Characterizing When Powers of a Caterpillar are Divisor Graphs*

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Abstract.

In this paper, we prove that for any tree T, T^2 is a divisor graph if and only if T is a caterpillar and the diameter of T is less than six. For any caterpillar T and a positive integer $k \geq 1$ with $\text{diam}(T) < 2k$, we show that T^k is a divisor graph. Moreover, for a caterpillar T and $k \geq 3$ with $\text{diam}(T) = 2k$ or $\text{diam}(T) = 2k + 1$, we show that T^k is a divisor graph if and only if the centers of T have degree two.

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Key words. Caterpillar, divisor graph, power of a graph.

1 Introduction

A graph G is called a tree, denoted by T, if it is connected and has no induced cycles. A vertex v in T is called a leaf or end vertex if $deg(v) = 1$, otherwise it is called an interior vertex. A tree T is called a caterpillar if the removal of its end vertices produces a path. Note that, a path is a trivial type of caterpillar.

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The distance between two vertices x and y, is the length of a shortest path between them. We denote this distance by $d_G(x,y)$. The diameter of a graph G is equal to $\sup\{d_G(x,y) : x, y \in V(G)\}$, denoted by d_G or diam(G). The power graph of G is denoted by G^k, where the vertex set of G^k is $V(G)$ and two vertices x, y are adjacent iff $d_G(x,y) \leq k$. The centers of a graph G are vertices having minimum eccentricity (The eccentricity of a vertex u in G, denoted by $e_G(u)$, is the distance from u to the vertex farthest from u in G, see [5] p.79).

Now, a graph G is called a divisor graph if there is a bijection $f : V(G) \rightarrow S$, for some finite nonempty set S of the positive integers such that $uv \in E(G)$ iff $\gcd(f(u), f(v)) = \min\{f(u), f(v)\}$ (This means $uv \in E(G)$ iff $f(u) \mid f(v)$ or $f(v) \mid f(u)$). The function f is called a divisor labeling of G.

Moreover, for a finite nonempty set S of the positive integers. The divisor graph $G(S)$ of S has S as its vertex set and two distinct vertices i and j are adjacent if $i \mid j$ or $j \mid i$. A graph G is a divisor graph if G is isomorphic to $G(S)$. While the divisor digraph $D(S)$ of S has a vertex set S and (i,j) is an arc of $D(S)$ iff i divides j. In a digraph D, a transmitter is a vertex having indegree 0, a receiver is a vertex having outdegree 0, while a vertex v is a transitive vertex if it has both positive outdegree and positive indegree such that $(u,w) \in E(D)$ whenever (u,v) and $(v,w) \in E(D)$. An orientation D of a graph G in which every vertex is a transmitter, a receiver, or a transitive vertex is called a divisor orientation of G.

The Length $g(n)$ of a longest path in the divisor graph whose divisor labeling has range $\{1, 2, \ldots, n\}$ was studied in [7], [9], and [10]. The concept of a divisor graph involving finite nonempty sets of integers rather than positive integers was introduced in [11]. It was shown in [11] that odd cycles of length greater than 3 are not divisor graphs, while even cycles and caterpillars are. Indeed, not only caterpillars, but also all bipartite graphs are divisor graphs, as shown in [6]. Since a tree is bipartite, then a tree is a divisor graph. Divisor graphs do not contain induced odd cycles of length greater than 3, but they may contain triangles, see [6]. For instance, the complete graphs are divisor graphs, see [6].

For undefined notions and terminology, the reader is referred to [5]. Also for more details on divisor graphs, the reader is referred to [1], [2], [3], [4], and [6].

2 Preliminaries

The following proposition was shown in [6].

Proposition 1. *Every induced subgraph of a divisor graph is a divisor graph.*

The following theorem was shown in [1], which determines precisely when a power of a path is a divisor graph.

Theorem 1. *For any integer $k \geq 2$, the graph P_n^k is a divisor graph iff $n \leq 2k+2$, where P_n is the path with n vertices.*

Hence, for any integer $k \geq 2$, the graph P_n^k is a divisor graph iff $\mathrm{diam}(P_n) \leq 2k+1$. Any graph G, with $\mathrm{diam}(G) \geq 2k+2$, has an induced subgraph which is isomorphic to P_{2k+3}. Thus, we have the following result, see [1].

Corollary 1. *For any integer $k \geq 2$, if G is a graph of diameter $d \geq 2k+2$, then G^k is not a divisor graph.*

We have the following theorem that characterizes divisor graphs, see [6].

Theorem 2. *Let G be a graph. Then G is a divisor graph if and only if G has a divisor orientation.*

For the centers in a tree T, we have the following theorem, see [5].

Theorem 3. *There are one or two centers in every tree T; in the latter case the centers are adjacent.*

3 When is a Power Graph of a Caterpillar a Divisor Graph

Let T be a caterpillar with diameter d. The vertices of T are:

- Interior vertices: say x_r, for $r = 1, 2, \ldots, d-1$. We have x_r is adjacent to x_{r+1}, where $r = 1, 2, \ldots, d-2$.
- End vertices:
  - If $\deg(x_r) > 2$ and $r \in \{2, 3, \ldots, d-2\}$, then $x_{r,i}$ for $i = 1, 2, \ldots, t_r$ are the end vertices that are adjacent to x_r.
  - If $\deg(x_r) = 2$ and $r \in \{2, 3, \ldots, d-2\}$, then there are no end vertices that are adjacent to x_r. Hence, we may assume $t_r = 0$
  - If $r = 1$ or $d-1$, then $x_{r,i}$ for $i = 1, 2, \ldots, t_r$ are the end vertices that are adjacent to x_r.

Observe that, t_r with $r \in \{1, 2, \ldots, d-1\}$ is the number of end vertices that are adjacent to x_r. So, the number of vertices of T is $n = d - 1 + \sum_{r=1}^{d-1} t_r$. The caterpillar graph is represented in Figure 1.

Now, we characterize which power of a caterpillar T is a divisor graph. We begin with T^2.

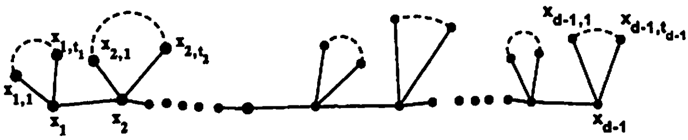

Figure 1: The sketch of a caterpillar T.

Theorem 4. *If T is a caterpillar with* $\text{diam}(T) = 5$, *then T^2 is a divisor graph.*

Proof. Let f be a divisor labeling of T^2 that is defined as follows:

$$\begin{aligned}
f(x_{1,i}) &= p^i \ ; \ 1 \le i \le t_1, \ i \in \mathbb{N}, \ f(x_1) = p^{t_1} q^{1+t_2}, \\
f(x_{2,i}) &= q^{1+i} \ ; \ 1 \le i \le t_2, \ i \in \mathbb{N}, \ f(x_2) = p^{t_1} q^{1+t_2} r^{1+t_3}, \\
f(x_3) &= q, \ f(x_{3,i}) = q r^{1+i} \ ; \ 1 \le i \le t_3, \ i \in \mathbb{N}, \\
f(x_4) &= qr, \ f(x_{4,i}) = qrs^i \ ; \ 1 \le i \le t_4, \ i \in \mathbb{N},
\end{aligned}$$

where, p, q, r, and s are distinct primes. Hence, T^2 is a divisor graph. □

The sketch of the divisor orientation of T^2, according to the divisor labeling f, is represented in Figure 2.

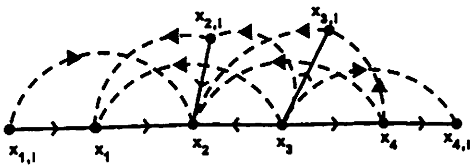

Figure 2: The sketch of the divisor orientation of T^2.

Now, by Proposition 1 and the previous theorem, we get the following corollary.

Corollary 2. *Let T be a caterpillar with* $\text{diam}(T) = 3$ *or* 4. *Then T^2 is a divisor graph.*

Moreover, by the previous theorem and Corollaries 1 and 2, we get the following corollary.

Corollary 3. *Let T be a caterpillar. Then T^2 is a divisor graph if and only if* $\text{diam}(T) \le 5$.

For the square of a tree we have the following theorem.

Theorem 5. *If T is a tree which is not a caterpillar, then T^2 is not a divisor graph.*

Proof. If T is a tree which is not a caterpillar, then T contains an induced subgraph which is isomorphic to T_1. Where T_1 is represented in Figure 3.

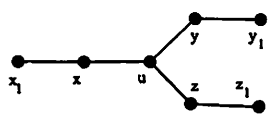

Figure 3: T_1.

Observe that, T_1 is the smallest tree which is not a caterpillar. So, it is enough to show that T_1^2 is not a divisor graph. The graph T_1^2 is represented in Figure 4.

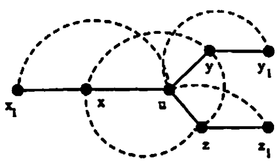

Figure 4: T_1^2.

Now, the set of vertices $\{x_1, y_1, z_1, x, y, z\}$ induces a graph in T_1^2 which is isomorphic to G_1, see Figure 5.

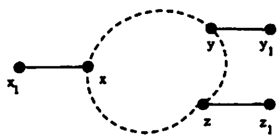

Figure 5: G_1.

It was shown that G_1 is not a divisor graph, see page 193 in [6]. Hence, T_1^2 is not a divisor graph. □

Theorem 5 and Corollary 3 characterize when T^2 is a divisor graph. This gives the following corollary.

Corollary 4. *Suppose T is a tree. Then T^2 is a divisor graph if and only if T is a caterpillar with* $\mathrm{diam}(T) \leq 5$.

The following theorem specifies when T^k is a divisor graph, for a caterpillar T with $\mathrm{diam}(T) \leq 2k-1$ and $k \geq 3$.

Theorem 6. *Suppose T is a caterpillar. If* $\mathrm{diam}(T) \leq 2k-1$*, where k is an integer with $k \geq 3$, then T^k is a divisor graph.*

Proof. Firstly, assume that $\mathrm{diam}(T) = 2k-1$, then to show that T^k is a divisor graph, we rename the vertices of T (we have talked about these vertices in the beginning of this section) as follows:

$z_l \in U = \{x_p : 1 \leq p \leq k-1,\ p \in \mathbb{N}\} \cup (\cup_{1 \leq p \leq k-1}\{x_{p,i};\ 1 \leq i \leq t_p,\ i \in \mathbb{N}\})$ where we set $z_1 = x_{1,1}$ and $z_l = x_{l-1}$ for $2 \leq l \leq k$, $l \in \mathbb{N}$.

$f_m \in W = \{x_{k+q} : 1 \leq q \leq k-2,\ q \in \mathbb{N}\} \cup (\cup_{0 \leq q \leq k-2}\{x_{k+q,i};\ 1 \leq i \leq t_{k+q}, i \in \mathbb{N}\})$, where we set $f_m = x_{k+m}$ for $1 \leq m \leq k-2$, $m \in \mathbb{N}$.

Observe that $V(T) = U \cup W \cup \{x_k\}$. Let D be an orientation of T^k, where $E(D) = A \cup B \cup C$ and A, B, & C are defined as follows:

$$A = \{(z_l, x_k) : 1 \leq l \leq |U|\} \cup \{(z_{l_1}, z_{l_2}) : d_T(z_{l_1}, x_k) > d_T(z_{l_2}, x_k) \text{ and } 1 \leq l_1,\ l_2 \leq |U|\} \cup \{(z_{l_1}, z_{l_2}) : d_T(z_{l_1}, x_k) = d_T(z_{l_2}, x_k) \text{ and } 1 \leq l_1 < l_2 \leq |U|\},$$

$$B = \{(f_{m_1}, f_{m_2}) : d_T(f_{m_1}, x_k) < d_T(f_{m_2}, x_k) \text{ and } 1 \leq m_1,\ m_2 \leq |W|\} \cup \{(f_{m_1}, f_{m_2}) : d_T(f_{m_1}, x_k) = d_T(f_{m_2}, x_k) \text{ and } k-2 < m_1 < m_2 \leq |W|\} \cup \{(f_{m_1}, f_{m_2}) : d_T(f_{m_1}, x_k) = d_T(f_{m_2}, x_k),\ 2 \leq m_2 \leq k-2, \text{ and } k-2 < m_1 \leq |W|\} \cup \{(f_1, f_{m_1}) : d_T(f_{m_1}, x_k) = 1 \text{ and } k-2 < m_1 \leq |W|\},$$

$$C = \{(f_m, x_k) : 1 \leq m \leq |W|\} \cup \{(f_m, z_l) : d_T(f_m, z_l) \leq k,\ 1 \leq m \leq |W|, \text{ and } 1 < l \leq |U|\}.$$

It is enough to show that every vertex of D is a transmitter, a receiver, or a transitive vertex. The sketch of the graph T is represented in Figure 1.

It is clear that x_k is a receiver, $x_{1,1}$ and x_{k+1} are transmitters. We want to show that all other vertices are transitive.
Let $g \in V(D) - \{x_k,\ x_{1,1},\ x_{k+1}\}$ and let agb be a directed path in D. Then, we have two cases to consider.

Case 1: $g \in U \setminus \{z_1\}$.

Since $(g,b) \in E(D)$, we have $b \in (U \setminus \{z_l\}) \cup \{x_k\}$ and $d_T(g,x_k) \geq d_T(b,x_k)$ according to the orientation of T^k which is defined before.

Subcase 1-1: If $a \in U \setminus \{z_k\}$

We obtain $d_T(a,x_k) \geq d_T(g,x_k)$ because of $(a,g) \in E(D)$. And therefore $d_T(a,x_k) \geq d_T(b,x_k)$. We will get $(a,b) \in A \subset E(D)$ by carefully checking.

Subcase 1-2: If $a \in W$

As $(a,g) \in C \subset E(D)$ and $(g,b) \in A \subset E(D)$, we derive $d_T(a,g) \leq k$ and $d_T(g,x_k) \geq d_T(b,x_k)$, respectively. Then from $d_T(a,b) = d_T(a,x_k) + d_T(x_k,b) \leq d_T(a,x_k) + d_T(x_k,g) = d_T(a,g) \leq k$, $(a,b) \in C \subset E(D)$.

Case 2: $g \in W \setminus \{f_1\}$.

Since $(a,g) \in E(D)$, we have $a \in (W)$ and $d_T(a,x_k) \leq d_T(g,x_k)$ according to the orientation of T^k which is defined before.

Subcase 2-1: If $b \in W \setminus \{f_1\}$

We obtain $d_T(b,x_k) \geq d_T(g,x_k)$ because of $(g,b) \in E(D)$. And therefore $d_T(a,x_k) \leq d_T(b,x_k)$. We will get $(a,b) \in B \subset E(D)$ by more carefully checking.

Subcase 2-2: If $b \in (U \setminus \{z_1\}) \cup \{x_k\}$

As $(a,g) \in B \subset E(D)$ and $(g,b) \in C \subset E(D)$, we derive $d_T(a,x_k) \leq d_T(g,x_k)$ and $d_T(g,b) \leq k$, respectively. Then from $d_T(a,b) = d_T(a,x_k) + d_T(x_k,b) \leq d_T(g,x_k) + d_T(x_k,b) = d_T(g,b) \leq k$, $(a,b) \in C \subset E(D)$.

Therefore, g is a transitive vertex in D. The sketch of the direction in D is represented in Figure 6.

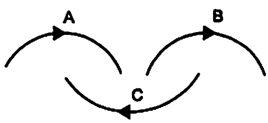

Figure 6: The sketch of the direction in D.

Thus, D is a divisor orientation of T^k. Hence by Theorem 2, T^k is a divisor graph with $\text{diam}(T) = 2k-1$.

Secondly, assume that $\text{diam}(T) < 2k-1$, then by above work and Proposition 1, T^k is a divisor graph. □

Note that, we can prove that T^k with $\text{diam}(T) = 2k - 1$ is a divisor graph (in the previous theorem) by using a divisor labeling approach.

Now, when $\text{diam}(T) = 2k$, we have the following theorem.

Theorem 7. *Suppose T is a caterpillar with* $\text{diam}(T) = 2k$ & $k \geq 3$. *Then T^k is a divisor graph if and only if the center of T has degree two.*

Proof. At first, since $\text{diam}(T) = 2k$ and by Theorem 3, we get x_k is the center of T. Suppose $\deg(x_k) = 2$, then $t_k = 0$. To show that T^k is a divisor graph, one can give an orientation as in the proof of Theorem 6. But, we give a divisor labeling of T^k. Let $s_0 = 0$, $s_{l-1} = 1 + s_{l-2} + t_{k+l-1}$ for $2 \leq l \leq k-1$, $l \in \mathbb{N}$ and $r_m = \sum_{i=1}^{m} t_i$, $m \in \mathbb{N}$. This divisor labeling f is as follows:

$$\begin{cases} f(x_{k+l}) &= p^{1+s_{l-1}} \text{ for } 1 \leq l \leq k-1,\ l \in \mathbb{N}; \\ f(x_{k+l,i}) &= p^{1+s_{l-1}+i} \text{ for } 1 \leq l \leq k-1,\ 1 \leq i \leq t_{k+l},\ l \,\&\, i \in \mathbb{N}; \end{cases}$$

$$\begin{cases} f(x_{1,i}) &= q^i \text{ for } 1 \leq i \leq t_1,\ i \in \mathbb{N}; \\ f(x_{l,i}) &= p^{1+s_{l-2}} q^{r_{l-1}+i} \text{ for } 1 \leq l \leq k-1,\ 1 \leq i \leq t_l,\ l \,\&\, i \in \mathbb{N}; \\ f(x_l) &= p^{1+s_{l-1}} q^{r_l} \text{ for } 1 \leq l \leq k-1,\ l \in \mathbb{N}; \\ f(x_k) &= p^{s_{k-1}} q^{r_{k-1}}, \end{cases}$$

where p and q are distinct primes. This divisor labeling shows that T^k is a divisor graph.

Conversely, assume that $\deg(x_k) > 2$. Then, we have two cases to consider.

- Case 1: If $k = 3$, then T induces a subgraph, say T_2 as in Figure 7.

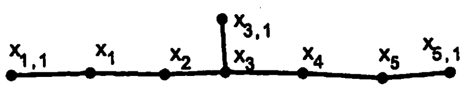

Figure 7: T_2.

Suppose that D is an orientation of T_2^3. We have two subcases to look at.

  - Subcase 1.1: Suppose that $(x_4,\ x_5) \in E(D)$. Since $x_1x_5 \notin E(T_2^3)$, we must have $(x_4,\ x_1) \in E(D)$. We get $(x_{1,1},\ x_1)$, $(x_4,\ x_{5,1}) \in E(D)$, because $x_4x_{1,1}$, $x_1x_{5,1} \notin E(T_2^3)$. We must have $(x_{3,1},\ x_1)$, $(x_4,\ x_2) \in E(D)$, because $x_{3,1}x_{1,1}$, $x_{5,1}x_2 \notin E(T_2^3)$. We have x_5x_1, $x_4x_{1,1} \notin E(T_2^3)$, which implies that $(x_{3,1},\ x_5)$, $(x_{1,1},\ x_2) \in E(D)$. Since $x_{3,1}x_{5,1}$, $x_{1,1}x_5 \notin E(T_2^3)$, we get $(x_{5,1},\ x_5)$, $(x_5,\ x_2) \in E(D)$ and which implies that $(x_{5,1},\ x_2) \in E(D)$. This leads a contradiction, since $x_2x_{5,1} \notin E(T_2^3)$.

  - Subcase 1.2: Suppose that $(x_5,\ x_4) \in E(D)$. Since $x_1x_5 \notin E(T_2^3)$, we must have $(x_1,\ x_4) \in E(D)$. We get $(x_1,\ x_{1,1})$, $(x_{5,1},\ x_4) \in E(D)$, because $x_4x_{1,1},\ x_1x_{5,1} \notin E(T_2^3)$. We must have $(x_1,\ x_{3,1})$, $(x_2,\ x_4) \in E(D)$, because $x_{3,1}x_{1,1},\ x_{5,1}x_2 \notin E(T_2^3)$. We have $x_5x_1,\ x_4x_{1,1} \notin E(T_2^3)$, which implies that $(x_5,\ x_{3,1})$, $(x_2,\ x_{1,1}) \in E(D)$. Since $x_{3,1}x_{5,1},\ x_{1,1}x_5 \notin E(T_2^3)$, we get $(x_5,\ x_{5,1})$, $(x_2,\ x_5) \in E(D)$ and which implies that $(x_2,\ x_{5,1}) \in E(D)$. This leads a contradiction, since $x_2x_{5,1} \notin E(T_2^3)$.

Hence, T_2^3 is not a divisor graph. Thus T^3 is not a divisor graph by Proposition 1.

- Case 2: If $k \geq 4$, then T induces a subgraph, say T_3 as in Figure 8.

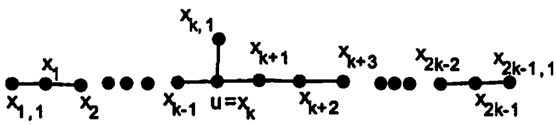

Figure 8: T_3

Then $\{x_{1,1},\ x_1,\ x_2,\ x_{k,1},\ x_{k+2},\ x_{k+3},\ x_{2k-1,1}\}$ induces in T^k a graph which is isomorphic to P_7^2. Thus by Proposition 1 and Theorem 1, T^k is not a divisor graph.

□

The previous theorem can be extended to have $\text{diam}(T) = 2k + 1$. We state that in the following theorem.

Theorem 8. *Suppose that T is a caterpillar with* $\text{diam}(T) = 2k + 1$ *and $k \geq 3$. Then T^k is a divisor graph if and only if the centers of T have degree two.*

Proof. At first, since $\text{diam}(T) = 2k+1$ and by Theorem 3, we get x_k, x_{k+1} are centers of T. suppose that $\deg(x_k) = 2 = \deg(x_{k+1})$, we have $t_k = 0 = t_{k+1}$. To show that T^k is a divisor graph, one can give an orientation as in the proof of Theorem 6. But, we give a divisor labeling of T^k. Let

$s_0 = 0$, $s_1 = 1$, $s_{l-1} = 1 + s_{l-2} + t_{k+l-1}$ for $2 \leq l \leq k$, $l \in \mathbb{N}$ and $r_m = \sum_{i=1}^{m} t_i$, $m \in \mathbb{N}$. This divisor labeling f is as follows:

$$\begin{cases} f(x_{k+l}) & = & p^{1+s_{l-1}} \text{ for } 1 \leq l \leq k,\ l \in \mathbb{N}; \\ f(x_{k+l,i}) & = & p^{1+s_{l-1}+i} \text{ for } 1 \leq l \leq k,\ 1 \leq i \leq t_{k+l},\ l \ \& \ i \in \mathbb{N}; \end{cases}$$

$$\begin{cases} f(x_{1,i}) & = & q^i \text{ for } 1 \leq i \leq t_1,\ i \in \mathbb{N}; \\ f(x_{l,i}) & = & p^{1+s_{l-2}} q^{r_{l-1}+i} \text{ for } 2 \leq l \leq k-1,\ 1 \leq i \leq t_l,\ l \ \& \ i \in \mathbb{N}; \\ f(x_l) & = & p^{1+s_{l-1}} q^{r_l} \text{ for } 1 \leq l \leq k-1,\ l \in \mathbb{N}; \\ f(x_k) & = & p^{s_{k-1}} q^{r_{k-1}}, \end{cases}$$

where p and q are distinct primes. This divisor labeling shows that T^k is a divisor graph.
Conversely, if at least one of the centers of T has degree greater than two, then T either induces a subgraph, say T_4, as in Figure 9 when $k = 3$ or induces a subgraph, say T_5, as in Figure 10 when $k \geq 4$. Using the similar arguments in Theorem 7, we can also show that T_4^3 and T_5^k both are not divisor graphs. Hence, by Proposition 1, T^k is not a divisor graph. □

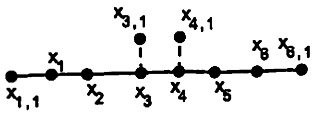

Figure 9: T_4, at least one of $x_3x_{3,1}$ or $x_4x_{4,1}$ is an edge.

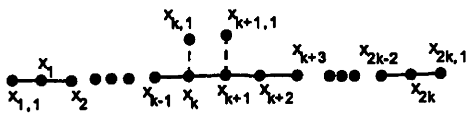

Figure 10: T_5, at least one of $x_kx_{k,1}$ or $x_{k+1}x_{k+1,1}$ is an edge.

Remark 1. *If* $\operatorname{diam}(T) > 2k + 1$ *with* $k \geq 3$, *then use Corollary* 1 *to get* T^k *is not a divisor graph.*

So, by previous remark, Corollary 4, Theorem 6, Theorem 7, and Theorem 8 we get a complete characterization of powers of caterpillars that are divisor graphs.

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