On distance Pell numbers and their connections with Fibonacci numbers

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Abstract

In this paper we introduce a new kind of generalized Pell numbers. This generalization is introduced in the distance sense. We give different interpretations and representations of these numbers. We present relations between distance Pell numbers and Fibonacci numbers. Moreover we describe graph interpretations of distance Pell numbers. These graphs interpretations in the natural way imply a new kind of generalized Jacobsthal numbers.

Keywords: Fibonacci numbers, Pell numbers, Jacobsthal numbers, matchings, Binet formula

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1 Introduction

The nth Fibonacci number F_n is defined by $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$. There are many numbers of the Fibonacci type given by the second order linear recurrence relations.

The nth Pell number P_n is defined recursively by the second order linear recurrence relation $P_n = 2P_{n-1} + P_{n-2}$, for $n \geq 2$ with the initial conditions $P_0 = 0$, $P_1 = 1$. The Pell numbers are well-known numbers in the number theory and they belong to the wide class of numbers of the Fibonacci type. They have some versions, one of them is the companion Pell numbers Q_n

defined as follows $Q_n = 2Q_{n-1} + Q_{n-2}$, for $n \geq 2$ and $Q_0 = Q_1 = 1$. The companion Pell numbers and the Pell numbers are in the relation $Q_n = P_{n-1} + P_n$.

The literature includes many papers with interesting generalizations of the Pell numbers, see [6], [8] for example.

In [12] the special distance generalization of the Pell numbers and the Pell-Lucas numbers were introduced in the context of their graph interpretations. We recall this generalizations. Let $k \geq 2$, $n \geq 0$ be integers. By nth generalized Pell numbers $P(k,n)$ we mean the number defined recursively as follows

$$P(k,n) = P(k,n-1) + P(k,n-k+1) + P(k,n-k), \text{ for } n \geq k+1$$

with the initial conditions $P(2,0) = 0$, $P(k,0) = 1$ for $k \geq 3$, $P(k,1) = 1$ for $k \geq 2$ and $P(k,n) = 2n-2$ for $2 \leq n \leq k$, see [12].

The graph interpretation of these numbers is closely related to the concept of k-independent sets in graphs (i.e. independent sets generalized in the distance sense). This type of generalized independent sets is studied intensively by H. Galeana-Sánchez, C. Hernández-Cruz and others in context of kernels in digraphs, see their interesting last papers [2], [3]. In [11] the graph interpretation of the generalized companion Pell numbers also in graphs is given, with respect to k-independent sets.

Our aim is to describe a new kind of generalized Pell numbers $Pd(k,n)$ and this generalization is directly related to the idea of numbers $P(k,n)$, given by I. Włoch in [12]. This article is a sequel of it.

2 Distance Pell numbers $Pd(k,n)$

Let $k \geq 1$, $n \geq 0$ be integers. The nth distance Pell numbers we define in the following way

$$Pd(k,n) = Pd(k,n-1) + Pd(k,n-2) + Pd(k,n-k), \text{ for } n \geq k$$

with the initial conditions

$Pd(k,0) = 0$,

$Pd(k,i) = 1$ for $k \leq i+2$, $i \geq 1$ and

$Pd(k,i) = 0$ for $k > i+2$, $i \geq 1$.

If $k = 1$ then this definition reduces to the classical Pell numbers P_n. If $k = 2$ then we obtain the Jacobsthal numbers J_n. If $k = 3$ then $Pd(3,n)$ gives the Tribonacci numbers T_n.

The following Table presents few initial distance Pell sequences

n	0	1	2	3	4	5	6	7	8	9	10	11
$Pd(1,n)$	0	1	2	5	12	29	70	169	408	985	2378	5741
$Pd(2,n)$	0	1	1	3	5	11	21	43	85	171	341	683
$Pd(3,n)$	0	1	1	2	4	7	13	24	44	81	149	274
$Pd(4,n)$	0	0	1	1	2	3	6	10	18	31	55	96
$Pd(5,n)$	0	0	0	1	1	2	3	5	9	15	26	44
$Pd(6,n)$	0	0	0	0	1	1	2	3	5	8	14	23

Table 1. The distance Pell numbers $Pd(k,n)$.

Interpretations of the Pell numbers are well-known, see e.g. The On-Line Encyclopedia of Integer Sequences. For distance Pell numbers $Pd(k,n)$ we give some combinatorial interpretations. Firstly, we apply this generalization for counting special families of subsets of the set of n integers.

Let $X = \{1,2,...,n\}$, $n \geq 1$ be the set of n integers. Assume that $k \geq 3$ be integer. Let $\mathcal{Y} = \{Y_t : t \in T\}$ be the family of pairwise disjoint subsets of the set X such that every subset Y_t, $t \in T$ contains consecutive integers and the following conditions hold

(a) $|Y_t| \in \{1,2,k\}$ for $t \in T$

(b) $\bigcup_{t\in T} Y_t = X$.

The family $\mathcal{Y}$ gives a $(1,2,k)$-decomposition of the set X.

Theorem 1. *Let $k \geq 3$, $n \geq 1$ be integers. Then the number of all $(1,2,k)$-decompositions of the set X is equal to $Pd(k, n+k-2)$.*

Proof. Let $k \geq 3$, $n \geq 1$ be integers and assume that $X = \{1,2,...,n\}$. Denote by $p(k,n)$ the number of all $(1,2,k)$-decompositions of the set X. If $n = 1,2,...,k-1$ then subsets Y_t have the cardinality 1 or 2 and the result immediately follows. If $n = k$ then additionally we have to consider decomposition $\{1,2,...,k\}$.

Assume now that $n \geq k+1$ and suppose that the Theorem is true for the cardinality of X equal to t, where $t < n$. Let $p_i(k,n)$ be the number of all $(1,2,k)$-decompositions $\mathcal{Y}$ of the set X such that $\{1,...,i\} \in \mathcal{Y}$, for $i = 1,2,k$, respectively. Then $p(k,n) = p_1(k,n)+p_2(k,n)+p_k(k,n)$. Clearly $p_i(k,n) = p(k,n-i)$, $i = 1,2,k$ because of $|X \setminus \{1,...,i\}| = n-i$. By induction hypothesis we obtain that

$$\begin{aligned} p(k,n) &= Pd(k,n-1+k-2)+Pd(k,n-2+k-2)+ \\ &\quad +Pd(k,n-k+k-2)= \\ &= Pd(k,n+k-3)+Pd(k,n+k-4)+Pd(k,n-2)= \\ &= Pd(k,n+k-2), \end{aligned}$$

from the definition of the distance Pell numbers. Thus the theorem is proved. □

This combinational interpretation of $Pd(k,n)$ immediately gives others. The distance Pell number $Pd(k,n+k-2)$ is equal to the number of all ordered composition of integer n into summands 1, 2 and k. It follows from the fact that every subset Y_t corresponds to a part n_t of $\sum_{t\in T} n_t = n$.

As example consider $n = 4$ and $k = 3$. Then all ordered composition of the number n on parts 1, 2, 3 are listed below

$$1+1+1+1,\ 2+1+1,\ 1+2+1,\ 1+1+2,\ 3+1,\ 1+3,\ 2+2$$

The number of all such ordered composition is equal to $Pd(3,5) = 7$.

Now we give the direct formula for the distance Pell numbers $Pd(k,n)$.

Theorem 2. *Let $n \geq 2$, $k \geq 3$ be integers. Then*

$$Pd(k,n+k-2) = \sum_{i=0}^{\lfloor \frac{n}{k} \rfloor} \sum_{j=0}^{\lfloor \frac{n-ik}{2} \rfloor} \binom{n-(k-1)i-j}{i} \cdot \binom{n-ki-j}{j}.$$

Proof. Consider the $(1,2,k)$-decomposition of the set $X = \{1,2,\ldots,n\}$, $n \geq 2$. Then it satisfies the conditions (a) and (b) and $n = n_1+n_2+\ldots+n_t$, where $1 \leq t \leq n$. Instead of this sum we can study a t-tuple $(n_1,n_2,\ldots,n_t)$, and assume that it has i words k and j words equal to 2. Then there are exactly $n-(i+j)$ words 1. Hence $t = i+j+n-ki-2j = n-(k-1)i-j$. Clearly $0 \leq i \leq \lfloor \frac{n}{k} \rfloor$. If a t-tuple has i words k then $0 \leq j \leq \lfloor \frac{n-ik}{2} \rfloor$. This immediately gives $\sum_{i=0}^{\lfloor \frac{n}{k} \rfloor} \sum_{j=0}^{\lfloor \frac{n-ik}{2} \rfloor} \binom{i+j+n-ki-2j}{i} \cdot \binom{j+n-ki-2j}{j} =$ $\sum_{i=0}^{\lfloor \frac{n}{k} \rfloor} \sum_{j=0}^{\lfloor \frac{n-ik}{2} \rfloor} \binom{n-(k-1)i-j}{i} \cdot \binom{n-ki-j}{j} = = Pd(k,n+k-2)$, by Theorem 1. □

3 Relations with $Pd(k,n)$ and F_n

In this section we derive some relations between distance Pell number $Pd(k,n)$ and the classical Fibonacci numbers.

This results are based on the research initiated by E. Kiliç (also with D. Tasci and P. Stanica), see [8], [7]. In these papers interesting results described relationships between another type of generalized Pell numbers and the classical Fibonacci numbers are studied. Analogous results are obtained in [11]. For the distance Pell numbers $Pd(k,n)$ we can prove the following result which immediately gives another.

Theorem 3. *Let $k \geq 3$, $n \geq k$ be integers. Then for fixed $1 \leq i \leq n-1$ holds*

$$\begin{aligned} Pd(k,n) = \; & F_i Pd(k,n-i) + F_{i-1} Pd(k,n-(i+1)) + \\ & + F_{i-1} Pd(k,n-k-(i-1)) + \\ & + \sum_{t=2}^{i} F_{i-t} Pd(k,n-k-(i-t)) . \end{aligned}$$

Proof. (by induction on i)
Let $i = 1$. Then

$$\begin{aligned} Pd(k,n) = \; & F_1 Pd(k,n-1) + F_0 Pd(k,n-2) + F_0 Pd(k,n-k) = \\ & = Pd(k,n-1) + Pd(k,n-2) + Pd(k,n-k) \end{aligned}$$

by definition of $Pd(k,n)$.

Assume that the formula is true for an arbitrary $i \geq 2$. We shall show that it holds for $i+1$, i.e.

$$\begin{aligned} Pd(k,n) = \; & F_{i+1} Pd(k,n-i-1) + F_i Pd(k,n-i-2) + \\ & + F_i Pd(k,n-k-i) + \\ & + \sum_{t=2}^{i+1} F_{i-t+1} Pd(k,n-k-(i+1-t)) . \end{aligned}$$

By calculating on the right side of this equation we have

$$\begin{aligned} & (F_{i-1} + F_i) Pd(k,n-i-1) + F_i Pd(k,n-i-2) + F_i Pd(k,n-k-i) + \\ & + F_{i-2+1} Pd(k,n-k-(i+1-2)) + F_{i-3+1} Pd(k,n-k-(i+1-3)) + \\ & + \ldots + F_{i-i+1} Pd(k,n-k-(i+1-i)) + \\ & + F_{i-(i+1)+1} Pd(k,n-k-(i+1-(i+1))) = \end{aligned}$$

$$\begin{aligned} & = F_{i-1} Pd(k,n-i-1) + F_i [Pd(k,n-i-1) + Pd(k,n-i-2) + \\ & + Pd(k,n-i-k)] + F_{i-1} Pd(k,n-k-(i-1)) + \\ & + F_{i-2} Pd(k,n-k-(i-2)) + \cdots + \\ & + F_1 Pd(k,n-k-1) + F_0 Pd(k,n-k) = \end{aligned}$$

$$\begin{aligned} & = F_{i-1} Pd(k,n-(i+1)) + F_i Pd(k,n-i) + F_{i-1} Pd(k,n-k-(i-1)) + \\ & + \sum_{t=2}^{i} F_{i-t} Pd(k,n-k-(i-t)) = Pd(k,n) \end{aligned}$$

by induction's assumption.

□

Using known relation $F_n = \frac{L_{n-1}+L_{n+1}}{5}$, where L_n is nth Lucas number defined by: $L_n = L_{n-1} + L_{n-2}$, $n \geq 2$ with $L_0 = 2$, $L_1 = 1$ we obtain the following result.

Corollary 4. *Let $k \geq 3$, $n \geq k$ be integers. Then*

$$\begin{aligned} 5Pd(k,n) = \; & (L_{i-1} + L_{i+1})Pd(k, n-i) + (L_{i-2} + L_i)Pd(k, n-(i+1)) + \\ & + (L_{i-2} + L_i)Pd(k, n-k-(i-1)) + \\ & + \sum_{t=2}^{i} (L_{i-t-1} + L_{i-t+1})Pd(k, n-k-(i-t)) \end{aligned}$$

for $1 \leq i \leq n-1$.

4 Graph interpretations of $Pd(k,n)$ and generalized Jacobsthal numbers

For graph concepts not defined here see [1]. In this section we give the graph interpretation of the numbers $Pd(k,n)$. It is worth to be noted that the graph interpretation of the number of the Fibonacci type was initiated by H. Prodinger and R. F. Tichy in [9]. This interpretation is closely related to the concept of independent sets in graphs and it gave an impetus for counting of independent set in graphs. Graph interpretations of the numbers of the Fibonacci type are studied recently in many papers, see e.g. [10]. This concept gives new tools for studying properties of these numbers. Graphs methods in this context are new and very useful. Our aim of this section is to show how graphs method can be applied for studying known sequences and their generalizations. In this paper we give a graph interpretation of $Pd(k,n)$ with respect to the number of $\mathcal{H}$-matchings. By $\mathbb{P}_n$ $n \geq 1$ we denote an n-vertex path and by K_n, $n \geq 2$ we mean a complete graph on n-vertices.

The graph interpretation of the classical Pell number is well-known in the context of the Hosoya index Z. This index is a graph parameter defined as the number of all matchings in G. The Hosoya index is a topological index introduced in 1971 by Japanese chemist H. Hosoya, see [5]. This structure descriptor is used for studying quantitative properties of a molecular graph. At the beginning it was studied for properties of alcanes. From a formal point of view the definition of the Merrifield-Simmons index σ of a molecular graph in analogous. It is defined as the number of all independent sets of a graph G. Many interesting papers related to these indices appear recently, see the last survey [4] and its references.

The Hosoya index and the Merrifield-Simmons index in some classes of graphs are given by the numbers of the Fibonacci type and for Pell numbers holds $P_n = Z(\mathbb{P}_n \circ K_1)$ where $\mathbb{P}_n \circ K_1$ is the corona of graph $\mathbb{P}_n$ and K_1.

The graph interpretation of $Pd(2, n)$ which is the nth Jacobsthal number is strictly connected with the Merrifield-Simmons index.

Firstly we give the graph interpretation of $Pd(2, n)$ which is nth Jacobsthal number. The Figure 1 gives the graph $P_n[K_2]$ for which the Merrifield-Simmons index σ is described by the Jacobsthal numbers.

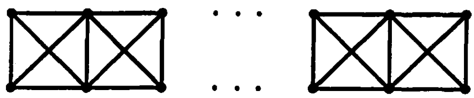

Fig.1. The graph $P_n[K_2]$.

Theorem 5. *Let $n \geq 1$ be integer. Then $\sigma(P_n[K_2]) = J_{n+2}$.*

This graph interpretation of the Jacobsthal numbers J_n in the natural way implies the generalization of the Jacobsthal numbers $J_{t,n}$.

Let $n \geq 0$, $t \geq 1$ be integers. The nth generalized Jacobsthal number $J_{t,n}$ is defined recursively as follows

$$J_{t,n} = J_{t,n-1} + t \cdot J_{t,n-2}, \text{ for } n \geq 2$$

with initial conditions $J_{t,0} = 0$ and $J_{t,1} = 1$. It is interesting to note that $J_{t,n}$ generalizes the Fibonacci numbers and the Jacobsthal numbers, simultaneously. If $t = 1$ then $J_{1,n} = F_{n+1}$ and for $t = 2$ holds $J_{2,n} = J_n$. The characteristic equation for the generalized Jacobsthal numbers $J_{t,n}$ has the form $r^2 - r - t = 0$ and by simple calculations we obtain the Binet formula of the form

$$J_{t,n} = \frac{1}{\sqrt{1+4t}}\left[\left(\frac{1+\sqrt{1+4t}}{2}\right)^n - \left(\frac{1-\sqrt{1+4t}}{2}\right)^n\right].$$

Evidently for $t = 1$ the above immediately gives the well-known Binet formula for the classical Fibonacci numbers.

The following Table gives generalized Jacobsthal sequences for few initial values of t.

n	0	1	2	3	4	5	6	7	8	9	10	11
$J_{1,n}$	0	1	1	2	3	5	8	13	21	34	55	89
$J_{2,n}$	0	1	1	3	5	11	21	43	85	171	341	683
$J_{3,n}$	0	1	1	4	7	19	40	97	217	508	1159	2683
$J_{4,n}$	0	1	1	5	9	29	65	181	441	1165	2929	7589
$J_{5,n}$	0	1	1	6	11	41	96	301	781	2286	6191	17621
$J_{6,n}$	0	1	1	7	13	55	133	463	1261	4039	11605	35839

Table 2. The nth generalized Jacobsthal number $J_{t,n}$.

The numbers $J_{t,n}$ have the graph interpretation directly related to the Merrifield-Simmons index σ.

Theorem 6. *Let $n \geq 1$, $t \geq 1$ be integers. Then*

$$\sigma\left(P_n[K_t]\right) = J_{t,n+2}.$$

Proof. Let n, t be as in the statement of the theorem and let $S \subset V\left(P_n[K_t]\right)$ be an arbitrary independent set of the graph $P_n[K_t]$. If $n = 1, 2$ then every independent set S of $P_n[K_t]$ has at most one vertex and the result is obvious.

Assume that $\sigma\left(P_m[K_t]\right) = J_{t,m}$ for $m < n$. Let $V(P_n) = \{x_1, \ldots, x_n\}$ and $V(K_t) = \{y_1, \ldots, y_t\}$ with the numbering of vertices in the natural fashion. We consider the following cases.

(1) $(x_n, y_i) \notin S$ for all $i = 1, \ldots, t$.
Then $S = S^*$ is an arbitrary independent set of the graph $P_{n-1}[K_t]$ and by induction's hypothesis we have $J_{t,n+1}$ independent sets in this case.

(2) There is $1 \leq i \leq t$ such that $(x_n, y_i) \in S$.
Then $S = S^{**} \cup \{(x_n, y_i)\}$ where S^{**} is an arbitrary independent set of the graph $P_{n-2}[K_t]$. Since the vertex (x_n, y_i) can be chosen on t ways hence by induction's hypothesis there are $t \cdot J_{t,n}$ independent sets in this way.

Consequently from the above cases we obtain that $\sigma\left(P_n[K_t]\right) = J_{t,n+1} + t \cdot J_{t,n}$ and by the definition of the generalized Jacobsthal number the result follows.

Thus the Theorem is proved. □

The graph interpretations of the numbers of the Fibonacci type give a new tool for proving identities. This method is very useful and convenient.

Theorem 7. *Let $n \geq 3$, $t \geq 1$ be integers. Then for $1 < i < n$*

$$J_{t,n} = J_{t,n-i+1} \cdot J_{t,n-i} + t \cdot J_{t,n-i-2} \cdot J_{t,n-i-1}.$$

Proof. By Theorem 6 we obtain that $J_{t,n} = \sigma\left(P_{n-2}[K_t]\right)$. Let $S \subset V\left(P_{n-2}[K_t]\right)$ be an independent set of the graph $P_{n-2}[K_t]$. Using the same method as in Theorem 6 by considering two cases: either $(x_i, y_p) \notin S$ or $(x_i, y_p) \in S$ for any $2 \leq i \leq n-1$ and $1 \leq p \leq t$ the result immediately follows. □

Corollary 8. *If $t = 1$ then from the above we obtain*

$$F_{n+1} = F_{n-i+2} \cdot F_{n-i+1} + F_{n-i-1} \cdot F_{n-i}.$$

If $t = 2$ then

$$J_n = J_{n-i+1} \cdot J_{n-i} + 2J_{n-i-2} \cdot J_{n-i-1}.$$

Additionally for $t = 1$ and $i = 2$ we have that

$$F_{n+1} = F_n \cdot F_{n-1} + F_{n-3} \cdot F_{n-2}$$

and for $t = 2$ and $i = 2$ we obtain

$$J_n = J_{n-1} \cdot J_{n-2} + 2J_{n-4} \cdot J_{n-3}.$$

For the generalized Jacobsthal number $J_{t,n}$ we can give also a combinatorial interpretation which is a consequence of it's graph interpretation.

Let $X = \{1, 2, \ldots, n\}$, $n \geq 1$ be the set of n consecutive integers and $X^{(t)} = X \times \{1, 2, \ldots, t\}$.

Let $\mathcal{Y} = \{Y \subset X^{(t)} : \ (i, j), (p, q) \in Y \text{ and } |i - p| \leq 2\}$.

Theorem 9. *Let $n \geq 1$, $t \geq 1$ be integers. Then the number of all families $\mathcal{Y}$ is equal to $J_{t,n+2}$.*

Proof. We use the graph interpretation of the number $J_{t,n+2}$. With the set $X^{(t)}$ we associate the vertex set of the graph $P_n[K_t]$ where $V(P_n) = \{x_1, \ldots, x_n\}$ and $V(K_t) = \{y_1, \ldots, y_t\}$ with the numbering the vertices in both of sets in the natural fashion. Then every subset $Y \in \mathcal{Y}$ corresponds to the independent set of the graph $P_n[K_t]$ and by Theorem 6 the result follows. □

To give the graph interpretation of $Pd(k, n)$ where $k \geq 3$ we need to introduce the concept of $\mathcal{H}$-matching in graphs.

Let $\mathcal{H} = \{H_1, ..., H_m\}$, $m \geq 1$ be a collection of m connected graphs. We say that a subgraph $M \subseteq G$ is an $\mathcal{H}$-matching of G if each connected component of M is isomorphic to same H_i, $1 \leq i \leq m$. Additionally if $V(M) = V(G)$ then we say that M is a perfect $\mathcal{H}$-matching of a graph G.

The Fig. 2 gives an example of perfect $\{K_2, C_3\}$-matching of a graph G.

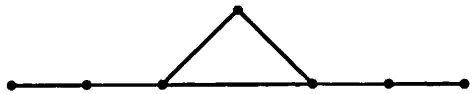

Fig.2. The perfect $\{K_2, C_3\}$-matching of a graph G.

Using this concept we can give the graph interpretation of the number $Pd(k, n)$, for $k \geq 3$.

Theorem 10. *Let $k \geq 3$, $n \geq 1$ be integers. Then the number of perfect $\{\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_k\}$-matchings of a graph $\mathbb{P}_n$ is equal to $Pd(k, n + k - 2)$.*

Proof. Assume that the set $X = \{1, 2, ..., n\}$ corresponds to the vertex set of a graph $\mathbb{P}_n$ with the numbering the vertex set in the natural fashion. Let $\mathcal{Y} = \{Y_t : t \in T\}$ be a family of the set X which gives a $(1, 2, k)$-decomposition of X. Then every Y_t corresponds to a subgraph $\mathbb{P}_{|Y_t|}$, $t \in T$ of $\mathbb{P}_n$. Then using Theorem 1 the result immediately follows. □

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