A NOTE ON THE IDENTITIES OF SPECIAL POLYNOMIALS

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ABSTRACT. Recently, Dere and Simsek have treated some applications of umbral algebra related to several special polynomials(see [8]). In this paper, we derive some new and interesting identities of special polynomials involving Bernoulli, Euler and Laguerre polynomials arising from umbral calculus.

1. INTRODUCTION

As is well known, the *Bernoulli polynomials* of order r are defined by the generating function to be

$$(1.1)\qquad \left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x)\frac{t^n}{n!},\ (|t|<2\pi,\ r\in\mathbb{R}),\ (\text{see [1-16]}).$$

In the special case, $x=0$, $B_n^{(r)}(0)=B_n^{(r)}$ are called the n-th Bernoulli numbers of order r. From (1.1), we have

$$(1.2)\qquad B_n^{(r)}(x)=\sum_{l=0}^{n}\binom{n}{l}B_{n-l}^{(r)}x^l,\ (\text{see [17-33]}).$$

The *Euler polynomials* of order r are also defined by

$$(1.3)\qquad \left(\frac{2}{e^t+1}\right)^r e^{xt}=\sum_{n=0}^{\infty}E_n^{(r)}(x)\frac{t^n}{n!},\ (|t|<\pi).$$

Let $x=0$. Then $E_n^{(r)}(0)=E_n^{(r)}$ are called the *n-th Euler numbers* of order r. By (1.3), we easily see that

$$(1.4)\qquad E_n^{(r)}(x)=\sum_{l=0}^{n}\binom{n}{l}E_{n-l}^{(r)}x^l,(\text{ see [14-20]}).$$

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For $\lambda(\neq 1) \in \mathbb{C}$, the *n-th Frobenius-Euler polynomials* of order r are also defined by the generating function to be

(1.5)
$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^r e^{xt} = \sum_{n=0}^{\infty} H_n^{(r)}(x|\lambda)\frac{t^n}{n!}, \quad (|-\log\lambda+t| < 2\pi,\ r \in \mathbb{R}), \quad \text{(see [1,14,19])}.$$

In the special case, $x = 0$, $H_n^{(r)}(0|\lambda) = H_n^{(r)}(\lambda)$ are called the *n-th Frobenius-Euler numbers* of order r.

As is well known, the *Hermite polynomials* are given by the generating function to be

(1.6)
$$e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}, \quad \text{(see [11,12,20])}.$$

In the special case, $x = 0$, $H_n(0) = H_n$ are called the *n-th Hermite numbers.* From (1.6), we have

(1.7)
$$H_n(x) = \sum_{l=0}^{n} \binom{n}{l} 2^l x^l H_{n-l}, \quad \text{(see [20])}.$$

Let $\mathcal{F}$ be the set of all formal power series in the variable t over $\mathbb{C}$ with

(1.8)
$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \ \middle|\ a_k \in \mathbb{C} \right\}.$$

Let $\mathbb{P}$ be the algebra of polynomials in the variable x over $\mathbb{C}$ and $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. $\langle L \mid p(x)\rangle$ denotes the action of a linear functional L on a polynomials $p(x)$ and we remind that the vector space structure on $\mathbb{P}^*$ are derived by

$$\langle L + M|p(x)\rangle = \langle L|p(x)\rangle + \langle M|p(x)\rangle,$$
$$\langle cL|p(x)\rangle = c\,\langle L|p(x)\rangle,$$

where c is a complex constant (see [11, 12, 25]).

The formal power series $f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \in \mathcal{F}$ defines a linear functional on $\mathbb{P}$ by setting

(1.9)
$$\langle f(t)|x^n\rangle = a_n, \quad \text{for all } n \geq 0.$$

By (1.9), we get

(1.10)
$$\langle t^k|x^n\rangle = n!\delta_{n,k}, \quad (n, k \geq 0), \quad \text{(see [8,11,12,25])},$$

where $\delta_{n,k}$ is the Kronecker symbol.

For $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k\rangle}{k!} t^k$, we have $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$. The map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. We shall call $\mathcal{F}$ the *umbral algebra.* The umbral calculus is the study of umbral algebra (see [11, 12, 25]).

The order $o(f(t))$ of the non-zero power series $f(t)$ is the smallest integer k for which the coefficient of t^k does not vanish. A series $f(t)$ having $o(f(t)) = 1$ is called a *delta series* and a series $f(t)$ having $o(f(t)) = 0$ is called an *invertible series*. Let $o(f(t)) = 1$ and $o(g(t)) = 0$. Then there exists a unique sequence $S_n(x)$ of polynomials such that $\langle g(t)f(t)^k | S_n(x) \rangle = n! \delta_{n,k}$ where $n, k \geq 0$. The sequence $S_n(x)$ is called *Sheffer sequence* for $(g(t), f(t))$, which is denoted by $S_n(x) \sim (g(t), f(t))$ (see [12, 25]). By (1.10), we see that $\langle e^{yt} | p(x) \rangle = p(y)$. For $f(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, we have

$$f(t) = \sum_{k=0}^{\infty} \frac{\langle f(t) | x^t \rangle}{k!} t^k, \quad p(x) = \sum_{k=0}^{\infty} \frac{\langle t^k | p(x) \rangle}{k!} x^k, \tag{1.11}$$

and, by (1.11), we get

$$p^{(k)}(0) = \langle t^k | p(x) \rangle, \quad \left\langle 1 \middle| p^{(k)}(x) \right\rangle = p^{(k)}(0). \tag{1.12}$$

Thus, from (1.12), we have

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k}, \quad (k \geq 0), \tag{1.13}$$

and

$$\langle f(t)g(t) | p(x) \rangle = \langle g(t) | f(t)p(x) \rangle = \langle f(t) | g(t)p(x) \rangle. \tag{1.14}$$

Let $S_n(x) \sim (g(t), f(t))$. Then we not that

$$\frac{1}{g(\bar{f}(t))} e^{y\bar{f}(t)} = \sum_{k=0}^{\infty} \frac{S_k(y)}{k!} t^k, \quad (y \in \mathbb{C}), \tag{1.15}$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$.

For $S_n(x) \sim (g(t), f(t))$ and $r_n(x) \sim (h(t), l(t))$, let us assume that

$$S_n(x) = \sum_{k=0}^{n} C_{n,k} r_k(x), \quad \text{(see [11,12,25])}. \tag{1.16}$$

Then, we have

$$C_{n,k} = \frac{1}{k!} \left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))} l(\bar{f}(t))^k \middle| x^n \right\rangle, \quad \text{(see [25])}. \tag{1.17}$$

The equation (1.16) and (1.17) are called the *alternative ways of Sheffer sequences.*

Umbral calculus has numerous applications in such diverse areas as applied mathematics, approximation theory, theoretical physics, combinatorics and statistics. Indeed, in [3] one can find more than five hundred old and new findings related to Sheffer sequences.

In this paper, we derive some new and interesting identities of special polynomials involving higher-order Bernoulli, Euler and Laguerre polynomials arising from umbral calculus.

2. Some identities of several special polynomials

For $\alpha \in \mathbb{R}$, the Laguerre polynomials of order α are given by

$$L_n^{(\alpha)}(x) \sim \left(\left(\frac{1}{1-t} \right)^{\alpha+1}, \frac{t}{t-1} \right), \text{ (see [15,25]).} \tag{2.1}$$

For $\alpha = -1$, let us define Laguerre polynomials as follows:

$$L_n(x) = L_n^{-1}(x) \sim \left(1, \frac{t}{t-1} \right), \text{ (see [15,25]),} \tag{2.2}$$

and

$$(1-t)^{-\alpha-1} L_n^{(\alpha)}(x) \sim \left(1, \frac{t}{t-1} \right). \tag{2.3}$$

Thus, by (2.2) and (2.3), we get

$$L_n^{(\alpha)}(x) = (1-t)^{\alpha+1} L_n(x). \tag{2.4}$$

Now, we suppose that

$$H_n^{(\alpha)}(x|\lambda) = \sum_{k=0}^{n} C_{n,k} L_k(x). \tag{2.5}$$

By (1.5), (1.15), (1.16) and (1.17), we get

(2.6)

$$\begin{aligned} C_{n,k} &= \frac{1}{k!} \left\langle \left(\frac{1-\lambda}{e^t - \lambda} \right)^{\alpha} \left(\frac{t}{t-1} \right)^k \middle| x^n \right\rangle \\ &= \frac{1}{k!} \left\langle \left(\frac{1-\lambda}{e^t - \lambda} \right)^{\alpha} (t-1)^{-k} \middle| t^k x^n \right\rangle \\ &= (-1)^k \binom{n}{k} \left\langle \left(\frac{1-\lambda}{e^t - \lambda} \right)^{\alpha} \middle| \sum_{l=0}^{\infty} \binom{k+l-1}{l} t^l x^{n-k} \right\rangle \\ &= (-1)^k \binom{n}{k} \sum_{l=0}^{n-k} \binom{k+l-1}{l} \frac{(n-k)!}{(n-k-l)!} \left\langle \left(\frac{1-\lambda}{e^t - \lambda} \right)^{\alpha} \middle| x^{n-k-l} \right\rangle \\ &= (-1)^k \binom{n}{k} \sum_{l=0}^{n-k} \binom{k+l-1}{l} \frac{(n-k)!}{(n-k-l)!} H_{n-k-l}^{(\alpha)}(\lambda). \end{aligned}$$

Therefore, by (2.5) and (2.6), we obtain the following theorem.

Theorem 2.1. *For $n \geq 0$, we have*

$$H_n^{(\alpha)}(x|\lambda) = n! \sum_{k=0}^{n} \sum_{l=0}^{n-k} \frac{(-1)^k H_{n-k-l}^{(\alpha)}(\lambda)}{(n-k-l)!k!} \binom{k+l-1}{l} L_k(x).$$

Note that

$$
\begin{aligned}
&(1-t)^{\alpha+1}H_n^{(\alpha)}(x|\lambda)\\
&=\sum_{k=0}^{n}\binom{n}{k}H_{n-k}^{(\alpha)}(\lambda)(1-t)^{\alpha+1}x^k\\
&=\sum_{k=0}^{n}\binom{n}{k}H_{n-k}^{(\alpha)}(\lambda)\sum_{l=0}^{\infty}\binom{\alpha+1}{l}(-1)^l t^l x^k\\
&=\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}H_{n-k}^{(\alpha)}(\lambda)\binom{\alpha+1}{l}(-1)^l\frac{k!}{(k-l)!}x^{k-l}\\
&=n!\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{\alpha+1}{l}(-1)^l H_{n-k}^{(\alpha)}(\lambda)\frac{1}{(n-k)!(k-l)!}x^{k-l}.
\end{aligned}
\tag{2.7}
$$

Therefore, by (2.4), (2.7) and Theorem 2.1, we obtain the following corollary.

Corollary 2.2. *For $n \geq 0$, we have*

$$
\begin{aligned}
&\sum_{k=0}^{n}\sum_{l=0}^{k}\frac{\binom{\alpha+1}{l}(-1)^l H_{n-k}^{(\alpha)}(\lambda)}{(n-k)!(k-l)!}x^{k-l}\\
&=\sum_{k=0}^{n}\sum_{l=0}^{n-k}\frac{(-1)^k H_{n-k-l}^{(\alpha)}(\lambda)}{(n-k-l)!k!}\binom{k+l-1}{l}L_k^{(\alpha)}(x).
\end{aligned}
$$

From (1.6) and (1.15), we have

$$
H_n(x) \sim \left(e^{\frac{1}{4}t^2}, \frac{1}{2}t\right). \tag{2.8}
$$

Let us assume that

$$
H_n(x) = \sum_{k=0}^{n} C_{n,k}L_k(x). \tag{2.9}
$$

By (1.17), (2.2) and (2.8), we get

(2.10)

$$\begin{aligned}
C_{n,k} &= \frac{1}{k!}\left\langle e^{-\frac{1}{4}(2t)^2}\left(\frac{2t}{2t-1}\right)^k \middle| x^n\right\rangle = \frac{1}{k!}\left\langle e^{-\frac{1}{4}t^2}\left(\frac{t}{t-1}\right)^k \middle| (2x)^n\right\rangle \\
&= \frac{2^n}{k!}\left\langle \left(\frac{1}{t-1}\right)^k e^{-\frac{1}{4}t^2} \middle| t^k x^n\right\rangle = 2^n\binom{n}{k}\left\langle \left(\frac{1}{t-1}\right)^k \middle| e^{-\frac{1}{4}t^2}x^{n-k}\right\rangle \\
&= 2^n\binom{n}{k}\sum_{l=0}^{[\frac{n-k}{2}]}\frac{(-1)^l}{4^l l!}\langle (t-1)^{-k}|t^{2l}x^{n-k}\rangle \\
&= 2^n\binom{n}{k}\sum_{l=0}^{[\frac{n-k}{2}]}\frac{(-1)^l(n-k)!}{4^l l!(n-k-2l)!}\langle (t-1)^{-k}|x^{n-k-2l}\rangle \\
&= 2^n\binom{n}{k}\sum_{l=0}^{[\frac{n-k}{2}]}\frac{(-1)^l(n-k)!}{4^l l!(n-k-2l)!}\binom{n-2l-1}{k-1}(-1)^k(n-k-2l)! \\
&= 2^n n!\sum_{l=0}^{[\frac{n-k}{2}]}\frac{(-1)^{l-k}}{2^{2l-n}l!k!}\binom{n-2l-1}{k-1}.
\end{aligned}$$

Therefore, by (2.9) and (2.10), we obtain the following theorem.

Theorem 2.3. *For $n \geq 0$, we have*

$$H_n(x) = n!\sum_{k=0}^{n}\sum_{l=0}^{[\frac{n-k}{2}]}\frac{(-1)^{l-k}}{2^{2l-n}l!k!}\binom{n-2l-1}{k-1}L_k(x).$$

From (1.7), we have

(2.11)

$$\begin{aligned}
(1-t)^{\alpha+1}H_n(x) &= \sum_{k=0}^{n}\binom{n}{k}2^k H_{n-k}(1-t)^{\alpha+1}x^k \\
&= \sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}\binom{\alpha+1}{l}2^k H_{n-k}(-1)^l t^l x^k \\
&= \sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}\binom{\alpha+1}{l}2^k H_{n-k}(-1)^l\frac{k!}{(k-l)!}x^{k-l} \\
&= n!\sum_{k=0}^{n}\sum_{l=0}^{k}\frac{2^k H_{n-k}(-1)^l}{(n-k)!(k-l)!}\binom{\alpha+1}{l}x^{k-l}.
\end{aligned}$$

Therefore, by (2.4) and Theorem 2.3 and (2.11), we obtain the following corollary.

Corollary 2.4. *For $n \geq 0$, we have*

$$\sum_{k=0}^{n}\sum_{l=0}^{k}\frac{2^k H_{n-k}(-1)^l}{(n-k)!(k-l)!}\binom{\alpha+1}{l}x^{k-l}$$
$$=\sum_{k=0}^{n}\sum_{l=0}^{[\frac{n-k}{2}]}\frac{(-1)^{l-k}}{2^{2l-n}l!k!}\binom{n-2l-1}{k-1}L_k^{(\alpha)}(x).$$

Let us assume that

$$B_n^{(\alpha)}(x)=\sum_{k=0}^{n}C_{n,k}L_k(x). \tag{2.12}$$

Them, from (1.1), (1.15), (1.17) and (2.2), we have

(2.13)

$$\begin{aligned}C_{n,k}&=\frac{1}{k!}\left\langle\left(\frac{t}{e^t-1}\right)^{\alpha}\left(\frac{t}{t-1}\right)^{k}\middle| x^n\right\rangle\\
&=\frac{1}{k!}\left\langle\left(\frac{t}{e^t-1}\right)^{\alpha}\left(\frac{1}{t-1}\right)^{k}\middle| t^k x^n\right\rangle\\
&=\binom{n}{k}\left\langle\left(\frac{t}{e^t-1}\right)^{\alpha}(t-1)^{-k}\middle| x^{n-k}\right\rangle\\
&=\binom{n}{k}\sum_{l=0}^{n-k}(-1)^k\binom{k+l-1}{l}\frac{(n-k)!}{(n-k-l)!}\left\langle\left(\frac{t}{e^t-1}\right)^{\alpha}\middle| x^{n-k-l}\right\rangle\\
&=\binom{n}{k}\sum_{l=0}^{n-k}\binom{k+l-1}{l}(-1)^k\frac{(n-k)!}{(n-k-l)!}B_{n-k-l}^{(\alpha)}.\end{aligned}$$

Therefore, by (2.12) and (2.13), we obtain the following theorem.

Theorem 2.5. *For $n \geq 0$, we have*

$$B_n^{(\alpha)}(x)=n!\sum_{k=0}^{n}\sum_{l=0}^{n-k}\frac{(-1)^k B_{n-k-l}^{(\alpha)}}{k!(n-k-l)!}\binom{k+l-1}{l}L_k(x).$$

By (1.2), we easily see that

$$\begin{aligned}(1-t)^{\alpha+1}B_n^{(\alpha)}(x)&=\sum_{k=0}^{n}\binom{n}{k}B_{n-k}^{(\alpha)}(1-t)^{\alpha+1}x^k\\
&=\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}B_{n-k}^{(\alpha)}\binom{\alpha+1}{l}(-1)^l\frac{k!}{(k-l)!}x^{k-l}\\
&=n!\sum_{k=0}^{n}\sum_{l=0}^{k}\frac{(-1)^l B_{n-k}^{(\alpha)}}{(n-k)!(k-l)!}\binom{\alpha+1}{l}x^{k-l}.\end{aligned} \tag{2.14}$$

Therefore, by (2.14) and Theorem 2.5, we obtain the following corollary.

Corollary 2.6. *For $n \geq 0$, we have*

$$\sum_{k=0}^{n}\sum_{l=0}^{k} \frac{(-1)^l B_{n-k}^{(\alpha)}}{(n-k)!(k-l)!}\binom{\alpha+1}{l} x^{k-l}$$
$$= \sum_{k=0}^{n}\sum_{l=0}^{n-k} \frac{(-1)^k B_{n-k-l}^{(\alpha)}}{k!(n-k-l)!}\binom{k+l-1}{l} L_k^{(\alpha)}(x).$$

Let us consider the following equation:

$$E_n^{(\alpha)}(x) = \sum_{k=0}^{n} C_{n,k} L_k(x), \text{ where } n \geq 0. \tag{2.15}$$

From (1.3), (1.15), (1.17) and (2.2), we have

$$\begin{aligned} C_{n,k} &= \frac{1}{k!}\left\langle \left(\frac{2}{e^t+1}\right)^\alpha \left(\frac{t}{t-1}\right)^k \middle| x^n \right\rangle \\ &= \binom{n}{k}\sum_{l=0}^{n-k}(-1)^k \binom{k+l-1}{l}\frac{(n-k)!}{(n-k-l)!}E_{n-k-l}^{(\alpha)}. \end{aligned} \tag{2.16}$$

Thus, by (2.15) and (2.16), we get

$$E_n^{(\alpha)}(x) = n! \sum_{k=0}^{n}\sum_{l=0}^{n-k} \frac{(-1)^k}{k!(n-k-l)!}\binom{k+l-1}{l} E_{n-k-l}^{(\alpha)} L_k(x). \tag{2.17}$$

By the same method of (2.14), we get

$$(1-t)^{\alpha+1} E_n^{(\alpha)}(x) = n! \sum_{k=0}^{n}\sum_{l=0}^{k} \frac{(-1)^l E_{n-k}^{(\alpha)}}{(n-k)!(k-l)!}\binom{\alpha+1}{l} x^{k-l}. \tag{2.18}$$

Thus, by (2.4), (2.17) and (2.18), we get

$$\sum_{k=0}^{n}\sum_{l=0}^{k} \frac{(-1)^k E_{n-k}^{(\alpha)}}{(n-k)!(k-l)!}\binom{\alpha+1}{l} x^{k-l}$$
$$= \sum_{k=0}^{n}\sum_{l=0}^{n-k} \frac{(-1)^k}{k!(n-k-l)!}\binom{k+l-1}{l} E_{n-k-l}^{(\alpha)} L_k^{(\alpha)}(x).$$

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REFERENCES

[1] S. Araci and M. Acikgoz, *A note on the Frobenius-Euler numbers and polynomials associated with Bernstein polynomials*, Adv. Styd. Contemp. Math. **22** (2012) 399-406.

[2] A. Bayad and T. Kim, *Identities involving values of Bernstein, q-Bernoulli, and q-Euler polynomials*, Russ. J. Math. Phys. 2010, **18** (2012), no. 2, 133-143.

[3] Di Bucchianico, A. and Loeb, D., *A selected survey of umbral calculus*, Electronic J. Combina. 2 (2000) #DS3.

[4] L. Carlitz, *Extended Bernoulli and Eulerian numbers*, Duke Math. J. **31** (1964), 667-689.

[5] M. Can, M. Cenkci, V. Kurt and Y. Simsek, *Twisted Dedekind type sums associated with Barnes' type multiple Frobenius-Eulerl-functions*, Adv. Stud. Contemp. Math. **18** (2009), no. 2, 135-160.

[6] J. Choi, D. S. Kim, T. Kim and Y. H. Kim, *Some arithmetic identities on Bernoulli and Euler numbers arising from the p-adic integrals on* $\mathbb{Z}_p$, Adv. Stud. Contemp. Math. **22** (2012) 239-247.

[7] D. Ding and J. Yang, *Some identities related to the Apostol-Euler and Apostol-Bernoulli polynomials*, Adv. Stud. Contemp. Math. **20** (2010) 7-21.

[8] R. Dere and Y. Simsek, *Applications of umbral algebra to some special polynomials*, Adv. Stud. Contemp. Math. **22** (2012), no. 1, 433-438.

[9] G. Kim, B. Kim and J. Choi, *The DC algorithm for computing sums of powers of consecutive integers and Bernoulli numbers*, Adv. Stud. Contemp. Math. **17** (2008), no. 1, 137-145.

[10] D. S. Kim, N. Lee, J. Na and K. H. Park, *Identities of symmetry for higher-order Euler polynomials in three variables (I)*, Adv. Stud. Contemp. Math. **22** (2012), 51-74.

[11] D. S. Kim and T. Kim, *Some identities of Frobenius-Euler polynomials arising from umbral calculus*, Adv. Difference Eq. 2012, **2012**:196.

[12] D. S. Kim and T. Kim, *Some new identities of Frobenius-Euler numbers and polynomials*, , Journal of Inequalities and Applications 2012, **2012**:307.

[13] D. S. Kim, T. Kim, S. H. Lee, *Poly-Cauchy numvbers and polynomials with umbral calculus viewpoint*, Int. Journal of Math. Analysis, Vol. 7, 2013.

[14] T. Kim, *Some identities for the Bernoulli, the Euler and the Genocchi numbers and polynomials*, Adv. Stud. Contemp. Math. **20** (2010), 23-28.

[15] T. Kim, S.-H. Rim, D. V. Dolgy and S.-H. Lee, *Some identities on Bernoulli and Euler polynomials arising from the orthogonality of Laguerre polynomials*, Adv. Difference Eq. 2012, **2012**:201.

[16] T. Kim, *Symmetry of power sum polynomials and multivariate fermionic p-adic invariant integral on* $\mathbb{Z}_p$, Russ. J. Math. Phys. **16** (2009), 93-96.

[17] T. Kim, *Some identities on the q-Euler polynomials of higher order and q-Stirling numbers by the fermionic p-adic integral on* $\mathbb{Z}_p$, Russ. J. Math. Phys. **16** (2009), 484-491.

[18] T. Kim and J. Choi, *A note on the product of Frobenius-Euler polynomials arising from the p-adic integral on* $\mathbb{Z}_p$, Adv. Stud. Contemp. Math. **22** (2012), 215-223.

[19] T. Kim, *An identity of the symmetry for the Frobenius-Euler polynomials associated with the fermionic p-adic invariant q-integrals on* $\mathbb{Z}_p$, Rocky Mountain J. Math. **41** (2011), 239-247.

[20] T. Kim, J. Choi and Y. H. Kim,, *On q-Bernstein and q-Hermite polynomials*, Proc. Jangjeon Math. Soc. **14** (2011), 215-221.

[21] Y.-H. Kim and K.-W. Hwang, *Symmetry of power sum and twisted Bernoulli polynomials*, Adv. Stud. Contemp. Math. **18** (2009), 127-133.
[22] A. Kudo, *A congruence of generalized Bernoulli number for the character of the first kind*, Adv. Stud. Contemp. Math. **2** (2000), 1-8.
[23] H. Ozden, I. N. Cangul and Y. Simsek, *Remarks on q-Bernoulli numbers associated with Daehee numbers*, Adv. Stud. Contemp. Math. **18** (2009), 41-48.
[24] S.-H. Rim and J. Jeong, *On the modified q-Euler numbers of higher order with weight*, Adv. Stud. Contemp. Math. **22** (2012), 93-98.
[25] S. Roman, *The umbral calculus*, Dover Publ. Inc. New York, 2005.
[26] S. Roman, G.-C. Rota, *The umbral calculus*, Advances in Math. **27** (1978), no. 2, 95–188.
[27] C. S. Ryoo and T. Kim, *A new identities on the q-Bernoulli numbers and polynomials*, Adv. Stud. Contemp. Math. **21** (2011), 161-169.
[28] C. S. Ryoo, *Some relations between twisted q-Euler numbers and Bernstein polynomials*, Adv. Stud. Contemp. Math. **21** (2011), 217-223.
[29] C. S. Ryoo, H. Song and R. P. Agarwal, *On the roots of the q-analogue of Euler-Barnes' polynomials*, Adv. Stud. Contemp. Math. **9** (2004), no. 2, 153-163.
[30] K. Shiratani and S. Yamamoto, *n a p-adic interpolation function for the Euler numbers and its derivatives*, Mem. Fac. Sci. Kyushu Univ. Ser. A. **9** (2004), 113-125.
[31] K. Shiratani and S. Yamamoto, *An application of p-adic convolutions*, Mem. Fac. Sci. Kyushu Univ. Ser. A. **36** (1982), 73-83.
[32] Y. Simsek, *Generating functions of the twisted Bernoulli numbers and polynomials associated with their interpolation functions*, Adv. Stud. Contemp. Math. **16** (2008), no. 2, 251-278.
[33] Y. Simsek, *Theorems on twisted L-function and twisted Bernoulli numbers*, Adv. Stud. Contemp. Math. **11** (2005), 205-218

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