The Crossing Numbers of Join of Path and Cycle with a Special Graph of Order Six *

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Abstract: Only few results concerning crossing numbers of join of some graphs are known. In the paper, for the special graph G on six vertices, we give the crossing numbers of $G \vee P_n$ and $G \vee C_n$, P_n and C_n are the path and cycle on n vertices, respectively.

Keywords: Graph, Drawing, Crossing number, Join product.

1 Introduction

Let G be a graph, whose vertex set and edge set are denoted by $V(G)$ and $E(G)$, respectively. A drawing of G is a representation of G in the plane such that its vertices are represented by distinct points and its edges by simple continuous arcs connecting the corresponding point pairs. We further require that no edge-arc passes through a point which represents a vertex. The *crossing number* $cr(G)$ of a graph G is the minimum number

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of edge crossings in any drawing. It is easy to see that a drawing with minimum number of crossings (an *optimal* drawing) is always a *good* drawing, meaning that no edge crosses itself, no two edges cross more than once, and no two edges incident with the same vertex cross.

The investigation on the crossing number of graphs is a classical and however very difficult problem. Garey and Johnson [1] proved that this problem is NP-complete. The exact values of crossing mumbers are known only for few specific families of graphs. The join product of two graphs G_1 and G_2, denoted by $G_1 \vee G_2$, is obtained from vertex-disjoint copies of G_1 and G_2 by adding all edges between $V(G_1)$ and $V(G_2)$. For $|V(G_1)| = m$, and $|V(G_2)| = n$, the edge set of $G_1 \vee G_2$ is the union of disjoint edge sets of the graph G_1, G_2 and the complete bipartite graph $K_{m,n}$. Kulli and Muddebihal [7] gave the characterization of all pairs of graphs which join is planar graph. Let P_n and C_n be the path and the cycle on n vertices, respectively. Using Kleitman's result [2], the crossing numbers for join of two paths, join of two cycles, and for join of path and cycle were studied in [3]. Moreover, the exact values for crossing numbers of $G \vee P_n$ and $G \vee C_n$ for all graphs G of order at most four are given. The crossing numbers of the graphs $G \vee P_n$ and $G \vee C_n$ are also known for very few graphs G of order five and six, see [4]-[6].

Let D be a good drawing of the graph G. We denote the number of crossings in D by $cr_D(G)$. Let G_i and G_j be edge-disjoint subgraphs of G. We denote by $cr_D(G_i, G_j)$ the number of crossings between the edges of G_i and the edges of G_j, and by $cr_D(G_i)$ the number of crossings among the edges of G_i in D. It is easy to see that for three edge-disjoint graphs G_i, G_j and G_k, the following equations hold:

$$\begin{aligned} cr_D(G_i \cup G_j) &= cr_D(G_i) + cr_D(G_j) + cr_D(G_i, G_j), \\ cr_D(G_i \cup G_j, G_k) &= cr_D(G_i, G_k) + cr_D(G_j, G_k). \end{aligned} \tag{1}$$

Let D be a good drawing of the graph G and G_i be a subgraph of the graph G, we denote the subdrawing of the subgraph G_i induced from D by $D(G_i)$.

In this paper, some proofs are based on Kleitman's result on crossing numbers of complete bipartite graphs. More precisely, he proved that

$$cr(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor, \text{ if } \min\{m, n\} \leq 6. \tag{2}$$

For convenience, the number $\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ is often denoted by $Z(m, n)$. In the proofs of the paper, we will often use the term "region" also in non-planar drawings. In this case, crossings are considered to be the vertices of the "map".

2 The graph $G \vee P_n$

The graph G in Fig.1 consists of one 6-cycle, denoted by $C_6(G)$ in the paper, and two edges which together with the edges of the 6-cycle form two 3-cycle $C_3(abf)$ and $C_3(def)$ and one 4-cycle $C_4(bcdf)$. The graph $G \vee nK_1$ consists of one copy of the graph G and n vertices $t_1, t_2, \cdots, t_n$, where every vertex t_i, $i = 1, 2, \cdots, n$, is adjacent to every vertex of G. For $i = 1, 2, \cdots, n$, let T^i denote the subgraph induced by six edges incident with the vertex t_i and let $F^i = G \cup T^i$. In the paper let G_n denote the graph $G \vee nK_1$.

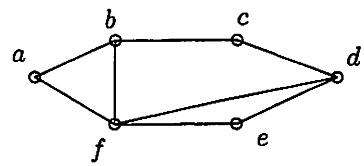

Fig.1. The graph G on six vertices.

In Fig.2 it is easily see that

$$G_n = G \vee nK_1 = G \cup K_{6,n} = G \cup \Big(\bigcup_{i=1}^{n} T^i\Big). \tag{3}$$

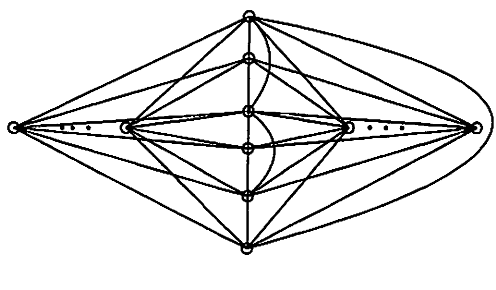

Fig.2. A good drawing D of the graph $G \vee nK_1$.

Lemma 2.1. *Let D be a good drawing of the graph $G \vee nK_1$ $(n \geq 3)$, for which every subdrawing $D(G \vee (n-2)K_1)$ has at least $6\lfloor \frac{n-2}{2} \rfloor \lfloor \frac{n-3}{2} \rfloor + 2\lfloor \frac{n-2}{2} \rfloor$ crossings. If for two different $i, j \in \{1, 2, \cdots, n\}$, $cr_D(T^i, T^j) = p$, $cr_D(G, T^i \cup T^j) = q$ and for every k, $k \neq i, j$, $cr_D(T^i \cup T^j, T^k) \geq 6$, then there are at least $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor + p + q - 2$ crossings in D.*

proof. Without loss of generality, let $cr_D(T^{n-1}, T^n) = p$. Since $G_n = G \vee nK_1 = G_{n-2} \cup (T^{n-1} \cup T^n)$ and $G_{n-2} = K_{6,n-2} \cup G$, using (1) and (3) we have

$$\begin{aligned} cr_D(G_n) &= cr_D(G_{n-2}) + cr_D(T^{n-1} \cup T^n) + cr_D(K_{6,n-2}, T^{n-1} \cup T^n) \\ &\quad + cr_D(G, T^{n-1} \cup T^n) \\ &\geq Z(6, n-2) + 2\left\lfloor \frac{n-2}{2} \right\rfloor + p + 6(n-2) + q \\ &= Z(6,n) + 2\left\lfloor \frac{n}{2} \right\rfloor + p + q - 2. \end{aligned}$$

□

Lemma 2.2. *Let D be a good drawing of the graph $G \vee nK_1$ in which for some i, $i \in \{1, 2, \cdots, n\}$, and for all $j = 1, 2, \cdots, n$, $j \neq i$, $cr_D(G \cup T^i, T^j) \geq 4$. If $cr_D(G \cup T^i, T^j) > 4$ for k different subgraphs T^j, then D has at least $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor + k$ crossings.*

proof. Assume, without loss of generality, that the edges of $F^n = G \cup T^n$ are crossed in D at least four times by the edges of every subgraph T^i, $i = 1, 2, \cdots, n-1$, and that k of the subgraphs T^i cross the edges of F^n more than four times. As $G_n = K_{6,n-1} \cup F^n$, we have

$$\begin{aligned} cr_D(G_n) &= cr_D(K_{6,n-1}) + cr_D(F^n) + cr_D(K_{6,n-1}, F^n) \\ &\geq 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 4(n-1) + k \geq Z(6,n) + 2\left\lfloor \frac{n}{2} \right\rfloor + k. \end{aligned}$$

□

In this section, we will use the next result which was proved in [9].

Theorem 2.3 ([9]). $cr(G \vee nK_1) = Z(6,n) + 2\lfloor \frac{n}{2} \rfloor$ *for* $n \geq 1$.

The graph $G \vee P_n$ contains $G \vee nK_1$ as a graph. For the subgraphs of the graph $G \vee P_n$ which are also subgraphs of the graph $G \vee nK_1$ we will use the same notation as above. Let P_n^* denote the path on n vertices of $G \vee P_n$ not belonging to the subgraph G. One can easily see that

$$G \vee P_n = G \cup K_{6,n} \cup P_n^* = G \cup \Big(\bigcup_{i=1}^{n} T^i\Big) \cup P_n^*.$$

It is easy to verify that for $n = 1$ the graph $G \vee P_1$ is planar. For $n \geq 2$, we have the next result.

Theorem 2.4. $cr(G \vee P_n) = 6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 2\lfloor\frac{n}{2}\rfloor + 1$ *for* $n \geq 2$.

proof. Fig.2 shows the drawing of the graph $G \vee nK_1$ with $Z(6,n) + \lfloor\frac{n}{2}\rfloor$ crossing. One can easily see that in this drawing it is possible to add $n-1$ edges which form the path P_n^* on the vertices of nK_1 in such a way that only one edge of P_n^* is crossed by an edge of G. Hence, $cr(G \vee P_n) \leq Z(6,n) + 2\lfloor\frac{n}{2}\rfloor + 1$.

To prove the reverse inequality we assume that there is a drawing of the graph $G \vee P_n$ with fewer than $Z(6,n) + 2\lfloor\frac{n}{2}\rfloor + 1$ crossings and let D be a such drawing. As the graph $G \vee P_n$ contains $G \vee nK_1$ as a subgraph, by Theorem 2.3, $cr(G \vee P_n) \geq Z(6,n) + 2\lfloor\frac{n}{2}\rfloor$ and therefore, no edge of the path P_n^* is crossed in D.

Claim. $cr(G \vee P_2) = 3$.

Proof: Assume r be the smallest nonnegative integer such that the removal of some r edges from the graph $G \vee P_2$ results in a planar subgraph $(G \vee P_2)_r$ of $G \vee P_2$. The graph $(G \vee P_2)_r$ is a connected spanning subgraph of the graph $G \vee P_2$ with eight vertices and $21-r$ edges. By Euler's formula, in any planar drawing of $(G \vee P_2)_r$, there are $15-r$ regions, Since $(G \vee P_2)_r$ has girth at least three, $3(15-r) \leq 2(21-r)$. So, $r \geq 3$ and $cr(G \vee P_2) \geq 3$. This contradiction proves the claim.

So, assume $n \geq 3$. For the drawing D we consider two possible cases.

Case 1. There is a subgraph T^i satisfying $cr_D(G, T^i) = 0$. Without loss of generality, let $cr_D(G, T^n) = 0$.

Regardless of wheather or not the edges of G cross each other, in the subdrawing $D(G)$ there is a region that six vertices of the subgraph G are on its boundary, say unbounded region and , in D, all vertices t_i, $i = 1, 2, \cdots, n$, are placed in this region. Since in the subdrawing $D(F^n)$ of $F^n = G \cup T^n$ there are exactly two vertices of G on the boundary of one region outside G, the edges of every subgraph T^i, $1 \leq i \leq n-1$, cross in D the edges of F^n at least four times. If there exists a T^i satisfying $cr_D(T^i, F^n) > 4$, then, by Lemma 2.2, in D there are at least $Z(6,n) + 2\lfloor\frac{n}{2}\rfloor + 1$ crossings. So all T^i, $i = 1, 2, \cdots, n-1$ satisfy $cr_D(T^i, F^n) = 4$ and in the same time one can easily see that $cr_D(T^i, G) \geq 2$. So the edges of G are crossed at least $2(n-1) > 2\lfloor \frac{n}{2} \rfloor$ times. On the other hand, as $G \vee P_n = G \cup K_{6,n} \cup P_n^*$ and as $cr(K_{6,n}) = Z(6,n)$, in D there are at most $2\lfloor \frac{n}{2} \rfloor$ crossings on the edges of G. This proves that the case 1 is impossible in the considered drawing D.

Case 2. Every subgraph T^i crosses G in D.

The case 2 is impossible for odd n, otherwise G has more than $\lfloor \frac{n}{2} \rfloor$ crossings. For even n, $cr_D(G, T^i) = 1$ for all $i = 1, 2, \cdots, n$, and no other crossings appear on the edges of G. Hence, $cr_D(G) = 0$. Up to the isomorphism, there are only two possible subdrawings of G induced by D. One of them is in Fig.1 and the other one we can obtain from the drawing in Fig.1 by replacing the vertex e into the 4-cycle $C_4(bcdf)$. In the second case, there are at most five vertices of G on the boundary of one region and, for fixed i, $i \in \{1, 2, \cdots, n\}$, T^i crosses G exactly once. As, in the view of the subdrawing $D(G)$, all vertices t_j, $j \neq i$, are placed in the same region as the vertex t_i, $cr_D(T^i, T^j) \geq 4$ and $cr_D(F^i, T^j) \geq 5$. This, together with Lemma 2.2, contradicts the assumption that D has at most $Z(6,n) + 2\lfloor \frac{n}{2} \rfloor$ crossings. So, the only possible subdrawing of G induced by D is shown in Fig.1 and, in D, all vertices t_i are placed in the region with six vertices of G on its boundary. If for all $i, j = 1, 2, \cdots, n$, $i \neq j$, $cr_D(T^i, T^j) \geq 3$, then in D there are at least $3\binom{n}{2} > Z(6,n) + 2\lfloor \frac{n}{2} \rfloor$ crossings. Thus, in D there are at least two subgraphs T^i and T^j for which $cr_D(T^i, T^j) \leq 2$.

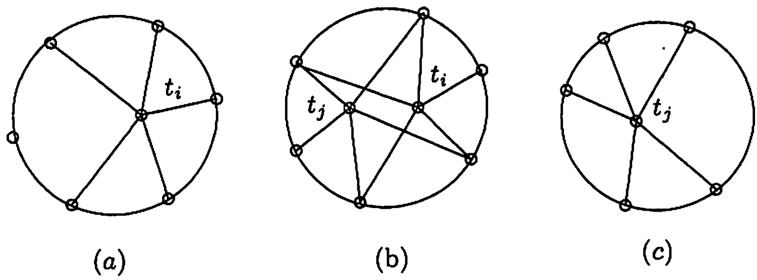

Fig.3. The possible placement of T^i and T^j inside $D(G)$.

Assume that the region with all six vertices of G is in $D(G)$ inside the cycle $C_6(G)$ and consider the vertex t_i placed in this region. The edges of T^i which do not cross G divides this region as shown in Fig.3(a). The vertex t_j for which $cr_D(T^i, T^j) \leq 2$ must be placed in D inside the region of $D(G \cup T^i)$ with three vertices of G on its boundary and the edges of T^j cross two times the edges of T^i as shown in Fig.3(b). Suppose that in D there is a subgraph T^k, $k \neq i, j$, for which $cr_D(T^i, T^k) < 3$. Then the vertex t_k is placed in the same region of $D(G \cup T^i)$ as the vertex t_j and the edge of T^k which crosses G is incident with the same vertex of G as the corresponding edge of T^j which also crosses G. So, $cr_D(T^i, T^k) = 2$. Fig.3(c) shows the region inside $C_6(H)$ divided by the edges of T^j which do not cross G. It is easy to verify that the edges of T^k incident with the same vertices(the edges which do not cross G) cross the edges of T^j at least four times. Hence, $cr_D(T^i, T^j) = 2$, $cr_D(G, T^i \cup T^j) = 2$ and $cr_D(T^i \cup T^j, T^k) \geq 6$. This, together with Lemma 2.1, contradicts the assumption that D has at most $Z(6,n) + 2\lfloor \frac{n}{2} \rfloor$ crossings. Up to the symmetry, the same contradiction is obtained if $cr_D(T^j, T^k) < 3$. If for every T^k, $k \neq i, j$, $cr_D(T^i, T^k) \geq 3$ and $cr_D(T^j, T^k) \geq 3$, we have again $cr_D(T^i, T^j) = 2$, $cr_D(G, T^i \cup T^j) = 2$ and $cr_D(T^i \cup T^j, T^k) \geq 6$ which, together with Lemma 2.1, contradicts the assumption. This completes the proof. □

3 The graph $G \vee C_n$

The graph $G \vee C_n$ consists of the graph $G \vee nK_1$ and n edges which induce the cycle $C_n = t_1 t_2 \cdots t_n t_1$. One can easy to see that $G \vee C_n$ contains $6K_1 \vee C_n$ as a subgraph. For $x = a, b, c, d, e, f$, let T^x denote the subgraph of $6K_1 \vee C_n$ induced on the edges incident with the vertex x and let C_n^* denote the cycle on n vertices of $G \vee C_n$ not belonging to the subgraph G. Thus,

$$G \vee C_n = G \cup K_{6,n} \cup C_n^* = G \cup \Big(\bigcup_{i=1}^{n} T^i\Big) \cup C_n^* = G \cup \Big(\bigcup_{x=a}^{f} T^x\Big) \cup C_n^*. \quad (4)$$

The main result of this section is based on the next lemma which was proved in [3] and also used in [6].

Lemma 3.1 ([3]). *Let D be a good drawing of the graph $mK_1 \vee C_n$, $m \geq 2$, $n \geq 3$, in which no edges of C_n^* is crossed, and C_n^* does not separate the* *other vertices of the graph. Then, for all $i, j = 1, 2, \cdots, m$, two different subgraphs T^i and T^j cross each other at least $\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ times in D.*

Assume now the edges of C_n^* can cross each other. Then, in the view of the subdrawing of the cycle C_n^*, there is only one region with all n vertices of C_n^* on its boundary. If, in this case, some subgraph T^i does not cross C_n^*, then it is placed in D in the considered region and the next corollary is obvious.

Corollary 3.2 ([6]). *Let D be a good drawing of the graph $mK_1 \vee C_n$, $m \geq 2$, $n \geq 3$, in which none of r subgraphs $T^{i_1}, T^{i_2}, \cdots, T^{i_r}$, $2 \leq r \leq m$, crosses the edges of C_n^*. Then, for all $j, k = 1, 2, \cdots, r$, two different subgraphs T^{i_j} and T^{i_k} cross each other at least $\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ times in D.*

Theorem 3.3. *$cr(G \vee C_n) = 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor + 3$ for $n \geq 3$.*

proof. In the drawing in Fig.2 it is possible to add n edges in such a way that they, together with the vertices $t_1, t_2, \cdots, t_n$ form the cycle C_n^* and that the edges of C_n^* were crossed only three times. Hence, $cr(G \vee C_n) \leq 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor + 3$. To prove the reverse inequality assume that there is a good drawing D of the graph $G \vee C_n$ with at most

$$6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor + 2 \tag{5}$$

crossings. Since $G \vee C_n = (G \vee nK_1) \cup C_n^*$, we have

$$\begin{aligned} cr_D(G \vee C_n) &= cr_D(G \vee nK_1) + cr_D(C_n^*) + cr_D(G \vee nK_1, C_n^*) \\ &\geq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor + cr_D(C_n^*) + cr_D(G \vee nK_1, C_n^*). \end{aligned} \tag{6}$$

Using (5) and (6), we have

$$cr_D(C_n^*) + cr_D(G \vee nK_1, C_n^*) \leq 2.$$

Moreover, by Theorem 2.4, no edge of C_n^* is crossed more than once, because otherwise deleting the crossed edge from C_n^* results in a drawing of the subgraph $G \vee P_n$ with fewer than $Z(6, n) + 2\lfloor \frac{n}{2} \rfloor + 1$ crossings.

Case 1. $cr_D(C_n^*) = 2$ and $cr_D(G \vee nK_1, C_n^*) = 0$.

In this case, none of the subgraphs T^x, $x \in \{a,b,c,d,e,f\}$, crosses the cycle C_n^*. So, in the view of the subdrawing $D(C_n^*)$, the cycle C_n^* does not separate the other vertices of the graph and, in $D(C_n^*)$, there exists only one region with all n vertices of C_n^* on its boundary. Thus, by Corollary 3.2, $cr_D(T^x, T^y) \geq \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ for every $x, y \in \{a,b,c,d,e,f\}$, $x \neq y$. So, in this case, the edges of $G \vee C_n$ cross each other at least $\binom{6}{2} \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor > Z(6,n) + 2\lfloor \frac{n}{2} \rfloor + 2$ times. This contradicts the assumption.

Case 2. $cr_D(C_n^*) = 1$ and $cr_D(G \vee nK_1, C_n^*) \leq 1$.

As $cr_D(C_n^*) = 1$, in the view of the subdrawing $D(C_n^*)$, the cycle C_n^* divided the plane into three regions which satisty (a) the boundary of every region contains at least two vertices of the cycle C_n^*, (b) there is a region denoted by δ that its boundary contains all vertices of the cycle C_n^*. Assume there exists one vertex x, $x \in \{a,b,c,d,e,f\}$, which is not placed in the region δ. One can easily verify that $cr_D(T^x, C_n^*) \geq 2$ by the above property (a). This contradicts $cr_D(G \vee nK_1, C_n^*) \leq 1$. So all vertices of the graph G must be placed in the region δ. At the same time, if the edges of the subgraph G cross the edges of C_n^*, then $cr_D(G, C_n^*) \geq 2$, a contradiction with $cr_D(G \vee nK_1, C_n^*) \leq 1$. So, the necessary condition for $cr_D(G \vee nK_1, C_n^*) \leq 1$ is $cr_D(\cup_{x=a}^{f} T^x, C_n^*) \leq 1$ and $cr_D(G, C_n^*) = 0$. If $cr_D(\cup_{x=a}^{f} T^x, C_n^*) = 0$, by Corollary 3.2, the graph $G \vee C_n$ has also at least $\binom{6}{2} \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor > Z(6,n) + 2\lfloor \frac{n}{2} \rfloor + 2$ crossings. This contradicts the assumption. So, we only need to discuss the case $cr_D(\cup_{x=a}^{f} T^x, C_n^*) = 1$. Without loss of generality, let one edge of T^a crosses one edge of C_n^*. In this case, none of the subgraphs T^b, T^c, T^d, T^e, and T^f crosses the cycle C_n^*. By Corollary 3.2, there are at least $\binom{5}{2} \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ crossings. So, in D, there are at least $\binom{5}{2} \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2 > Z(6,n) + 2\lfloor \frac{n}{2} \rfloor + 2$ crossings. This contradicts the assumption of D.

Case 3. $cr_D(C_n^*) = 0$ and $cr_D(G \vee nK_1, C_n^*) \leq 2$.

Case 3.1. $cr_D(G, C_n^*) \neq 0$.

As, in the graph G, no two vertices of degree two are adjacent, by the restriction $cr_D(G \vee nK_1, C_n^*) \leq 2$, the cycle C_n^* separates one vertex of degree two from the other vertices of G. In this case $cr_D(G, C_n^*) = 2$. So, by Corollary 3.2, in D there are at least $\binom{5}{2} \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2 > Z(6,n) + 2\lfloor \frac{n}{2} \rfloor + 2$

crossings. This contradicts the assumption of D.

Case 3.2. $cr_D(G, C_n^*) = 0$.

In this case, all vertices are placed in the same region in the view of the subdrawing $D(C_n^*)$, say external region. If $cr_D(\cup_{x=a}^{f} T^x, C_n^*) = 2$, there are the following two cases: (a) there is a subgraph T^x, $x \in \{a, b, c, d, e, f\}$, with $cr_D(T^x, C_n^*) = 2$, (b)there are two subgraphs T^x and T^y, $x, y \in \{a, b, c, d, e, f\}$, with $cr_D(T^x, C_n^*) = 1$ and $cr_D(T^y, C_n^*) = 1$. For the case (a) none of the subgraph T^k, $k \neq x$, crosses the edges of C_n^*. So, by Corollary 3.2, in D there are at least $\binom{5}{2}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 2 > Z(6, n) + 2\lfloor\frac{n}{2}\rfloor + 2$ crossings. For the case (b), by Corollary 3.2, the edges of T^x and T^y which do not cross C_n^* cross each other at least $\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor$ times. As every of T^x and T^y crosses T^k, $k \neq x, y$, at least $\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor$ times, in D there are at least $\binom{4}{2}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 8\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor + \lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor + 2 > Z(6, n) + 2\lfloor\frac{n}{2}\rfloor + 2$ crossings. If $cr_D(\cup_{x=a}^{f} T^x, C_n^*) = 1$, similar to the above discussion, we obtain that in D there are at least $\binom{5}{2}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 5\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor + 1 > Z(6, n) + 2\lfloor\frac{n}{2}\rfloor + 2$ crossings. If $cr_D(\cup_{x=a}^{f} T^x, C_n^*) = 0$, by Corollary 3.2, we immediately may obtain at least $\binom{6}{2}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor > Z(6, n) + 2\lfloor\frac{n}{2}\rfloor + 2$ crossings in D. These contradictions complete the proof.

□

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