b-1)n+|X|+bn'+2m-1}{a+b-1}$. Let $b = a \geq 1$, $n' \geq 0$, $t \geq 3$ be integers, and $\frac{(t-2)a+2m-1}{2}$ is an integer. Since $a=b$, we have $g(x)=a=b=f(x)$ for all $x \in V(G)$. Let $G = K_{(b-1)t+n'+2m+1} \vee (2aK_1 \cup \frac{(t-2)a+2m-1}{2}K_2)$. Then $n=(a+b-1)t+n'+4m$. Let $X=V(2aK_1)$. So,

$$\begin{aligned}
\delta(G) &= (b-1)t+n'+2m+1 \\
&= \frac{(a+b-1)((b-1)t+n'+2m+1)}{a+b-1} \\
&= \frac{(b-1)n+a+b+bn'+2m-1}{a+b-1} \\
&> \frac{(b-1)n+a+b+bn'+2m-2}{a+b-1},
\end{aligned}$$

and

$$\begin{aligned}
|N_G(X)| &= (b-1)t+n'+2m+1 \\
&= \frac{(b-1)n+a+b+bn'+2m-1}{a+b-1} \\
&= \frac{(b-1)n+2a+bn'+2m-1}{a+b-1} \\
&= \frac{(b-1)n+|X|+bn'+2m-1}{a+b-1}.
\end{aligned}$$

Therefore, $|N_G(X)| \geq \frac{(b-1)n+|X|+bn'+2m-1}{a+b-1}$ holds for any non-empty independent set $X \subseteq V(G)$. Let $S = V(K_{(b-1)t+n'+2m+1})$, $T = V(2aK_1 \cup \frac{(t-2)a+2m-1}{2}K_2)$, H is a subgraph of $G[T]$ with m edges. Then $|S| = (b-1)t+n'+2m+1 > n'$, $|T| = at+2m-1$, $d_{G-S}(T) = (t-2)a+2m-1$, and $\sum_{x\in T} d_H(x) - e_H(S,T) = 2m$. Thus,

$$\begin{aligned}
& f(S)+d_{G-S}(T)-g(T) \\
=\ & b|S|+d_{G-S}(T)-a|T| \\
=\ & b((b-1)t+n'+2m+1)+(t-2)a+2m-1-a(at+2m-1) \\
=\ & b((b-1)t+n'+2m+1)+(t-2)b+2m-1-b(bt+2m-1) \\
=\ & bn'+2m-1 \\
<\ & \max_{U\subseteq S, |U|=n', H\subseteq E(G-U), |H|=m} \{f(U)+\sum_{x\in T} d_H(x) - e_H(T,S)\}.
\end{aligned}$$

Namely, G is not a fractional (g,f,n',m)-critical deleted graph.

Now, we begin to prove Theorem 5.

Proof of Theorem 5. Suppose that G satisfies the conditions of Theorem 5, but is not a fractional (g,f,n',m)-critical deleted graph. Obviously, $T \neq \emptyset$. By Lemma 1, there exist disjoint subsets S and T satisfying

$$f(S)-g(T)+d_{G-S}(T) \leq bn'+2m-1, \tag{13}$$

where $|S| \geq n'$. We choose S and T such that $|T|$ is minimum. Then $d_{G-S}(x) \leq g(x) - 1 \leq b - 1$ for each $x \in T$.

For each $X \subseteq V(G)$, $X \neq \emptyset$ and $N_G(X) \neq V(G)$. Let $Y = V(G) \setminus N_G(X)$. Clearly, $\emptyset \neq Y \subseteq V(G)$.

Claim 3 $X \cap N_G(Y) = \emptyset$.

Proof of Claim 3. Assume that $X \cap N_G(Y) \neq \emptyset$, say $x \in X \cap N_G(Y)$. By $x \in N_G(Y)$, we have $y \in Y$ and $xy \in E(G)$. Thus, $y \in N_G(x) \subseteq N_G(X)$, contradicting $y \in Y = V(G) \setminus N_G(X)$. □

Claim 4 $|N_G(X)| > \frac{(b-1)n+|X|+bn'+2m-1}{a+b-1}$.

Proof of Claim 4. Using Claim 3, we have

$$|X| + |N_G(Y)| \leq n \tag{14}$$

and

$$N_G(Y) \neq V(G). \tag{15}$$

According to (14), (15) and the definition of bind(G), we get

$$\text{bind}(G) \leq \frac{|N_G(Y)|}{|Y|} \leq \frac{n-|X|}{|V(G) \setminus N_G(X)|} = \frac{n-|X|}{n-|N_G(X)|}. \tag{16}$$

From (16), we have

$$|N_G(X)| \geq n - \frac{n-|X|}{\text{bind}(G)}. \tag{17}$$

Let $F(t) = n - \frac{n-|X|}{t}$. Then, by $X \subseteq V(G)$, we obtain

$$F'(t) = \frac{n-|X|}{t^2} \geq 0.$$

Combining this with $\text{bind}(G) \geq \frac{(a+b-1)(n-1)}{a(n-1)-bn'-2m}$, we get

$$F(\text{bind}(G)) \geq F(\frac{(a+b-1)(n-1)}{a(n-1)-bn'-2m}).$$

Thus,

$$n - \frac{n-|X|}{\text{bind}(G)} \geq n - \frac{n-|X|}{\frac{(a+b-1)(n-1)}{a(n-1)-bn'-2m}} = n - \frac{(n-|X|)(a(n-1)-bn'-2m)}{(a+b-1)(n-1)}. \tag{18}$$

By (17), (18) and $n \geq \frac{(a+b-1)(a+b-2)-2}{a} + \frac{bn'+2m}{a-1}$, we obtain

$$\begin{aligned}
& |N_G(X)| \\
\geq\ & n - \frac{n-|X|}{bind(G)} \geq n - \frac{(n-|X|)(a(n-1)-bn'-2m)}{(a+b-1)(n-1)} \\
=\ & \frac{(b-1)(n-1)n + (a(n-1)-bn'-2m)|X| + (bn'+2m)n}{(a+b-1)(n-1)} \\
=\ & \frac{(b-1)(n-1)n + (n-1)|X| + ((a-1)(n-1)-bn'-2m)|X|}{(a+b-1)(n-1)} \\
& + \frac{(bn'+2m)n}{(a+b-1)(n-1)} \\
\geq\ & \frac{(b-1)(n-1)n + (n-1)|X| + ((a-1)(n-1)-bn'-2m)}{(a+b-1)(n-1)} \\
& + \frac{(bn'+2m)n}{(a+b-1)(n-1)} \\
=\ & \frac{(b-1)(n-1)n + (n-1)|X| + (a-1)(n-1) + (bn'+2m)(n-1)}{(a+b-1)(n-1)} \\
=\ & \frac{(b-1)n + |X| + bn' + 2m + a - 1}{a+b-1} \\
>\ & \frac{(b-1)n + |X| + bn' + 2m - 1}{a+b-1}
\end{aligned}$$

Therefore, Claim 4 holds. □

Since each $\emptyset \neq X \subseteq V(G)$ satisfies $|N_G(X)| \geq \frac{(b-1)n+|X|+bn'+2m+a-1}{a+b-1}$, we get

$$\delta(G) \geq \frac{(b-1)n + a + bn' + 2m}{a+b-1}. \tag{19}$$

Claim 5 $\delta(G) > \frac{(b-1)n+a+b+bn'+2m-2}{a+b-1}$.

Proof of Claim 5. Suppose that $\delta(G) \leq \frac{(b-1)n+a+b+bn'+2m-2}{a+b-1}$. By(19),

$$\lceil \frac{(b-1)n + a + bn' + 2m}{a+b-1} \rceil \leq \delta(G) \leq \lfloor \frac{(b-1)n + a + b + bn' + 2m - 2}{a+b-1} \rfloor.$$

That is,

$$\lceil \frac{(b-1)n + a + bn' + 2m}{a+b-1} \rceil = \delta(G) = \lfloor \frac{(b-1)n + a + b + bn' + 2m - 2}{a+b-1} \rfloor.$$

This contradicts the condition of Theorem 5. □

Now, the result follows from Claim 4, Claim 5 and Lemma 2. □

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