Conditional Fault Hamiltonicity of the Star Graph

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Abstract

Fault tolerance is an important property on network performance. A graph G is *k-edge-fault conditional hamiltonian* if $G - F$ is hamiltonian for every $F \subset E(G)$ with $|F| \leq k$ and $\delta(G - F) \geq 2$. In this paper we show that for $n \geq 4$ the n-dimensional star graph S_n is $(3n - 10)$-edge-fault conditional hamiltonian.

Keywords: hamiltonian, star graphs, fault-tolerant.

1 Basic Definitions

An interconnection network connects the processors of parallel computers. Its architecture can be represented as a graph in which the vertices correspond to processors and the edges correspond to connections. Hence we use graphs and networks interchangeably. There are many mutually conflicting requirements in designing the topology for computer networks. The n-cube is one of the most popular topologies [21]. The n-dimensional star network S_n was proposed in [1] as "an attractive alternative to the n-cube" topology for interconnecting processors in parallel computers. Since

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its introduction, the network S_n has received considerable attention. The star graphs are bipartite, vertex transitive and edge transitive, and several classes of graphs can be embedded into them, e.g. grids [18], trees [3,5,9], and hypercubes [27]. Cycle embeddings and path embeddings are studied in [12,14–16,18,22,28]. The diameter and fault diameters of star graphs were computed in [1,20,29]. Some other interesting properties of star graphs are studied in [8,11,23,24].

For graph definitions and notations we follow [4]. In a *graph* $G = (V, E)$, V is a finite set of *vertices* and E is a set of *edges* consisting of unordered pairs of V. If F is a set of edges in G, the graph obtained by deleting the edges of F from G is denoted by $G - F$. A graph G is *vertex transitive* if for every two vertices u and v of G there is a graph isomorphism $f : V \to V$ such that $f(u) = v$. A graph G is *edge transitive* if for every two edges (u, v) and (x, y) of G there is a graph isomorphism $f : V \to V$ such that $(f(u), f(v)) = (x, y)$. Vertex u is called a *neighbor* of vertex v whenever $(u, v) \in E$, and the *neighborhood* of a vertex u, denoted by $N_G(u)$, is the set of its neighbors: $\{v \mid (u, v) \in E\}$. The *degree* of a vertex u is $\deg_G(u) = |N_G(u)|$. A graph G is *k-regular* if $\deg_G(u) = k$ for all vertices $u \in V$. We use $\delta(G)$ to denote the minimum degree in the graph G. A *path* P is a sequence of adjacent vertices, written as $\langle v_1, v_2, \ldots, v_k \rangle$, in which the vertices $v_1, v_2, \ldots, v_k$ are distinct except that possibly $v_1 = v_k$. We use P^{-1} to denote the path $\langle v_k, v_{k-1}, \ldots, v_2, v_1 \rangle$. The *length* of a path Q, denoted by $l(Q)$, is the number of edges in Q. We also write the path $\langle v_1, v_2, \ldots, v_k \rangle$ as $\langle v_1, Q_1, v_i, v_{i+1}, \ldots, v_j, Q_2, v_t, \ldots, v_k \rangle$ if Q_1 is the path $\langle v_1, v_2, \ldots, v_i \rangle$ and Q_2 is the path $\langle v_j, v_{j+1}, \ldots, v_t \rangle$. Hence it is possible to write a path as $\langle v_1, Q, v_1, v_2, \ldots, v_k \rangle$ if $l(Q) = 0$. A path is a *hamiltonian path* if it contains all vertices of G. A graph G is *hamiltonian connected* if for every two distinct vertices of G there is a hamiltonian path of G between them. A *cycle* is a path with at least three vertices such that the first vertex is the same as the last vertex. A *hamiltonian cycle* of G is a cycle that traverses every vertex of G exactly once. A graph is *hamiltonian* if it has a hamiltonian cycle.

Fault tolerance is an important property on network performance. The *edge fault-tolerant hamiltonicity* was proposed by Hsieh et al. [13] to measure the tolerance of the hamiltonian property to faults in networks. A graph G is *k-edge-fault hamiltonian* if $G - F$ remains hamiltonian for every $F \subseteq E(G)$ with $|F| \leq k$. A graph G is *bipartite* if its vertex set can be partitioned into two subsets V_1 and V_2 such that every edge joins a vertex of V_1 to a vertex of V_2. A bipartite graph G is *k-edge-fault hamiltonian* if $G - F$ is hamiltonian for every set of edges F with $|F| \leq k$. Latifi et al. [19] showed that the n-dimensional hypercube Q_n is $(n-2)$-edge-fault hamiltonian. Li et al. [22] showed that the n-dimensional star graph S_n is $(n-3)$-edge-fault hamiltonian.

In any graph $\delta(G) - 2$ is an immediate upper bound on the edge-fault hamiltonicity, since if all but one edge incident to a vertex are faulty, the graph cannot have a hamiltonian cycle. Hence Chen and Lee [6] studied the existence of a hamiltonian cycle in the n-dimensional hypercube when each vertex is incident to at least two non-faulty edges. A graph G is called *k-edge-fault conditional hamiltonian* if $G-F$ is hamiltonian for every $F \subseteq E(G)$ with $|F| \leq k$ and $\delta(G-F) \geq 2$. Clearly, whenever a graph is k-edge-fault hamiltonian, it is also k-edge-fault conditional hamiltonian, but the conditional hamiltonicity could be much larger. Chen and Lee [6] showed that the n-dimensional hypercube is $(2n-5)$-edge-fault conditional hamiltonian. Fu [10] showed that the n-dimensional star graph is $(2n-7)$-edge-fault conditional hamiltonian.

In Section 2 we give the definition of the star graph and introduce some of their basic properties. In Section 3 we show that for $n \geq 4$ the n-dimensional star graph S_n is $(3n-10)$-edge-fault conditional hamiltonian. Finally, some conclusions are given in Section 4.

2 The star graphs and their properties

We use $\langle n \rangle$ to denote the set $\{1, 2, \ldots, n\}$, where n is a positive integer. A *permutation* on $\langle n \rangle$ is a sequence $u_1u_2 \ldots u_i \ldots u_n$ of n distinct elements $u_i \in \langle n \rangle$. The *$n$-dimensional star graph*, denoted by S_n, is a graph with vertex set $V(S_n) = \{u_1u_2 \ldots u_n \mid u_i \in \langle n \rangle \text{ and } u_i \neq u_j \text{ for } i \neq j\}$. The edges of S_n are specified as follows: the permutation $u_1u_2 \ldots u_i \ldots u_n$ is adjacent to the permutation $v_1v_2 \ldots v_i \ldots v_n$ by an edge in dimension i with $2 \leq i \leq n$ if $v_1 = u_i$, $v_i = u_1$, and $v_j = u_j$ for $j \notin \{1, i\}$. By definition, S_n is an $(n-1)$-regular graph with $n!$ vertices. Moreover, it is vertex transitive and edge transitive [1]. The star graphs S_2, S_3, and S_4 are shown in Figure 1 for illustration.

We use boldface to denote vertices in S_n, hence $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}$ denotes a sequence of vertices in S_n. Obviously, S_n is a bipartite graph with one partite set containing vertices corresponding to odd permutations and the other partite set containing vertices corresponding to even permutations. We will use white vertices to represent the even permutation vertices and use black vertices to represent the odd permutation vertices. Let $\mathbf{u} = u_1u_2 \ldots u_n$ be any vertex of the star graph S_n. We say that u_i is the i-th *coordinate* of $\mathbf{u}$, denoted by $(\mathbf{u})_i$ for $1 \leq i \leq n$. By the definition of S_n, there is exactly one neighbor $\mathbf{v}$ of $\mathbf{u}$ such that $\mathbf{u}$ and $\mathbf{v}$ are adjacent through an i-dimensional edge with $2 \leq i \leq n$. For this reason, we use $(\mathbf{u})^i$ to denote the unique i-neighbor of $\mathbf{u}$. Obviously, $((\mathbf{u})^i)^i = \mathbf{u}$. For $1 \leq i \leq n$, let $S_n^{\{i\}}$ denote the subgraph of S_n induced by those vertices $\mathbf{u}$ with $(\mathbf{u})_n = i$. It is easy to see that the vertices of S_n can be decomposed

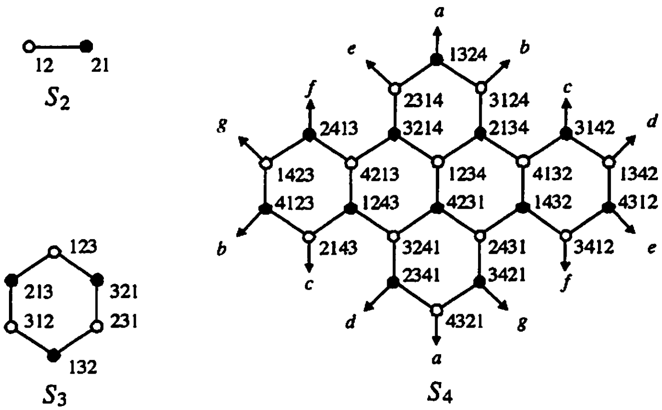

Figure 1: The star graphs S_2, S_3, and S_4.

into n subgraphs $S_n^{\{i\}}$, $1 \le i \le n$, and each $S_n^{\{i\}}$ is isomorphic to S_{n-1}. Thus, the star graph can be constructed recursively. Obviously, $\mathbf{u} \in S_n^{\{(\mathbf{u})_n\}}$ and $(\mathbf{u})^n \in S_n^{\{(\mathbf{u})_1\}}$. Let $I \subseteq \langle n \rangle$. We use S_n^I to denote the subgraph of S_n induced by $\cup_{i \in I} V(S_n^{\{i\}})$. For any two distinct elements i and j in $\langle n \rangle$, we use $E^{i,j}$ to denote the set of edges between $S_n^{\{i\}}$ and $S_n^{\{j\}}$.

The following are easy properties describing the structure of S_n for $n \ge 4$:

(I) $|E^{i,j}| = (n-2)!$ for every two distinct i and j in $\langle n \rangle$, and there are $\frac{(n-2)!}{2}$ edges joining black vertices of $S_n^{\{i\}}$ to white vertices of $S_n^{\{j\}}$. Moreover, edges of $E^{i,j}$ are independent, i.e., their endpoints are all different.

(II) Let $\mathbf{u}$ and $\mathbf{v}$ be any two distinct vertices in S_n with distance at most 2. Then $(\mathbf{u})_1 \ne (\mathbf{v})_1$, and $\{((\mathbf{u})^i)_1 \mid 2 \le i \le n-1\} = \langle n \rangle - \{(\mathbf{u})_1, (\mathbf{u})_n\}$.

The following theorems of [22] and [25] will be useful in proving our results:

Theorem 1. *[22] If $n \ge 4$, then $S_n - F$ is hamiltonian laceable for every set of edges $F \subseteq E(S_n)$ with $|F| \le n-3$.*

Theorem 2. *[25] Let $I = \{i_1, i_2, \ldots, i_t\}$ be a nonempty subset of $\langle n \rangle$ with $n \ge 5$. Then S_n^I is hamiltonian laceable.*

Theorem 3. *[25] Let $\mathbf{r}$ and $\mathbf{s}$ be two adjacent vertices of S_n with $n \ge 4$. Then for every white vertex $\mathbf{u}$ in $S_n - \{\mathbf{r}, \mathbf{s}\}$ and for every $i \in \langle n \rangle$, there* *exists a hamiltonian path* P *of* $S_n - \{\mathbf{r}, \mathbf{s}\}$ *joining* $\mathbf{u}$ *to a black vertex* $\mathbf{v}$ *with* $(\mathbf{v})_1 = i$.

Theorem 2 can be generalized when faulty edges are present in the graph:

Theorem 4. *Let* $I = \{i_1, i_2, \ldots, i_t\}$ *be a nonempty subset of* $\langle n \rangle$ *with* $n \geq 5$, *and let* F *be a set of edges of* S_n *such that* $|F \cap E(S_n^{\{i_k\}})| \leq n - 4$ *for every* $k \in \langle t \rangle$, *and* $|F \cap E^{i_k, i_{k+1}}| \leq \frac{(n-2)!}{2} - 1$ *for every* $k \in \langle t - 1 \rangle$. *If* $\mathbf{u}$ *is a white vertex of* $S_n^{\{i_1\}}$ *and* $\mathbf{v}$ *is a black vertex of* $S_n^{\{i_t\}}$, *then there is a hamiltonian path* P *of* $S_n^I - F$ *joining* $\mathbf{u}$ *to* $\mathbf{v}$.

Proof. Since $S_n^{\{i_k\}}$ is isomorphic to S_{n-1} for every $k \in \langle t \rangle$, Theorem 1 implies that this statement holds for $t = 1$. If $2 \leq t \leq n$, then we set $\mathbf{x_1} = \mathbf{u}$ and $\mathbf{y_t} = \mathbf{v}$. By (I), there are $\frac{(n-2)!}{2}$ edges joining black vertices of $S_n^{\{i\}}$ to white vertices of $S_n^{\{j\}}$ for every two distinct elements $i, j \in \langle n \rangle$. Since $\frac{(n-2)!}{2} > \frac{(n-2)!}{2} - 1$ if $n \geq 5$, for every $k \in \langle t-1 \rangle$ we can choose a black vertex $\mathbf{y_k}$ in $S_n^{\{i_k\}}$ such that $(\mathbf{y_k})^n \in S_n^{\{i_{k+1}\}}$ and $(\mathbf{y_k}, (\mathbf{y_k})^n) \notin F$. Set $\mathbf{x_{k+1}} = (\mathbf{y_k})^n$ for every $k \in \langle t-1 \rangle$. Obviously, $\mathbf{x_k}$ is a white vertex for every $k \in \langle t \rangle$. By Theorem 1, there is a hamiltonian path H_k of $S_n^{\{i_k\}} - F$ joining $\mathbf{x_k}$ to $\mathbf{y_k}$ for every $k \in \langle t \rangle$. Hence $\langle \mathbf{u} = \mathbf{x_1}, H_1, \mathbf{y_1}, \mathbf{x_2}, H_2, \mathbf{y_2}, \ldots, \mathbf{x_t}, H_t, \mathbf{y_t} = \mathbf{v} \rangle$ is a hamiltonian path of $S_n^I - F$ joining $\mathbf{u}$ to $\mathbf{v}$. □

The role of white and black vertices in Theorems 3 and 4 can be switched as well.

3 Conditional Fault Hamiltonicity

In this section we prove that we can delete up to $3n - 10$ edges from the star graph S_n for $n \geq 4$, and it will still be hamiltonian, provided that every vertex still has degree at least 2. This will show that S_n is $(3n-10)$-edge-fault conditional hamiltonian.

Before we prove our main theorem on the conditional fault hamiltonicity of the star graphs, we will discuss the proof technique. The proof is by induction with case analysis. Moreover, additional lemmas are provided to simplipfy the proof. The cases depend on the distribution of faults. Although none of the cases is unmanageable, care must be taken and the cases require a careful use of the recursive nature of the star graphs. These types of long but systematic and well-organized case analyses are common in this area of research. Moreover, the symmetry of the graphs is used to reduce the number of cases. A source of difficulty is some arguments are not applicable to small cases and hence need to be addressed separately.

We start with two lemmas that will be helpful in proving the main result. The first lemma gives an upper bound on the number of vertices of degree 2 that we can have in S_n with at most $3n-10$ faulty edges present:

Lemma 5. *Let F be a subset of $E(S_n)$ with $|F| \leq 3n-10$ such that $\delta(S_n - F) \geq 2$, for $n \geq 5$. If A is the set of vertices of degree 2 in $S_n - F$, then $|A| \leq 4$. Moreover, if $n \geq 6$, then $|A| \leq 3$.*

Proof. Suppose first that $X = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5\} \subseteq A$, and let H be the subgraph with vertex set X containing those edges of F that join two vertices in X. Since S_n is bipartite, so is H. At least one partite set can be chosen such that it has at least three vertices. If we count the number of edges in F incident to these three vertices, then no edges of F is counted twice, so we get that $|F| \geq 3(n-3) \geq 3n-9$, contradiction. Thus $|A| \leq 4$.

Next assume that $|A| = 4$. Again consider the subgraph H with vertex set A containing those edges of F that join two vertices in A. Since H contains no cycles, it is a forest, so it has at most 3 edges. Counting the number of edges in F at each vertex of A, we count each edge of H twice, so we get that $|F| \geq 4(n-3) - 3 = 4n - 15$. Since $|F| \leq 3n - 10$, this implies $n \leq 5$, and the lemma is proven. □

The second lemma will help us extend a hamiltonian cycle in $S_n^{\{i\}}$ to the rest of the graph:

Lemma 6. *Let $\mathbf{x}$, $\mathbf{y}$, $\mathbf{p}$, and $\mathbf{q}$ be four distinct vertices of $S_n^{\{i\}}$ for some $i \in \langle n \rangle$ with $(\mathbf{x}, \mathbf{y}) \in E(S_n)$ and $(\mathbf{p}, \mathbf{q}) \in E(S_n)$, for $n \geq 5$. Then there are two disjoint paths P_1 and P_2 of $S_n^{\langle n \rangle - \{i\}}$ such that*

(1) P_1 joins $(\mathbf{x})^n$ to $(\mathbf{y})^n$,

(2) P_2 joins $(\mathbf{p})^n$ to $(\mathbf{q})^n$, and

(3) $P_1 \cup P_2$ spans $S_n^{\langle n \rangle - \{i\}}$.

Proof. Without loss of generality, we may assume that $\{\mathbf{x}, \mathbf{y}, \mathbf{p}, \mathbf{q}\} \subset S_n^{\{n\}}$. By (II), $(\mathbf{x})_1 \neq (\mathbf{y})_1$ and $(\mathbf{p})_1 \neq (\mathbf{q})_1$. We have the following cases depending on the location of the neighbors of $\mathbf{x}, \mathbf{y}, \mathbf{p}, \mathbf{q}$ outside $S_n^{\{n\}}$:

Case 1: $|\{(\mathbf{x})_1, (\mathbf{y})_1\} \cap \{(\mathbf{p})_1, (\mathbf{q})_1\}| = 0$.

By Theorem 4, there is a hamiltonian path P_1 of $S_n^{\{(\mathbf{x})_1, (\mathbf{y})_1\}}$ joining $(\mathbf{y})^n$ to $(\mathbf{x})^n$. Similarly, there is a hamiltonian path P_2 of $S_n^{\langle n-1 \rangle - \{(\mathbf{x})_1, (\mathbf{y})_1\}}$ joining $(\mathbf{p})^n$ to $(\mathbf{q})^n$. Then P_1 and P_2 are the desired paths.

Case 2: $|\{(\mathbf{x})_1, (\mathbf{y})_1\} \cap \{(\mathbf{p})_1, (\mathbf{q})_1\}| = 1$.

By symmetry, we may assume that $(\mathbf{x})_1 \notin \{(\mathbf{p})_1, (\mathbf{q})_1\}$ and $(\mathbf{y})_1 = (\mathbf{p})_1$. Since $((\mathbf{y})^n)_1 = ((\mathbf{p})^n)_1 = n$, by (II) there is a vertex $\mathbf{z}$ in $S_n^{\{(\mathbf{p})_1\}}$ such that $\mathbf{z}$ is a neighbor of $(\mathbf{y})^n$, and $(\mathbf{z})_1 = (\mathbf{x})_1$. By Theorem 4, there is a hamiltonian path H of $S_n^{\{(\mathbf{x})_1\}}$ joining $(\mathbf{x})^n$ to $(\mathbf{z})^n$. Let t

be any integer in $\langle n-1\rangle - \{(\mathbf{x})_1, (\mathbf{p})_1, (\mathbf{q})_1\}$. By Theorem 3, there is a hamiltonian path R of $S_n^{\{(\mathbf{p})_1\}} - \{(\mathbf{y})^n, \mathbf{z}\}$ joining $(\mathbf{p})^n$ to a vertex $\mathbf{w}$ such that $(\mathbf{w})_1 = t$. By Theorem 4, there is a hamiltonian path Q of $S_n^{\langle n-1\rangle - \{(\mathbf{x})_1, (\mathbf{p})_1\}}$ joining $(\mathbf{w})^n$ to $(\mathbf{q})^n$. Then $P_1 = \langle (\mathbf{x})^n, H, (\mathbf{z})^n, \mathbf{z}, (\mathbf{y})^n\rangle$ and $P_2 = \langle (\mathbf{p})^n, R, \mathbf{w}, (\mathbf{w})^n, Q, (\mathbf{q})^n\rangle$ are the desired paths.

Case 3: $|\{(\mathbf{x})_1, (\mathbf{y})_1\} \cap \{(\mathbf{p})_1, (\mathbf{q})_1\}| = 2$.

By symmetry, we may assume that $(\mathbf{x})_1 = (\mathbf{p})_1$ and $(\mathbf{y})_1 = (\mathbf{q})_1$. Let t and s be any two distinct integers in $\langle n-1\rangle - \{(\mathbf{p})_1, (\mathbf{q})_1\}$. Since $((\mathbf{x})^n)_1 = n$ and $((\mathbf{p})^n)_1 = n$, by (II) there is a vertex $\mathbf{w}$ in $S_n^{\{(\mathbf{p})_1\}}$ such that $\mathbf{w}$ is a neighbor of $(\mathbf{x})^n$, and $(\mathbf{w})_1 = t$. Similarly, there is a vertex $\mathbf{z}$ in $S_n^{\{(\mathbf{q})_1\}}$ such that $\mathbf{z}$ is a neighbor of $(\mathbf{y})^n$, and $(\mathbf{z})_1 = t$. By Theorem 4, there is a hamiltonian path H of $S_n^{\{t\}}$ joining $(\mathbf{w})^n$ to $(\mathbf{z})^n$. By Theorem 3, there is a hamiltonian path R_1 of $S_n^{\{(\mathbf{p})_1\}} - \{(\mathbf{x})^n, \mathbf{w}\}$ joining $(\mathbf{p})^n$ to a vertex $\mathbf{u}$ such that $(\mathbf{u})_1 = s$. By Theorem 3, there is a hamiltonian path R_2 of $S_n^{\{(\mathbf{q})_1\}} - \{(\mathbf{y})^n, \mathbf{z}\}$ joining a vertex $\mathbf{v}$ such that $(\mathbf{v})_1 = s$ to $(\mathbf{q})^n$. By Theorem 2 there is a hamiltonian path Q of $S_n^{\langle n-1\rangle - \{t, (\mathbf{p})_1, (\mathbf{q})_1\}}$ joining $(\mathbf{u})^n$ to $(\mathbf{v})^n$. Then $P_1 = \langle (\mathbf{x})^n, \mathbf{w}, (\mathbf{w})^n, H, (\mathbf{z})^n, \mathbf{z}, (\mathbf{y})^n\rangle$ and $P_2 = \langle (\mathbf{p})^n, R_1, \mathbf{u}, (\mathbf{u})^n, Q, (\mathbf{v})^n, \mathbf{v}, R_2, (\mathbf{q})^n\rangle$ are the desired paths. □

Now we can prove our main theorem in this section:

Theorem 7. *Let F be a subset of $E(S_n)$ with $|F| \le 3n-10$ and $\delta(S_n - F) \ge 2$, for $n \ge 4$. Then $S_n - F$ is hamiltonian.*

Proof. We prove this statement by induction on n.

For $n = 4$, clearly $\delta(S_4 - F) = 2$, and without loss of generality we may assume that $|F| = 2$. Since S_4 is vertex transitive and edge transitive, we may also assume that $(1234, 4231) \in F$. Let f be the other edge in $F - \{(1234, 4231)\}$. Obviously, f can't be any of the edges in $\{(1234, 4231), (1234, 2134), (1234, 3214), (4231, 2431), (4231, 3241)\}$. The required hamiltonian cycles of $S_4 - \{(1234, 4231), f\}$ are listed below for the remaining possibilities:

$f \in \{(1324, 2314), (1423, 2413), (1243, 3241), (1342, 2341),$ $(2143, 3142), (3214, 4213), (3124, 4123), (3412, 4312), (3421, 4321)\}$ $\langle 1234, 2134, 3124, 1324, 4321, 2341, 3241, 4231, 2431, 3421, 1423, 4123, 2143,$ $1243, 4213, 2413, 3413, 1432, 4132, 3142, 1342, 4312, 2314, 3214, 1234\rangle$
$f \in \{(1324, 3124), (1342, 3142), (1423, 3421), (1432, 2431),$ $(2341, 4321), (2143, 4123), (2413, 3412), (2314, 4312), (2134, 4132)\}$ $\langle 1234, 2134, 3124, 4123, 1423, 2413, 4213, 1243, 2143, 3142, 4132, 1432, 3412,$ $3412, 1342, 2341, 3241, 4231, 2431, 3421, 4321, 1324, 2314, 3214, 1234\rangle$
$f \in \{(1324, 4321), (1342, 4312), (3142, 4132), (1432, 3412), (2431, 3421), (3241, 2341), (4213, 1243)\}$ $\langle 1234, 2134, 4132, 1432, 2431, 4231, 3241, 1243, 2143, 3142, 1342, 2341, 4321,$ $3421, 1423, 4123, 3124, 1324, 2314, 4312, 3412, 2413, 4213, 3214, 1234\rangle$
$f \in \{(1423, 4123), (1432, 4132), (2314, 3214), (2134, 3124), (2413, 4213), (2143, 1243)\}$ $\langle 1234, 2134, 4132, 3142, 2143, 4123, 3124, 1324, 2314, 4312, 1342, 2341, 4321,$ $3421, 1423, 2413, 3412, 1432, 2431, 4231, 3241, 1243, 4213, 3214, 1234\rangle$

Suppose now that the statement holds for S_k for every $4 \le k \le n-1$, where $n \ge 5$. Let $A = \{\mathbf{u} \mid \deg_{S_n - F}(\mathbf{u}) = 2\}$. Since $n \ge 5$, by Lemma 5 we have $|A| \leq 4$. We set $F_i = F \cap E(S_n^{\{i\}})$ for every $i \in \langle n \rangle$ and $F_{j,k} = F \cap E^{j,k}$ for every two distinct elements $j, k \in \langle n \rangle$. We consider cases depending on the size of A. Some subcases will require checking the claim specifically for $n = 5$. Including these would increase the number of subcases without being applicable to the general case. Hence we will omit them here to simplify the proof, and instead include them in the Appendix.

Case 1: $|A| = 4$.

By Lemma 5, $n = 5$. This case is discussed in the Appendix.

Case 2: $2 \leq |A| \leq 3$.

We consider cases depending on whether there is an edge of F joining two vertices in A.

Case 2.1: There are vertices $\mathbf{u}, \mathbf{v} \in A$ such that $(\mathbf{u}, \mathbf{v}) \in F$.

Since S_n is vertex transitive and edge transitive, we may assume that $\mathbf{u} \in S_n^{\{n\}}$ and $\mathbf{v} \in S_n^{\{n-1\}}$. Clearly, $|F_n| \geq n-4$, $|F_{n-1}| \geq n-4$, and $|F_i| \leq n-3$ for every $i \in \langle n-2 \rangle$, and $|E^{i,j}| \leq n-2$ for every distinct $i, j \in \langle n \rangle$.

Case 2.1.1: $\delta(S_n^{\{i\}} - F_i) \geq 2$ for every $i \in \langle n \rangle$.

Since $|F| \leq 3n-10$, $|F_i| \geq n-3$ can occur for at most two different $i \in \langle n \rangle$. Let a, b be integers in $\langle n \rangle$ such that $|F_a| \geq |F_i|$ for every $i \in \langle n \rangle$, and $|F_b| \geq |F_j|$ for every $j \in \langle n \rangle - \{a\}$. By induction, there is a hamiltonian cycle C_1 of $S_n^{\{a\}} - F_a$, and there is a hamiltonian cycle C_2 of $S_n^{\{b\}} - F_b$. For every vertex $\mathbf{p} \in S_n^{\{a\}}$ with $(\mathbf{p})_1 = b$, we set $A(\mathbf{p}) = \{\mathbf{p}\} \cup N_{C_1}(\mathbf{p}) \cup N_{C_2}((\mathbf{p})^n)$ and $B(\mathbf{p}) = \{(\mathbf{q}, (\mathbf{q})^n) \mid \mathbf{q} \in A(\mathbf{p})\}$. Since $|E^{a,b}| = (n-2)! > n-2$ if $n \geq 5$, there is a vertex $\mathbf{z} \in S_n^{\{a\}}$ with $(\mathbf{z})_1 = b$ such that $B(\mathbf{z}) \cap F = \emptyset$. Let $\mathbf{p}$ be a neighbor of $\mathbf{z}$ on C_1. Then by (II), the two neighbors of $(\mathbf{z})^n$ on C_2 have different first coordinates, so at least one of them is different from $(\mathbf{p})_1$. Let $\mathbf{q}$ be such a vertex. Then we have $C_1 = \langle \mathbf{z}, R_1, \mathbf{p}, \mathbf{z} \rangle$ and $C_2 = \langle \mathbf{q}, R_2, (\mathbf{z})^n, \mathbf{q} \rangle$ and $(\mathbf{p})^n \neq (\mathbf{q})^n$. For $n \geq 6$ we also have $\frac{(n-2)!}{2} > n-2$, hence by Theorem 4, there is a hamiltonian path H of $S_n^{\langle n \rangle - \{a,b\}} - F$ joining $(\mathbf{p})^n$ to $(\mathbf{q})^n$, and then $\langle \mathbf{z}, R_1, \mathbf{p}, (\mathbf{p})^n, H, (\mathbf{q})^n, \mathbf{q}, R_2, (\mathbf{z})^n, \mathbf{z} \rangle$ is a hamiltonian cycle of $S_n - F$. The case $n = 5$ is discussed in the Appendix.

Case 2.1.2: $\delta(S_n^{\{t\}} - F_t) = 1$ for some $t \in \langle n \rangle$.

Let $\mathbf{y}$ be the vertex in $S_n^{\{t\}}$ with $\deg_{S_n^{\{t\}} - F_t}(\mathbf{y}) = 1$. Obviously, $\mathbf{y}$ is neither $\mathbf{u}$ nor $\mathbf{v}$, and $(\mathbf{y}, (\mathbf{y})^n) \notin F$. Moreover, $|A| = 3$, and we have identified all but one edge in F.

Suppose first that $\mathbf{y}$ is adjacent to neither $\mathbf{u}$ nor $\mathbf{v}$. Then we have identified all edges of F, hence $|F_i| \leq n-4$ for every $i \in \langle n \rangle - \{t\}$. Since $\deg_{S_n^{\{t\}} - F_t}(\mathbf{y}) = 1$ and $n \geq 5$, we can choose an edge $(\mathbf{y}, \mathbf{z}) \in F_t$ such that $\mathbf{z}$ is neither $\mathbf{u}$ nor $\mathbf{v}$. By induction, there is a hamiltonian cycle C of $S_n^{\{t\}} - (F_t - \{(\mathbf{y}, \mathbf{z})\})$. Clearly, edge $(\mathbf{y}, \mathbf{z})$ must be in C, so we can write $C = \langle \mathbf{y}, R, \mathbf{z}, \mathbf{y} \rangle$. By Theorem 4, there is a hamiltonian path H of

$S_n^{\langle n\rangle - \{t\}} - F$ joining $(\mathbf{z})^n$ to $(\mathbf{y})^n$. Then $\langle \mathbf{y}, R, \mathbf{z}, (\mathbf{z})^n, H, (\mathbf{y})^n, \mathbf{y}\rangle$ forms a hamiltonian cycle of $S_n - F$.

Secondly, if $\mathbf{y}$ is adjacent to either $\mathbf{u}$ or $\mathbf{v}$, then without loss of generality we may assume that $(\mathbf{y}, \mathbf{u}) \in E(S_n)$, so $\mathbf{y} \in S_n^{\{n\}}$. Since all but one edge of F have been identified so far, out of the $n-3$ faulty edges incident to y in $S_n^{\{n\}}$, at most one can go to a vertex $\mathbf{z} \neq \mathbf{u}$ such that $(\mathbf{z}, (\mathbf{z})^n) \in F$. If $n = 5$ and this happens, then we have four vertices of degree 2 in $S_n - F$, so we get Case 1. If $n \geq 6$, we can choose an edge $(\mathbf{y}, \mathbf{z}) \in F_n$ such that $(\mathbf{z}, (\mathbf{z})^n) \notin F$. By induction, there is a hamiltonian cycle C of $S_n^{\{n\}} - (F_n - \{(\mathbf{y}, \mathbf{z})\})$. Clearly, edge $(\mathbf{y}, \mathbf{z})$ must be in C, so we can write $C = \langle \mathbf{y}, R, \mathbf{z}, \mathbf{y}\rangle$. If the so far unidentified edge of F is not in F_{n-1}, then $|F_{n-1}| = n-4$, and by Theorem 4, there is a hamiltonian path H of $S_n^{\langle n-1\rangle} - F$ joining $(\mathbf{z})^n$ to $(\mathbf{y})^n$, hence $\langle \mathbf{y}, R, \mathbf{z}, (\mathbf{z})^n, H, (\mathbf{y})^n, \mathbf{y}\rangle$ forms a hamiltonian cycle of $S_n - F$. On the other hand, if the last unidentified edge of F is in F_{n-1}, then $|F_{n-1}| = n-3$, and we have $|F_i| = 0$ for every $i \in \langle n-1\rangle$ and $|F_{i,j}| = 0$ for every two different $i, j \in \langle n-1\rangle$. Since $(\mathbf{u})_1 = n-1$, and $\mathbf{y}$ is adjacent to both $\mathbf{u}$ and $\mathbf{z}$, (II) implies that $(\mathbf{y})_1 \neq (\mathbf{z})_1$, $(\mathbf{y})_1 \neq n-1$, and $(\mathbf{z})_1 \neq n-1$. Then we can choose a vertex $\mathbf{w}$ in $S_n^{\{n-1\}}$ such that its color is different from the color of $\mathbf{y}$, and $(\mathbf{w})_1 = (\mathbf{y})_1$. Since $n-3 \leq 3n-13$ for $n \geq 5$ and $\delta(S_n^{\{n-1\}} - F_{n-1}) = 2$, by induction, there is a hamiltonian cycle C' in $S_n^{\{n-1\}} - F_{n-1}$. We can write $C' = \langle \mathbf{w}, \mathbf{p}, P, \mathbf{q}, \mathbf{w}\rangle$. By (II), $(\mathbf{p})_1$, $(\mathbf{w})_1$, and $(\mathbf{q})_1$ are all different, so without loss of generality, we may assume that $(\mathbf{p})_1 \neq n$. By Theorem 2, there is a hamiltonian path H_1 of $S_n^{\{(\mathbf{y})_1\}}$ joining $(\mathbf{w})^n$ to $(\mathbf{y})^n$, and there is a hamiltonian path H_2 of $S_n^{\langle n-2\rangle - \{(\mathbf{y})_1\}}$ joining $(\mathbf{z})^n$ to $(\mathbf{p})^n$. Then $\langle \mathbf{y}, R, \mathbf{z}, (\mathbf{z})^n, H_2, (\mathbf{p})^n, \mathbf{p}, P, \mathbf{q}, \mathbf{w}, (\mathbf{w})^n, H_1, (\mathbf{y})^n, \mathbf{y}\rangle$ forms a hamiltonian cycle of $S_n - F$.

Case 2.2: $(\mathbf{u}, \mathbf{v}) \notin F$ for every two distinct vertices $\mathbf{u}, \mathbf{v} \in A$.

Counting the edges of F at each vertex of A, we get that $|A|(n-3) \leq F \leq 3n-10$, hence $|A| \leq 2$. Thus $|A| = 2$, so let $A = \{\mathbf{u}, \mathbf{v}\}$. Since S_n is vertex transitive and edge transitive, we may assume that $(\mathbf{u})_n = n$ and $(\mathbf{u}, (\mathbf{u})^n) \in F$. Now consider cases depending on the location of $\mathbf{v}$.

Case 2.2.1: $(\mathbf{v})_n = n$ and $(\mathbf{v}, (\mathbf{v})^n) \in F$.

Clearly, $2n-8 \leq |F_n| \leq 3n-12$, so there can be at most $n-4$ edges of F outside $S_n^{\{n\}}$ different from $(\mathbf{u}, (\mathbf{u})^n)$ and $(\mathbf{v}, (\mathbf{v})^n)$. If $|F_n| \leq 3n-13$, then by induction, there is a hamiltonian cycle C of $S_n^{\{n\}} - F_n$. Since $|V(S_n^{\{n\}})| = (n-1)! > 2(n-2)$, there is an edge $(\mathbf{p}, \mathbf{q}) \in E(C)$ such that $(\mathbf{p}, (\mathbf{p})^n) \notin F$ and $(\mathbf{q}, (\mathbf{q})^n) \notin F$. Then we can write $C = \langle \mathbf{p}, H, \mathbf{q}, \mathbf{p}\rangle$. Since $n - 4 < \frac{(n-2)!}{2}$ for $n \geq 5$, Theorem 4 implies that there is a hamiltonian path R of $S_n^{\langle n-1\rangle}$ joining $(\mathbf{q})^n$ to $(\mathbf{p})^n$, and then $\langle \mathbf{p}, H, \mathbf{q}, (\mathbf{q})^n, R, (\mathbf{p})^n, \mathbf{p}\rangle$ is a hamiltonian cycle of $S_n - F$.

On the other hand, if $|F_n| = 3n-12$, then every edge in F is located.

Since $2(n-4) < 3n-12$, there is an edge $f \in F_n$ such that f is incident to neither $\mathbf{u}$ nor $\mathbf{v}$. By induction, there is a hamiltonian cycle C of $S_n^{\{n\}} - (F_n - \{f\})$. However, f may be not in C. If $f \in C$, let $f = (\mathbf{p}, \mathbf{q})$, otherwise pick an edge $(\mathbf{p}, \mathbf{q})$ in C such that $\{\mathbf{p}, \mathbf{q}\} \cap \{\mathbf{u}, \mathbf{v}\} = \emptyset$. Then we can write $C = \langle \mathbf{p}, H, \mathbf{q}, \mathbf{p} \rangle$. By Theorem 4, there is a hamiltonian path R of $S_n^{\langle n-1 \rangle}$ joining $(\mathbf{q})^n$ to $(\mathbf{p})^n$, hence $\langle \mathbf{p}, H, \mathbf{q}, (\mathbf{q})^n, R, (\mathbf{p})^n, \mathbf{p} \rangle$ is a hamiltonian cycle of $S_n - F$.

Case 2.2.2: $(\mathbf{v})_n = n$ and $(\mathbf{v}, (\mathbf{v})^n) \notin F$.

Within $S_n^{\{n\}}$, there are $n-4$ faulty edges incident to $\mathbf{u}$, and $n-3$ faulty edges incident to $\mathbf{v}$. Since $(n-3) + (n-4) > n-2$ for $n \geq 6$, there must be an integer $i \in \langle n \rangle - \{1\}$ such that $(\mathbf{u}, (\mathbf{u})^i) \in F$ and $(\mathbf{v}, (\mathbf{v})^i) \in F$. Using vertices of having the same i-th coordinate instead of the same n-th coordinate to define the sets $S_n^{\{j\}}$ for $j \in \langle n \rangle$ will change this case to Case 2.2.1. The case $n = 5$ is discussed in the Appendix.

Case 2.2.3: $(\mathbf{v})_n \neq n$ and $(\mathbf{v}, (\mathbf{v})^n) \in F$.

Without loss of generality we may assume that $(\mathbf{v})_n = n-1$. By induction, there is a hamiltonian cycle C_1 of $S_n^{\{n\}} - F_n$, and there is a hamiltonian cycle C_2 of $S_n^{\{n-1\}} - F_{n-1}$. For every vertex $\mathbf{p}$ in $S_n^{\{n\}}$ with $(\mathbf{p})_1 = n-1$, we set $A(\mathbf{p}) = \{\mathbf{p}\} \cup N_{C_1}(\mathbf{p}) \cup N_{C_2}((\mathbf{p})^n)$ and $B(\mathbf{p}) = \{(\mathbf{q}, (\mathbf{q})^n) \mid \mathbf{q} \in A(\mathbf{p})\}$. Since $|E^{n-1,n}| = (n-2)! > n-4$ if $n \geq 5$, there is a vertex $\mathbf{z} \in S_n^{\{n\}}$ with $(\mathbf{z})_1 = n-1$ such that $B(\mathbf{z}) \cap F = \emptyset$. Let $\mathbf{p}$ be a neighbor of $\mathbf{z}$ on C_1. Then by (II), the two neighbors of $(\mathbf{z})^n$ on C_2 have different first coordinates, so at least one of them is different from $(\mathbf{p})_1$. Let $\mathbf{q}$ be such a vertex. Then we can write $C_1 = \langle \mathbf{z}, R_1, \mathbf{p}, \mathbf{z} \rangle$ and $C_2 = \langle \mathbf{q}, R_2, (\mathbf{z})^n, \mathbf{q} \rangle$. By Theorem 4, there is a hamiltonian path H of $S_n^{\langle n-2 \rangle} - F$ joining $(\mathbf{p})^n$ to $(\mathbf{q})^n$, and then $\langle \mathbf{z}, R_1, \mathbf{p}, (\mathbf{p})^n, H, (\mathbf{q})^n, \mathbf{q}, R_2, (\mathbf{z})^n, \mathbf{z} \rangle$ is a hamiltonian cycle of $S_n - F$.

Case 2.2.4: $(\mathbf{v})_n \neq n$ and $(\mathbf{v}, (\mathbf{v})^n) \notin F$.

There are $n-3$ faulty edges incident to both $\mathbf{u}$ and $\mathbf{v}$. Since $2(n-3) > n-1$ for $n \geq 6$, there must be an integer $i \in \langle n \rangle - \{1\}$ such that $(\mathbf{u}, (\mathbf{u})^i) \in F$ and $(\mathbf{v}, (\mathbf{v})^i) \in F$. Using vertices of having the same i-th coordinate instead of the same n-th coordinate to define the sets $S_n^{\{j\}}$ for $j \in \langle n \rangle$ will change this case to either Case 2.2.1 (if $\mathbf{u}$ and $\mathbf{v}$ have the same i-th coordinate) or Case 2.2.3 (if $\mathbf{u}$ and $\mathbf{v}$ have different i-th coordinates). If $n = 5$ and we can't reduce this case to one of the previous cases, we can do the same that we did for Case 2.2.2 in the Appendix.

Case 3: $0 \leq |A| \leq 1$.

Let $\mathbf{x}$ be a vertex in $S_n - F$ of minimum degree. Since S_n is edge transitive, we may assume that $(\mathbf{x}, (\mathbf{x})^n) \in F$, thus $\delta(S_n^{\{i\}} - F_i) \geq 2$ and $|F_i| \leq 3n-11$ for every $i \in \langle n \rangle$. Without loss of generality, we may assume that $|F_n| \geq |F_{n-1}| \geq \ldots \geq |F_1|$. Thus $|F_i| \leq n-4$ for every $i \in \langle n-2 \rangle$. Now look at cases depending on the sizes of F_n and F_{n-1}.

Case 3.1: $|F_n| \leq 3n-13$ and $|F_{n-1}| \geq n-3$.

By induction, there is a hamiltonian cycle C_1 of $S_n^{\{n\}} - F_n$, and there is a hamiltonian cycle C_2 of $S_n^{\{n-1\}} - F_{n-1}$. For every vertex $\mathbf{w} \in S_n^{\{n\}}$ with $(\mathbf{w})_1 = n-1$, let $\mathbf{w}_1$ and $\mathbf{w}_2$ be its two neighbors in $S_n^{\{n\}}$ on C_1, and and let $\mathbf{w}_3$ and $\mathbf{w}_4$ be the two neighbors of $(\mathbf{w})^n$ in C_2. Since $(n-2)! > n-4$ for $n \geq 6$, there is a vertex $\mathbf{z}$ in $S_n^{\{n\}}$ such that $(\mathbf{z})_1 = n-1$, $(\mathbf{z}, (\mathbf{z})^n) \notin F$, and $(\mathbf{z}_i, (\mathbf{z}_i)^n) \notin F$ for every $1 \leq i \leq 4$. By II, $(\mathbf{z}_1)_1 \neq (\mathbf{z}_2)_1$ and $(\mathbf{z}_3)_1 \neq (\mathbf{z}_4)_1$. Without loss of generality, we may assume that $(\mathbf{z}_2)_1 \neq (\mathbf{z}_3)_1$. Then we can write $C_1 = \langle \mathbf{z}, \mathbf{z}_1, R_1, \mathbf{z}_2, \mathbf{z} \rangle$ and $C_2 = \langle (\mathbf{z})^n, \mathbf{z}_3, R_2, \mathbf{z}_4, (\mathbf{z})^n \rangle$. By Theorem 4, there is a hamiltonian path H of $S_n^{\langle n-2 \rangle} - F$ joining $(\mathbf{z}_2)^n$ to $(\mathbf{z}_3)^n$, and then $\langle \mathbf{z}, \mathbf{z}_1, R_1, \mathbf{z}_2, (\mathbf{z}_2)^n, H, (\mathbf{z}_3)^n, \mathbf{z}_3, R_2, \mathbf{z}_4, (\mathbf{z})^n, \mathbf{z} \rangle$ is a hamiltonian cycle of $S_n - F$.

Case 3.2: $|F_n| \leq 3n-13$ and $|F_{n-1}| \leq n-4$.

If $|F_{i,j}| \leq \frac{(n-2)!}{2} - 1$ for every distinct $i, j \in \langle n-1 \rangle$, then by induction, there is a hamiltonian cycle C of $S_n^{\{n\}} - F_n$. Since $|V(S_n^{\{n\}})| = (n-1)! > 2(3n-10)$ if $n \geq 5$, there is an edge $(\mathbf{u}, \mathbf{v}) \in C$ such that $(\mathbf{u}, (\mathbf{u})^n) \notin F$ and $(\mathbf{v}, (\mathbf{v})^n) \notin F$. Then we can write $C = \langle \mathbf{u}, H, \mathbf{v}, \mathbf{u} \rangle$. By Theorem 4, there is a hamiltonian path R of $S_n^{\langle n-1 \rangle} - F$ joining $(\mathbf{v})^n$ to $(\mathbf{u})^n$, and then $\langle \mathbf{u}, H, \mathbf{v}, (\mathbf{v})^n, R, (\mathbf{u})^n, \mathbf{u} \rangle$ is a hamiltonian cycle of $S_n - F$.

On the other hand, if $|F_{i,j}| > \frac{(n-2)!}{2} - 1$ for some $i, j \in \langle n-1 \rangle$, then since $|F| \leq 3n-10$ and $\frac{(n-2)!}{2} - 1 > 3n-10$ if $n \geq 6$, we get $n = 5$. The case $n = 5$ is discussed in the Appendix.

Case 3.3: $|F_n| = 3n-12$.

Since $|F_n| = 3n-12 > 2(n-4)$ if $n \geq 5$, there is an edge $(\mathbf{p}, \mathbf{q}) \in F_n$ such that edges $(\mathbf{p}, (\mathbf{p})^n)$ and $(\mathbf{q}, (\mathbf{q})^n)$ are not in F. By induction, there is a hamiltonian cycle C of $S_n^{\{n\}} - (F_n - \{(\mathbf{p}, \mathbf{q})\})$. We can write $C = \langle \mathbf{u}, H, \mathbf{v}, \mathbf{u} \rangle$, where $\{\mathbf{u}, \mathbf{v}\} = \{\mathbf{p}, \mathbf{q}\}$ if edge $(\mathbf{p}, \mathbf{q})$ is part of C, otherwise $(\mathbf{u}, \mathbf{v})$ is an arbitrary edge of C such that edges $(\mathbf{u}, (\mathbf{u})^n)$ and $(\mathbf{v}, (\mathbf{v})^n)$ are not in F. By Theorem 4, there is a hamiltonian path R of $S_n^{\langle n-1 \rangle} - F$ joining $(\mathbf{v})^n$ to $(\mathbf{u})^n$, and then $\langle \mathbf{u}, H, \mathbf{v}, (\mathbf{v})^n, R, (\mathbf{u})^n, \mathbf{u} \rangle$ is a hamiltonian cycle of $S_n - F$.

Case 3.4: $|F_n| = 3n-11$ and $\deg_{S_n - F}(\mathbf{x}) = 2$.

Clearly, $\mathbf{x} \in S_n^{\{n\}}$, and all edges of F are accounted for. Since $|F_n| = 3n-11 > 2(n-4)$ if $n \geq 5$, there are two edges $(\mathbf{u}, \mathbf{v})$ and $(\mathbf{p}, \mathbf{q})$ in F_n such that $(\mathbf{u}, (\mathbf{u})^n) \notin F$, $(\mathbf{v}, (\mathbf{v})^n) \notin F$, $(\mathbf{p}, (\mathbf{p})^n) \notin F$, $(\mathbf{q}, (\mathbf{q})^n) \notin F$, and vertices $\mathbf{u}, \mathbf{v}, \mathbf{p}, \mathbf{q}$ are all different. By induction, there is a hamiltonian cycle C in $S_n^{\{n\}} - (F_n - \{(\mathbf{u}, \mathbf{v}), (\mathbf{p}, \mathbf{q})\})$. If either $(\mathbf{u}, \mathbf{v})$ or $(\mathbf{p}, \mathbf{q})$ is not in C, then without loss of generality, we may assume that $(\mathbf{p}, \mathbf{q}) \notin C$. We can write $C = \langle \mathbf{w}, R, \mathbf{z}, \mathbf{w} \rangle$ where $\{\mathbf{w}, \mathbf{z}\} = \{\mathbf{u}, \mathbf{v}\}$ if $(\mathbf{u}, \mathbf{v}) \in E(C)$, otherwise $(\mathbf{w}, \mathbf{z})$ is chosen to be an arbitrary edge of C such that $(\mathbf{w}, (\mathbf{w})^n) \notin F$ and

$(\mathbf{z}, (\mathbf{z})^n) \notin F$. By Theorem 4, there is a hamiltonian path H of $S_n^{\langle n-1\rangle}$ joining $(\mathbf{z})^n$ to $(\mathbf{w})^n$, and then $\langle \mathbf{w}, R, \mathbf{z}, (\mathbf{z})^n, H, (\mathbf{w})^n, \mathbf{w}\rangle$ is a hamiltonian cycle of $S_n - F$.

On the other hand, if both $(\mathbf{u}, \mathbf{v})$ and $(\mathbf{p}, \mathbf{q})$ are in C, then without loss of generality, we can write $C = \langle \mathbf{u}, R_1, \mathbf{p}, \mathbf{q}, R_2, \mathbf{v}, \mathbf{u}\rangle$. By Lemma 6, there are two disjoint paths Q_1 and Q_2 of $S_n^{\langle n-1\rangle}$ such that (1) Q_1 joins $(\mathbf{v})^n$ to $(\mathbf{u})^n$, (2) Q_2 joins $(\mathbf{p})^n$ to $(\mathbf{q})^n$, and (3) $Q_1 \cup Q_2$ spans $S_n^{\langle n-1\rangle}$. Then $\langle \mathbf{u}, R_1, \mathbf{p}, (\mathbf{p})^n, Q_2, (\mathbf{q})^n, \mathbf{q}, R_2, \mathbf{v}, (\mathbf{v})^n, Q_1, (\mathbf{u})^n, \mathbf{u}\rangle$ is a hamiltonian cycle of $S_n - F$.

Case 3.5: $|F_n| = 3n - 11$ and $\deg_{S_n - F}(\mathbf{x}) \geq 3$.

Since $\lceil \frac{3n-11}{n-2} \rceil \geq 2$ if $n \geq 5$, there is an integer $2 \leq i \leq n-1$ such that there are at least two faulty edges among the edges of dimension i in S_n. Define the sets $S_n^{\{j\}}$ for $j \in \langle n\rangle$ using vertices of having the same i-th coordinate instead of the same n-th coordinate. Then $\deg_{S_n - F}(\mathbf{x}) \geq 3$ will imply that the minimum degree in $S_n^{\{j\}} - F_j$ is at least 2 and $|F_j| \leq 3n - 12$ for every $j \in \langle n\rangle$. Hence this case reduces to one of Cases 3.1–3.3.

This finishes the proof of the theorem. □

In the next section we give an example showing that Theorem 7 is sharp.

4 Conclusions

Since reliability of networks is an important measure, it is important to study the fault tolerance on networks. In the previous section we examined the fault tolerance in the context of conditional hamiltonicity. Now we give an example showing that our result is sharp.

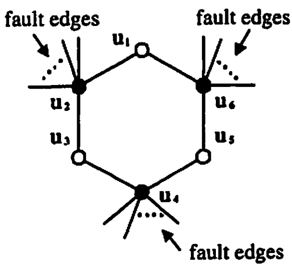

Figure 2: Illustration of S_n with $(3n - 9)$ edge faults

First consider conditional hamiltonicity. Let $n \geq 4$, and let $H = \langle \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_6, \mathbf{u}_1\rangle$ be a cycle with six vertices in S_n. We set the faulty set F to be all the edges incident with $\mathbf{u}_2$, $\mathbf{u}_4$, and $\mathbf{u}_6$ that are not edges of H shown in Figure 2. Obviously, $|F| = 3n - 9$. Since $\deg_{S_n - F}(\mathbf{v}) = 2$ for every vertex in $\{\mathbf{u_2}, \mathbf{u_4}, \mathbf{u_6}\}$, $S_n - F$ is not hamiltonian.

The example show that the result of Theorem 7, namely, that the n-dimensional star graph S_n is $(3n - 10)$-edge-fault conditional hamiltonian for $n \geq 4$ are sharp.

5 Appendix

Here we give proof for the cases omitted from Theorem 7 when $n = 5$:

Case 1: $|A| = 4$.

Suppose that $A = \{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}\}$, and let H be the subgraph with vertex set A containing those edges of F that join two vertices in A. From Lemma 5 we get that H must have exactly three edges, so it is a tree. Since S_5 is 4-regular and $\delta(S_5 - F) \geq 2$, H must be a path of length 3. Without loss of generality, we may assume that $(\mathbf{u_i}, \mathbf{u_{i+1}}) \in F$ for every $i \in \langle 3 \rangle$. Since in $S_5 - F$ the degree of every vertex of A is 2, there is a vertex $\mathbf{u_0} \in V(S_5 - A)$ such that $(\mathbf{u_0}, \mathbf{u_1}) \in F$ and there is a vertex $\mathbf{u_5} \in V(S_5 - A)$ such that $(\mathbf{u_4}, \mathbf{u_5}) \in F$. Since S_5 does not contain any cycle of length 5, $\mathbf{u_0} \neq \mathbf{u_5}$, and we have identified all edges of F.

For each $i \in \langle 5 \rangle$ let α_i be the dimension of edge $(\mathbf{u_{i-1}}, \mathbf{u_i})$, so that $\mathbf{u_i} = (\mathbf{u_{i-1}})^{\alpha_i}$. By the definition of S_5, $\alpha_i \neq \alpha_{i+1}$ for every $i \in \langle 5 \rangle$. Notice that since $2 \leq \alpha_i \leq 5$ for all $i \in \langle 5 \rangle$, the integers α_1, $\alpha_2, \ldots$, α_5 are not all different. By symmetry, either α_1 or α_2 is equal to at least one of the remaining others. Hence we have the following cases depending on which of the α_i's are equal:

Case 1.1: $\alpha_1 = \alpha_3 = \alpha_5$.

Since S_5 is edge transitive, we may assume that $\alpha_1 = 5$. Then by (II), $|F_i| \leq 1$ for every $i \in \langle 5 \rangle$ and $|F_{j,k}| \leq 1$ for every two distinct elements $j, k \in \langle 5 \rangle$. Since $|F_{1,5}| \leq 1$ and $|E^{1,5}| = 6 > 1$, we can choose two vertices $\mathbf{u} \in S_5^{\{1\}}$ and $\mathbf{v} \in S_5^{\{5\}}$ with $(\mathbf{u}, \mathbf{v}) \in E^{1,5} - F_{1,5}$. By Theorem 4, there is a hamiltonian path P of $S_5 - F$ joining $\mathbf{u}$ to $\mathbf{v}$. Then $\langle \mathbf{u}, P, \mathbf{v}, \mathbf{u} \rangle$ is a hamiltonian cycle of $S_5 - F$.

Case 1.2: $\alpha_1 = \alpha_3$ and $\alpha_1 \neq \alpha_5$.

Since S_5 is vertex transitive and edge transitive, we may assume that $\mathbf{u_3} \in S_5^{\{5\}}$ and $\alpha_1 = 5$. Then $|F_i| \leq 1$ for every $i \in \langle 4 \rangle$, $F_5 = \{(\mathbf{u_3}, \mathbf{u_4}), (\mathbf{u_4}, \mathbf{u_5})\}$, and $|F_{j,k}| \leq 1$ for every two distinct elements $j, k \in \langle 5 \rangle$. By Theorem 1, there is a hamiltonian path P of $S_5^{\{5\}} - \{(\mathbf{u_3}, \mathbf{u_4})\}$ joining $\mathbf{u_4}$ to $\mathbf{u_5}$, and by Theorem 4, there is a hamiltonian path Q of $S_5^{\langle 4 \rangle} - F$ joining $(\mathbf{u_5})^5$ to $(\mathbf{u_4})^5$. Then $\langle \mathbf{u_4}, P, \mathbf{u_5}, (\mathbf{u_5})^5, Q, (\mathbf{u_4})^5, \mathbf{u_4} \rangle$ is a hamiltonian cycle of $S_5 - F$.

Case 1.3: $\alpha_1 = \alpha_4$.

Again, we may assume that $\mathbf{u}_1 \in S_5^{\{5\}}$ and $\alpha_1 = 5$. Then by (II), $|F_i| \leq 1$ for every $i \in \langle 4 \rangle$, $F_5 = \{(\mathbf{u}_1, \mathbf{u}_2), (\mathbf{u}_2, \mathbf{u}_3)\}$, and $|F_{j,k}| \leq 1$ for every two distinct elements $j, k \in \langle 5 \rangle$. By Theorem 1, there is a hamiltonian path P of $S_5^{\{5\}} - \{(\mathbf{u}_1, \mathbf{u}_2)\}$ joining $\mathbf{u}_2$ to $\mathbf{u}_3$. Since in $S_5^{\{5\}}$ every vertex has degree 3, $\mathbf{u}_3$ has a neighbor $\mathbf{x}$ different from the two vertices adjacent to it on P. Thus we can write P as $\langle \mathbf{u}_2, P_1, \mathbf{x}, \mathbf{y}, P_2, \mathbf{u}_3 \rangle$, where $\mathbf{y} \neq \mathbf{u}_3$ and $\mathbf{y} \neq \mathbf{u}_1$. By Theorem 4, there is a hamiltonian path Q of $S_5^{\langle 4 \rangle} - F$ joining $(\mathbf{y})^5$ to $(\mathbf{u}_2)^5$, hence $\langle \mathbf{u}_2, P_1, \mathbf{x}, \mathbf{u}_3, P_2^{-1}, \mathbf{y}, (\mathbf{y})^5, Q, (\mathbf{u}_2)^5, \mathbf{u}_2 \rangle$ is a hamiltonian cycle of $S_5 - F$.

Case 1.4: $\alpha_1 = \alpha_5$ and $\alpha_1 \neq \alpha_3$.

Again, we may assume that $\mathbf{u}_1 \in S_5^{\{5\}}$ and $\alpha_1 = 5$. Then $|F_i| \leq 1$ for every $i \in \langle 4 \rangle$, $F_5 = \{(\mathbf{u}_1, \mathbf{u}_2), (\mathbf{u}_2, \mathbf{u}_3), (\mathbf{u}_3, \mathbf{u}_4)\}$, and $|F_{j,k}| \leq 2$ for every two distinct $j, k \in \langle 5 \rangle$. Since $S_n^{\{5\}}$ is isomorphic to S_4, the case $n = 4$ in Theorem 7 implies that there is a hamiltonian cycle C of $S_5^{\{5\}} - \{(\mathbf{u}_1, \mathbf{u}_2), (\mathbf{u}_3, \mathbf{u}_4)\}$. Obviously, edge $(\mathbf{u}_2, \mathbf{u}_3)$ is part of C, so we can write C as $\langle \mathbf{u}_2, P, \mathbf{u}_3, \mathbf{u}_2 \rangle$. By Theorem 4, there is a hamiltonian path Q of $S_5^{\langle 4 \rangle} - F$ joining $(\mathbf{u}_3)^5$ to $(\mathbf{u}_2)^5$. Then $\langle \mathbf{u}_2, P, \mathbf{u}_3, (\mathbf{u}_3)^5, Q, (\mathbf{u}_2)^5, \mathbf{u}_2 \rangle$ is a hamiltonian cycle of $S_5 - F$.

Case 1.5: $\alpha_2 = \alpha_4$.

Again, we may assume that $\alpha_2 = 5$. By (II), we get $|F_i| \leq 1$ for every $i \in \langle 5 \rangle$ and $|F_{j,k}| \leq 1$ for every two distinct $j, k \in \langle 5 \rangle$. The rest of the proof is the same as that of Case 1.1.

Case 2.1.1 $\delta(S_n^{\{i\}} - F_i) \geq 2$ for every $i \in \langle n \rangle$

When $n = 5$, we have $\frac{(n-2)!}{2} = 3 = n - 2$, hence there may be a pair $i, j \in \langle 5 \rangle$ such that $|F_{i,j}| = \frac{(n-2)!}{2} = 3$. However, in that case $|F_4| = |F_5| = 1$ and $|F_{4,5}| \geq 1$, so $|F| = 5$ and $|F_{i,j}| = 3$ implies that $|F_{4,5}| = 3$. Hence $\{a, b\} = \{4, 5\}$ and $|F_{i,j}| = 0$ for every distinct $i, j \in \langle 5 \rangle - \{a, b\}$, and the argument for $n \geq 6$ applies to this case as well.

Case 2.2.2: $(\mathbf{v})_n = n$ and $(\mathbf{v}, (\mathbf{v})^n) \notin F$.

If $n = 5$, and there is no integer $2 \leq i \leq 5$ such that $(\mathbf{u}, (\mathbf{u})^i) \in F$ and $(\mathbf{v}, (\mathbf{v})^i) \in F$, then the four edges of F incident to $\mathbf{u}$ and $\mathbf{v}$ have all different dimensions. Let i be the dimension of the fifth edge in F, and define the sets $S_5^{\{j\}}$ for $j \in \langle n \rangle$ using vertices of having the same i-th coordinate instead of the same n-th coordinate. Then exactly one of $\mathbf{u}$ and $\mathbf{v}$ will have its edge incident to it having dimension i belong to F. By symmetry, we can assume it is $\mathbf{u}$. Let $t = (\mathbf{v})_i$. Then $(\mathbf{u}, (\mathbf{u})^i) \in F$ and $(\mathbf{v}, (\mathbf{v})^i) \notin F$, so $\delta(S_5^{\{j\}} - F_j) \geq 2$ and $|F_j| \leq 1$ for $j \in \langle 5 \rangle - \{t\}$, $|F_t| \leq 3$, and $|F_{j,k}| \leq 2$ for every distinct $j, k \in \langle 5 \rangle$. We can choose an edge $(\mathbf{v}, \mathbf{z}) \in F_t$, and then $\mathbf{z} \neq \mathbf{u}$ since $|A| = 2$. By induction, there is a hamiltonian cycle C of $S_5^{\{t\}} - (F_t - \{(\mathbf{v}, \mathbf{z})\})$. Clearly, edge $(\mathbf{v}, \mathbf{z})$ must be in C, so we can write $C = \langle \mathbf{v}, R, \mathbf{z}, \mathbf{v} \rangle$. By Theorem 4, there is a hamiltonian path H of

$S_5^{\langle 5\rangle-\{t\}} - F$ joining $(\mathbf{z})^i$ to $(\mathbf{v})^i$, and then $\langle \mathbf{v}, R, \mathbf{z}, (\mathbf{z})^i, H, (\mathbf{v})^i, \mathbf{v}\rangle$ forms a hamiltonian cycle of $S_5 - F$.

Case 3.2: $|F_n| \leq 3n - 13$ and $|F_{n-1}| \leq n - 4$.

We have $|F_{i,j}| \geq 3$ for some $i, j \in \langle 4\rangle$. Clearly, $3 \leq |F_{i,j}| \leq 5$ and $|F_{s,t}| \leq 2$ for every two distinct elements $s, t \in \langle 5\rangle$ such that $\{s, t\} \neq \{i, j\}$.

If $|F_5| \leq 1$, then by (I), we can choose an edge $(\mathbf{u}, \mathbf{v})$ in $E^{i,j} - F_{i,j}$. By Theorem 4, there is a hamiltonian path P of $S_5 - F$ joining $\mathbf{u}$ to $\mathbf{v}$. Thus, $\langle \mathbf{u}, P, \mathbf{v}, \mathbf{u}\rangle$ is a hamiltonian cycle of $S_5 - F$.

On the other hand, if $|F_5| = 2$, then every edge of F is accounted for. By induction, there is a hamiltonian cycle C of $S_5^{\{5\}} - F_5$. Without loss of generality, we can write $C = \langle \mathbf{u}, \mathbf{v}, P, \mathbf{w}, \mathbf{u}\rangle$ where $\mathbf{u}$ is a white vertex with $(\mathbf{u})_1 = i$. By (II), $(\mathbf{v})_1 \neq i$, $(\mathbf{w})_1 \neq i$, and $(\mathbf{v})_1 \neq (\mathbf{w})_1$. Without loss of generality, we may assume that $(\mathbf{w})_1 \neq j$. By Theorem 1, there is a hamiltonian path Q of $S_5^{\{i\}}$ joining the black vertex $(\mathbf{u})^5$ to a white vertex $\mathbf{z}$ with $(\mathbf{z})_1 \in \langle 4\rangle - \{i, j, (\mathbf{z})_1\}$. By Theorem 4, there is a hamiltonian path R of $S_5^{\langle 4\rangle-\{i\}} - F$ joining the black vertex $(\mathbf{z})^5$ to the white vertex $(\mathbf{w})^5$. Then $\langle \mathbf{u}, \mathbf{v}, P, \mathbf{w}, (\mathbf{w})^5, R^{-1}, (\mathbf{z})^5, \mathbf{z}, Q^{-1}, (\mathbf{u})^5, \mathbf{u}\rangle$ is a hamiltonian cycle of $S_5 - F$.

References

[1] S. B. Akers and B. Krishnamurthy, A group-theoretic model for symmetric interconnection networks, IEEE Transactions on Computers 38 (1989), 555–566.

[2] M. Albert, R. E. L. Aldred, D. Holton, and J. Sheehan, On 3*-connected graphs, Australasian Journal of Combinatorics 24 (2001), 193–207.

[3] N. Bagherzadeh, M. Dowd, and N. Nassif, Embedding an arbitrary binary tree into the star graph, IEEE Transactions on Computers 45 (1996), 475–481.

[4] J. A. Bondy and U. S. R. Murty, *Graph theory with applications*, North Holland, New York, 1980.

[5] A. Bouabdallah, M. C. Heydemann, J. Opatrny, and D. Sotteau, Embedding complete binary trees into star and pancake graphs, Theory of Computing Systems 31 (1998), 279–305.

[6] M. Y. Chan and S.-J. Lee, On the existence of Hamiltonian circuits in faulty hypercubes, SIAM Journal on Discrete Mathematics 4 (1991), 511–527.

[7] C.-H. Chang, C.-K. Lin, H.-M. Huang, and L.-H. Hsu, The super laceability of the hypercube, Information Processing Letters 92 (2004), 15–21.

[8] P. Fragopoulou and S. G. Akl, Optimal communication algorithms on the star graphs using spanning tree constructions, Journal of Parallel and Distributed Computing 24 (1995), 55–71.

[9] P. Fragopoulou and S. G. Akl, Edge-disjoint spanning trees on the star networks with applications to fault tolerance, IEEE Transactions on Computers 45 (1996), 174–185.

[10] J.-S. Fu, Conditional fault-tolerant hamiltonicity of star graphs, Parallel Computing (2007), doi: 10.1016/j.parco.2007.02.007.

[11] Q. P. Gu and S. Peng, Node-to-node cluster fault tolerant routing in star graphs, Information Processing Letters 56 (1995), 29–35.

[12] S.-Y. Hsieh, Embedding longest fault-free paths onto star graphs with more vertex faults, Theoretical Computer Science 337 (2005), 370–378.

[13] S.-Y. Hsieh, G.-H. Chen, C.-W. Ho, Fault-free hamiltonian cycles in faulty arrangement graphs, IEEE Transactions on Parallel and Distributed Systems 10 (32) (1999) 223–237.

[14] S.-Y. Hsieh, G.-H. Chen, and C.-W. Ho, Hamiltonian-laceability of star graphs, Networks 36 (2000), 225–232.

[15] S.-Y. Hsieh, G.-H. Chen, and C.-W. Ho, Longest fault-free paths in star graphs with vertex faults, Theoretical Computer Science 262 (2001), 215–227.

[16] S.-Y. Hsieh, G.-H. Chen, and C.-W. Ho, Longest fault-free paths in star graphs with edge faults, IEEE Transactions on Computers 50 (2001), 960–971.

[17] D. F. Hsu, On container width and length in graphs, groups, and networks, IEICE Transactions Fundamentals E77-A (1994), 668–680.

[18] J. S. Jwo, S. Lakshmivarahan, and S. K. Dhall, Embedding of cycles and grids in star graphs, Journal of Circuits, Systems, and Computers 1 (1991), 43–74.

[19] S. Latifi, S. Q. Zheng, and N. Bagherzadeh, Optimal ring embedding in hypercubes with faulty links, in Proceedings of the IEEE Symposium on Fault-Tolerant Computing (1992), 178–184.

[20] S. Latifi, On the fault-diameter of the star graph, Information Processing Letters 46 (1993), 143–150.

[21] F. T. Leighton, *Introduction to parallel algorithms and architectures: arrays · trees · hypercubes*, Morgan Kaufmann, CA, San Mateo, 1992.

[22] T.-K. Li, J. J. M. Tan, and L.-H. Hsu, Hyper hamiltonian laceability on edge fault star graph, Information Sciences 165 (2004), 59–71.

[23] C.-K. Lin, H.-M. Huang, and L.-H. Hsu, The super connectivity of the pancake graphs and the super laceability of the star graphs, Theoretical Computer Science 339 (2005), 257–271.

[24] C.-K. Lin, H.-M. Huang, L.-H. Hsu, and S. Bau, Mutually Independent Hamiltonian Paths in Star Networks, Networks 46 (2005), 100–117.

[25] C.-K. Lin, H.-M. Huang, D. F. Hsu, and L.-H. Hsu, On the spanning w-wide diameter of the star graph, Networks 48 (2006), 235–249.

[26] K. Menger, Zur allgemeinen Kurventheorie, Fundamentale Mathematik 10 (1927), 95–115.

[27] Z. Miller, D. Pritikin, and I. H. Sudborough, Near embeddings of hypercubes into Cayley graphs on the symmetric group, IEEE Transactions on Computers 43 (1994), 13–22.

[28] J. H. Park and H. C. Kim, Longest paths and cycles in faulty star graphs, Journal of Parallel and Distributed Computing 64 (2004), 1286–1296.

[29] Y. Rouskov, S. Latifi, and P. K. Srimani, Conditional fault diameter of star graph networks, Journal of Parallel and Distributed Computing 33 (1996), 91–97.

[30] C.-H. Tsai, J. J. M. Tan, and L.-H. Hsu, The super-connected property of recursive circulant graphs, Information Processing Letters 91 (2004), 293–298.