On optimizing m-restricted edge connectivity of generalized permutation graphs

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Abstract To construct a large graph from two smaller ones that have same order, one can add an arbitrary perfect matching between their vertex-sets. The topologies of many networks are special cases of these graphs. An interesting and important problem is how to persist or even improve their link reliability and link fault-tolerance. Traditionally, this may be done by optimizing the edge connectivity of their topologies, a more accurate method is to improve their m-restricted edge connectivity. This work presents schemes for optimizing m-restricted edge connectivity of these graphs, some well-known results are direct consequences of our observations.

Keywords Restricted edge connectivity; generalized permutation graph; network reliability; fault-tolerance
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1 Introduction

For constructing a large graph from two smaller ones G_1 and G_2 that have same order, one can add an arbitrary perfect matching M between their vertex-sets [5]. The resulting graph is denoted by $G(G_1, G_2; M)$ and is called a generalized permutation graph since permutation graphs (or generalized prisms) are special cases of this kind graphs [1,7,9]. The topologies of many networks are these graphs, an interesting and important problem is how to persist or even improve their link reliability and link fault-tolerance. Traditionally, this may be done by optimizing the edge connectivity of their topologies, a more accurate method is to improve their m-restricted edge connectivity [2,6,8,10,14,15].

An m-restricted edge cut of a connected graph is an edge cut whose removal separates this graph into components of order at least m [4,12]. When $m = 2$, it is the so-called restricted edge cut [6]; when $m = 1$, it is the traditional edge cut. The minimum cardinality $\lambda_m(G)$ over all m-restricted edge cuts of graph G is called its m-restricted edge connectivity. It is known that $\lambda_m(G) \leq \xi_m(G)$ holds for almost any graph G that contains m-restricted edge cuts (in view of probability) [4,6,11,13], where $\xi_m(G) = \min\{\partial(X) : X$ is a vertex induced subgraph of order $m\}$ and $\partial(X)$ is the number of edges with only one end in X. Graph G is called maximally m-restricted edge connected if $\lambda_m(G) = \xi_m(G)$, and super m-restricted edge connected if every minimum m-restricted edge cut separates a component of order m. In all these concepts, 2-restricted is simplified as restricted.

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It is known that networks with maximal m-restricted edge connectivity are locally more reliable when $m \leq 3$ [10,15], which seems also true when $m \geq 4$. And so, the optimization of m-restricted edge connectivity is of its importance in the design of most reliable networks. This work presents schemes for optimizing m-restricted edge connectivity of generalized permutation graphs. Many known results are direct consequences of our observations.

For two subsets or subgraphs of $V(G)$ of a graph G, let $[X, Y]$ denote the set of edges of G with one end in X and the other in Y. For other symbols and terminology not specified herein, we follow that of [3].

2 Restricted edge connectivity

Theorem 2.1. Let G_1 and G_2 be two maximally restricted edge connected k-regular graphs with same order. Then

1. $G(G_1, G_2; M)$ is maximally restricted edge connected if and only if $|G_1| = |G_2| \geq 2k$.
2. When $k \geq 3$, $G(G_1, G_2; M)$ is super restricted edge connected if and only if $|G_1| = |G_2| > 2k$.

Proof. Let $t = |G_1|$. Since G_1 and G_2 are k-regular maximally restricted edge connected graphs, it follows that $t \geq \max\{4, k+1\}$. If $t < 2k$, then M is a restricted edge cut of $G(G_1, G_2; M)$ since its removal results in components G_1 and G_2. And so, $\lambda_2(G) \leq |M| = t < 2k = \min\{d(u) + d(v) - 2 : uv \in E(G)\} = \xi_2(G)$. This observation shows that $G(G_1, G_2; M)$ is not maximally restricted edge connected in this case.

Now consider the case when $t \geq 2k$. Let S be an arbitrary minimum restricted edge cut of $G(G_1, G_2; M)$. We shall show at first that G is maximally restricted edge connected. To this end, it suffices to show that $|S| = 2k$, since if so then $2k = \xi_2(G) \geq \lambda_2(G) = |S| = 2k$ implies that $\lambda_2(G) = \xi_2(G) = 2k$.

Suppose on the contrary that $|S| \leq 2k - 1$. Then M cannot be a minimum restricted edge cut of G, and so $G_1 - S \cap E(G_1)$ or $G_2 - S \cap E(G_2)$ is disconnected. Since G_1 and G_2 has restricted edge connectivity $\lambda_2(G_1) = \lambda_2(G_2) = 2k-2$, $k \geq 2$ and $|S| = \lambda_2(G) \leq \xi_2(G) = 2k$, it follows that $G_1 - S \cap E(G_1)$ or $G_2 - S \cap E(G_2)$ is connected. Assume without loss of generality that $G_1 - S \cap E(G_1)$ is connected and $G_2 - S \cap E(G_2)$ is disconnected with X_2, Y_2 being two of its components such that $1 \leq |X_2| \leq |Y_2|$. Noticing that either $[X_2, Y_2]$ is a restricted edge cut of G_2 or $|X_2| = 1$ and Y_2 is a component of $G - S$, we deduce that either

$$|S| \geq |[X_2, Y_2]| + |[X_2, G_1]| \geq \lambda_2(G_2) + |X_2| \geq 2k$$

or

$$|S| \geq \lambda(G_2) + |G_2| - 1 = k + t - 1 > 2k,$$

where $\lambda(G_2)$ denotes the edge connectivity of graph G_2. The previous contradictions implies that $|S| = 2k$, and so the first statement follows.

Continue to show that $G - S$ contains an isolated edge when $k \geq 3$ and $t > 2k$. Let X and Y be the two components of $G - S$ with $2 \leq |X| \leq |Y|$. Let

$$X \cap G_1 = X_1, X \cap G_2 = X_2, Y \cap G_1 = Y_1, Y \cap G_2 = Y_2.$$

Then

$$[X_1, Y_1] \cup [X_1, Y_2] \cup [X_2, Y_2] \cup [X_2, Y_1] \subseteq S.$$

Consider at first the case when none of X_1, X_2, Y_1, Y_2 is empty. In this case, $[X_i, Y_i]$ is an edge cut of G_i, $i = 1, 2$. By the first statement of this theorem, we have

$$\begin{aligned} 2k &= |S| \geq |[X_1, Y_1] \cup [X_1, Y_2] \cup [X_2, Y_2] \cup [X_2, Y_1]| \\ &\geq \lambda(G_1) + |[X_1, Y_2]| + \lambda(G_2) + |[X_2, Y_1]| \\ &= k + |[X_1, Y_2]| + k + |[X_2, Y_1]| \geq 2k. \end{aligned}$$

The inequalities in above formula must become equalities. Hence, $|[X_1, Y_2]| = |[X_2, Y_1]| = 0$ and $|[X_1, Y_1]| = |[X_2, Y_2]| = k$. Since G_1 and G_2 are maximally restricted edge connected with $k \geq 3$, it follows that $[X_i, Y_i]$ separates an isolated vertex from G_i for all $i \in \{1, 2\}$. Therefore, $|X_1| = |X_2| = 1$ or $|Y_1| = |Y_2| = 1$.

Consider secondly the case when at least one of X_1, X_2, Y_1, Y_2, say X_1, is empty. Since $t > 2k$, M cannot be a minimum restricted edge cut of G. And so, $[X_2, Y_2]$ forms an edge cut of G_2 and $[X_2, Y_2] \cup [X_2, Y_1] \subseteq S$. Noticing that $|X_2| \geq 2$, if $|Y_2| \geq 2$ then $[X_2, Y_2]$ is a restricted edge cut of G_2, and so

$$2k = |S| \geq 2k - 2 + |[X_2, Y_1]| = 2k - 2 + |X_2| \geq 2k,$$

which implies that X_2 is an isolated edge of $G - S$; if otherwise $|Y_2| = 1$ then

$$2k = |S| \geq k + |[X_2, Y_1]| = k + |G_2| - 1 = k + t - 1 > k + 2k - 1 > 2k.$$

The sufficiency of the second statement follows from this contradiction. If $t \leq 2k$, then the perfect matching M forms a restricted edge cut, which separates G into components of order at least three. And so, the theorem follows. □

Remark 1. Since the Cartesian product of graph H and K_2 is a special case of $G(G_1, G_2; M)$, our observation can be employed to optimize restricted edge connectivity of some Cartesian product graphs. For example, binary hypercube Q_n is one of a most popular topology [6], it can be recursively defined as $Q_1 = K_2$ and Q_n is the Cartesian product of Q_{n-1} and K_2. Since Q_3 is maximally restricted edge connected, by Theorem 2.1, Q_n is super restricted edge connected whenever $n \geq 4$. This observation is also obtained in [6].

Remark 2. The permutation graph over graph H is obtained by adding an arbitrary perfect matching between two disjoint copies of H [7], which is also called generalized prisms [9] and is obviously a special case of $G(G_1, G_2; M)$. In [1, corollary 3.3], the authors show that if G is a connected triangle-free graph with minimum vertex degree $\delta(G) \geq 2$ and $\lambda_2(G) \geq \xi_2(G) + 2 - \delta(G)$ then the permutation graph over G is maximally restricted edge connected. For the case when G is a k-regular graph, this results is obviously a directed consequence of Theorem 2.1.

To optimize the m-restricted edge connectivity of $G(G_1, G_2; M)$ for any $m \geq 3$, we need introduce another parameter at first. Let $r = \max\{|[X \cap G_1, X \cap G_2]| : X \text{ is a connected subgraph of order } m \text{ of } G(G_1, G_2; M)\}$. If $m \geq 3$, then $r \geq 1$. Let $\xi_m = \xi_m(G(G_1, G_2; M))$. With these conventions, we obtain the following observation.

Lemma 2.2. Let G_1 and G_2 be two maximally m-restricted edge connected k-regular graphs with $m \geq 3$. If they have girth at least $m+1$, then $\xi_m = (k-1)m + 4 - 2r$.

Proof. Let X be a connected vertex-induced subgraph graph of order m of $G(G_1, G_2; M)$ such that $|[X, X^c]| = \xi_m$. If $X \subseteq G_1$ or $X \subseteq G_2$, then X is a tree since G_1 and G_2 have girth at least $m+1$. Noticing that $r \geq 1$, we deduce in this case that $\xi_m = |[X, X^c]| = (k+1)m - 2(m-1) = (k-1)m + 2 \geq (k-1)m + 4 - 2r$. If $X \cap G_1 \neq \emptyset \neq X \cap G_2$, then

$$\begin{aligned}
\xi_m &= |[X, X^c]| = (k+1)m - 2|E(X)| \\
&= (k+1)m - 2(|E(X \cap G_1)| + |E(X \cap G_2)| + |[X \cap G_1, X \cap G_2]|) \\
&\geq (k+1)m - 2(|X \cap G_1| + |X \cap G_2| - 2 + r) \\
&= (k+1)m - 2(|X| - 2 + r) = (k-1)m + 4 - 2r.
\end{aligned}$$

The inequality in above formula becomes equality if and only if $|[X \cap G_1, X \cap G_2]| = r$. The lemma follows from above discussion. □

Theorem 2.3. Let G_1 and G_2 be two maximally m-restricted edge connected k-regular graphs with $k, m \geq 3$ and girth at least $m+1$. Then $G(G_1, G_2; M)$ is maximally m-restricted edge connected if and only if $|G_1| = |G_2| \geq \xi_m$.

Proof. Let $S = [X, Y]$ be a minimum m-restricted edge cut of $G(G_1, G_2; M)$ with $|X| \leq |Y|$. If $|G_1| < \xi_m$, then $\lambda_m(G) = |S| \leq |M| = |G_1| < \xi_m$ and the necessity follows. Assume in what follows that that $|G_1| = |G_2| \geq \xi_m$. To prove the sufficiency, we shall show at first that $|S| \geq \xi_m$.

Let us consider at first the case when $X \subseteq G_1$ or $X \subseteq G_2$, say $X \subseteq G_1$. If one component of $Y \cap G_1$ has order at least m, then $[X, Y \cap G_1]$ contains an

m-restricted edge cut of G_1. And so, $|S| \geq \lambda_m(G_1) + |X| = \xi_m(G_1) + |X| = (k-2)m + 2 + |X| \geq mk - m + 2 \geq \xi_m$. If $|Y \cap G_1| \geq m$ but every component W_i of $Y \cap G_1$ has order at most $m-1$, $i = 1, 2, ..., \omega$, then $Y \cap G_1$ is a forest. And so,

$$\begin{aligned} |S| &\geq \sum_{i=1}^{\omega}(k|W_i| - 2|W_i| + 2) + |X| \\ &= k|Y \cap G_1| - 2|Y \cap G_1| + 2\omega + |X| \\ &\geq (k-2)m + 2 + |X| \geq (k-1)m + 2 \\ &\geq (k-1)m + 4 - 2r = \xi_m. \end{aligned} \tag{1}$$

If $|Y \cap G_1| \leq m - 1$, then

$$\begin{aligned} |S| &\geq k|Y \cap G_1| - 2|Y \cap G_1| + 2\omega + |X| \\ &\geq (k-3)|Y \cap G_1| + |Y \cap G_1| + 2\omega + |X| \\ &= (k-3)|Y \cap G_1| + 2\omega + |G_1| \geq |G_1| \geq \xi_m \end{aligned} \tag{2}$$

Now consider the case when $X \cap G_1 \neq \emptyset \neq X \cap G_2$ and $Y \cap G_1 \neq \emptyset \neq Y \cap G_2$. Define X_1, X_2, Y_1, Y_2 as in the proof of Theorem 2.1 and assume without loss of generality that $|X_1| \leq |X_2|$. Then $|Y_2| \leq |Y_1|$. Since G_1 and G_2 are maximally restricted edge connected, it follows that $|X_1| + |Y_1| = |G_1| = |G_2| = |X_2| + |Y_2| \geq 2m$. And so, at least one of X_1 and Y_1 has order at least m, as well as X_2 and Y_2. Hence, there are only three diffierent cases.

Case 1. $|X_1|, |X_2|, |Y_1|, |Y_2| \geq m$.

If both X_1 and Y_1 contain a component of order at least m, then $|[X_1, Y_1]| \geq \lambda_m(G_1) = km - 2m + 2$; if one of X_1 and Y_1, say X_1, consists of components $W_1, W_2, \cdots, W_\omega$ with $|W_i| \leq m - 1$, $i = 1, 2, \cdots, \omega$, then $|[X_1, Y_1]| \geq \sum_{i=1}^{\omega}(k|W_i| - 2|W_i| + 2) = k|X_1| - 2|X_1| + 2\omega \geq km - 2m + 2$. In any case, we have $[X_1, Y_1]| \geq km - 2m + 2$. Similarly, $|[X_2, Y_2]| \geq km - 2m + 2$. And so, $|S| \geq |[X_1, Y_1]| + |[X_2, Y_2]| \geq (k-1)m + 4 + m(k-3) \geq \xi_m + 2$ when case 1 occurs.

Case 2. Only one of X_1, X_2, Y_1 and Y_2 has order at most $m - 1$.

Assume without loss of generalizty that $|X_1| \leq m - 1$. Then $|[X_1, Y_1]| \geq k|X_1| - 2|X_1| + 2$. Since $|X_2|, |Y_2| \geq m$, as is shown in the proof of case 1 we have $|[X_2, Y_2]| \geq km - 2m + 2$. Recalling that $|X_2| \geq |X_1|$, we deduce that $|[X_2, Y_1]| \geq |X_2| - |X_1|$. And so,

$$\begin{aligned} |S| &\geq |[X_1, Y_1]| + [X_2, Y_2]| + |[X_2, Y_1]| + |[X_1, Y_2]| \\ &\geq k|X_1| - 2|X_1| + 2 + mk - 2m + 2 + |X_2| - |X_1| \\ &= (k-1)m + 4 + (k-3)|X_1| + |X_2| - m \\ &\geq (k-1)m + 4 \\ &\geq \xi_m + 2. \end{aligned}$$

Case 3. Exactly two of X_1, X_2, Y_1, Y_2 have order at most $m-1$.

In this case, either $|X_1|, |Y_2| \le m-1$ or $|X_1|, |X_2| \le m-1$. Consider at first the subcase when $|X_1|, |Y_2| \le m-1$. We claim at first that $|X_1| \le |Y_2| \le |X_2| \le |Y_1|$ in this subcase. Suppose on the contrary that $|X_1| > |Y_2|$. Since $|X_1| + |X_2| = |X| \le |Y| = |Y_1| + Y_2|$, it follows that $|Y_1| > |X_2|$. And so, $|G_1| = |X_1| + |Y_1| > |Y_2| + |X_2| = |G_2|$. This contradition implies that $|X_1| \le |Y_2|$. Since $|X_2| \ge m$, it follows that $|X_1| \le |Y_2| \le |X_2|$. Similarly, $|Y_1| < |X_2|$ implies that $|Y_2| > |X_1|$ since $|Y| \ge |X|$, and so $|G_1| < |G_2|$. This contradiction shows that $|X_2| \le |Y_1|$. Hence, our claim follows. Now

$$\begin{aligned}
|S| &\ge k|X_1| - 2|X_1| + 2 + k|Y_2| - 2|Y_2| + 2 + |[X_2, Y_1]| \\
&\ge k(|X_1| + |Y_2|) - 2(|X_1| + |Y_2|) + 4 + |X_2| - |X_1| \\
&= k(|X_1| + |Y_2|) - 2(|X_1| + |Y_2|) + 4 + |X_2| + |Y_2| - |Y_2| - |X_1| \\
&= |X_2| + |Y_2| + (k-3)(|X_1| + |Y_2|) + 4 \ge |G_2| + 4 \ge \xi_m + 4.
\end{aligned}$$

Continue to consider the subcase when $|X_1|, |X_2| \le m-1$. In this subcase, we have

$$\begin{aligned}
|S| &\ge k|X_1| - 2|X_1| + 2 + k|X_2| - 2|X_2| + 2 + |[X_1, Y_2]| + |[X_2, Y_1]| \\
&\ge k|X_1| - 2|X_1| + 2 + k|X_2| - 2|X_2| + 2 + |[X_1, Y_2]| + |[X_2, Y_1]| \\
&= (k-2)(|X_1| + |X_2|) + 4 + |X_2| - |[X_1, X_2]| + |X_1| - |[X_1, X_2]| \\
&= (k-1)(|X_1| + |X_2|) + 4 - 2|[X_1, X_2]| \\
&\ge (k-1)|X| + 4 - 2r \ge \xi_m.
\end{aligned}$$

These discussions show that $\lambda_m(G) \ge \xi_m(G)$ whenever $|G_1| = |G_2| \ge \xi_m$.

In what follows we shall show that $\lambda_m(G) \le \xi_m(G)$, and so the sufficiency follows. Let X be a vertex-induced connected subgraph of $G(G_1, G_2; M)$ of order m with $\partial(X) = \xi_m(G)$. Let $X_1 = X \cap G_1$, $X_2 = X \cap G_2$ and assume without loss of generality that $|X_1| \le |X_2|$. Then $|X_1| \le m/2$.

Suppose on the contrary that $G_1 - X_1$ contains no components of order at least m. Since G_1 has girth at least $m+1$, it follows that for every component H_i of $G_1 - X_1$ we have $|[H_i, G_1 - H_i]| = k|H_i| - 2(|H_i| - 1) = (k-2)|H_i| + 2$. If $G_1 - X_1$ has ω_1 components, then

$$\begin{aligned}
|[G_1 - X_1, X_1]| &= \sum_{i=1}^{\omega_1} |[H_i, G_1 - H_i]| = \sum_{i=1}^{\omega_1} (k|H_i| - 2(|H_i| - 1)) \\
&= (k-2)(|G_1| - |X_1|) + 2\omega_1.
\end{aligned}$$

Similarly, if X_1 has ω_2 components then $|[X_1, G_1 - X_1]| = (k-2)|X_1| + 2\omega_2$. Since $|[G_1 - X_1, X_1]| = |[X_1, G_1 - X_1]|$, $|X_1| \le m/2$ and $|G_1 - X_1| \ge 3m/2$, it follows that $2(\omega_2 - \omega_1) = (k-2)(|G_1| - 2|X_1|) \ge (k-2)(2m - m) \ge m(k-2)$. Since $G_1 - X_1$ contains no components of order at least m, it follows that $\omega_1 \ge 2$

and $\omega_2 \le m/2$. And so, $m/2-2 \ge \omega_2 - \omega_1 \ge m(k-2)/2$. Recalling that $k \ge 3$, the previous observation implies that $m/2-2 \ge m/2$. This contradiction shows that $G_1 - X_1$ contains at least one component of order at least m. So, $[X, G-X]$ contains m-restricted edge cut and $\lambda_m(G) \le |[X, G-X]| = \partial(X) = \xi_m(G)$. □

Theorem 2.4. Let G_1 and G_2 be two maximally m-restricted edge connected k-regular graphs with $k, m \ge 3$ and $g \ge m+1$. Then $G(G_1, G_2; M)$ is super m-restricted edge connected if and only if $|G_1| = |G_2| \ge \xi_m + 1$.

Proof If $|G_1| \le \xi_m$, then the perfect matching M is an m-restricted edge cut of $G(G_1, G_2; M)$ of size no more than ξ_m. Furthermore, $G - M$ consists of two components of order at least $2m$. And so, the necessity follows.

Suppose on the contrary that $G(G_1, G_2; M)$ is not super m-restricted edge connected. Then there is a minimum m-restricted edge cut $S = [X, Y]$ with $|Y| \ge |X| \ge m+1$.

Define X_1, X_2, Y_1 and Y_2 as in the proof of theorem 2.1. If $X \subseteq G_1$ or $X \subseteq G_2$, say $X \subseteq G_1$, then, as shown in formulas (1) and (2), either $|S| \ge (k-2)m+2+|X|$ or $|S| \ge |G_1|$. And so, $|S| \ge \xi_m + 1$ in this case. If none of X_1, X_2, Y_1 and Y_2 is empty, then, as is pointed out in the proof of Theorem 2.3 (refer to case 1, 2 and 3), either $|S| \ge \xi_m + 2$ or $|S| \ge (k-1)|X| + 4 - 2r \ge (k-1)(m+1) + 4 - 2r > \xi_m$. The theorem follows from these contradictions. □

Remark 3. The lower bound on k of Theorem 2.1 and Theorem 2.4 is best possible. If $k = 2$, the two graphs G_1 and G_2 are isomorphic cycles. When they have order at least $2m$, it is not difficult to see that there is a perfect matching M such that $G(G_1, G_2; M)$ is not super m-restricted edge connected for every integer $m \ge 1$.

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