On Super $(a, 2)$-Edge-Antimagic Total Labeling of Disconnected Graphs

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Abstract

A *labeling* of a graph is a mapping that carries some set of graph elements into numbers (usually the positive integers). An (a, d)-*edge-antimagic total labeling* of a graph with p vertices and q edges is a one-to-one mapping that takes the vertices and edges onto the integers $1, 2, \dots, p+q$, such that the sums of the label on the edges and the labels of their end points form an arithmetic sequence starting from a and having a common difference d. Such a labeling is called *super* if the smallest possible labels appear on the vertices.

In this paper we study the super $(a, 2)$-edge-antimagic total labelings of disconnected graphs. We also present some necessary conditions for the existence of (a, d)-edge-antimagic total labelings for d even.

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1 Introduction

We consider finite undirected graphs without loops and multiple edges. If G is a graph, then $V(G)$ and $E(G)$ stand for the vertex set and edge set of G, respectively.

For a (p,q)-graph G with p vertices and q edges, a bijective function $f : V(G) \cup E(G) \to \{1, 2, \dots, p+q\}$ is called a *total labeling* of G and the associated edge-weight is $w_f(uv) = f(u) + f(uv) + f(v)$ for $uv \in E(G)$.

An *(a,d)-edge-antimagic total labeling* (*(a,d)-EAT* for short) of G is a total labeling with the property that the edge-weights form an arithmetic sequence starting from a and having common difference d, where $a > 0$ and $d \geq 0$ are two given integers. The definition of (a,d)-EAT labeling was introduced by Simanjuntak, Bertault and Miller in [9] as a natural extension of *magic valuation* which are also known as *edge-magic labelings* defined by Kotzig and Rosa in [7]. Kotzig and Rosa [7] showed that all caterpillars have magic valuations and conjectured that all trees have magic valuations. An (a,d)-EAT labeling is called *super* if the smallest possible labels appear on the vertices. For more information on edge-magic and super edge-magic labelings, please see [4] and [11].

A graph that admits an (a,d)-EAT labeling or a super (a,d)-EAT labeling is called an (a,d)-EAT graph or super (a,d)-EAT graph, respectively.

Let a (p,q)-graph be super (a,d)-EAT. It is easy to see that the minimum possible edge-weight is at least $p+4$ and the maximum possible edge-weight is no more than $3p+q-1$. Thus

$$a + (q-1)d \leq 3p + q - 1 \quad \text{and} \quad d \leq \frac{2p+q-5}{q-1}.$$

For any connected (p,q)-graph where $p-1 \leq q$ we have that $d \leq 3$.

In this paper we mainly investigate the existence of super (a,d)-EAT labelings for disconnected graphs. We concentrate on the following problem: If a graph G is (super) $(a,2)$-EAT, is the disjoint union of m copies of the graph G (denoted by mG) (super) $(a,2)$-EAT as well?

2 $(a,0)$-EAT labeling

A vertex labeling $f : V(G) \to \{1, 2, \dots, p\}$ of a (p,q)-graph G is *(a,d)-edge-antimagic vertex* (in short, (a,d)-EAV) if the set of all the edge-weights is $\{a, a+d, a+2d, \dots, a+(q-1)d\}$, for two integers $a > 0$ and $d \geq 0$. A graph that admits an (a,d)-EAV labeling is called an (a,d)-EAV graph. The definition of (a,d)-EAV labeling was given by Simanjuntak, Bertault and Miller [9]. This labeling is called the (a,d)-indexable labeling in [5].

Figueroa-Centeno, Ichishima and Muntaner-Batle in [2] showed that a graph G admits a super edge-magic labeling, in our terminology G is super $(a,0)$-EAT, if and only if G admits a $(a,1)$-EAV labeling. Moreover, in [1], it has been proved that

Proposition 1. *If G has an (a,d)-EAV labeling then*

i) G has a super $(a+|V|+1, d+1)$-EAT labeling,

ii) G has a super $(a+|V|+|E|, d-1)$-EAT labeling.

Proposition 1 allows us to extend the previous known results on super edge-magic labelings onto super $(a-q+1, 2)$-EAT labeling. However, the condition in Proposition 1 is only sufficient for the existence of a super $(a, 2)$-EAT labeling from the existence of a super $(a, 0)$-EAT labeling of a graph. For example, let us consider two copies of a path on three vertices. In [3] it is proved that $2P_3$ is not super $(a, 0)$-EAT, but it is super $(a, 2)$-EAT, see Figure 1. Note, that $2P_3$ is $(a, 0)$-EAT, see Figure 2.

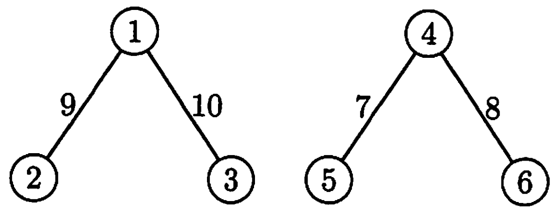

Figure 1: Super $(12, 2)$-EAT labeling of $2P_3$.

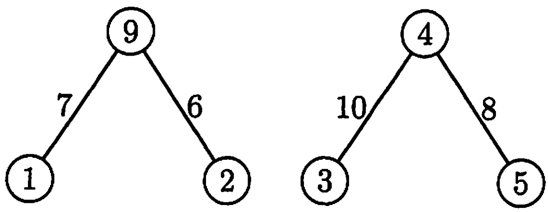

Figure 2: The $(17, 0)$-EAT labeling of $2P_3$.

3 $(a, 2)$-EAT labeling

In [3] Figueroa-Centeno, Ichishima and Muntaner-Batle proved

Proposition 2. *If G is a (super) edge-magic bipartite or tripartite graph and m is odd, then mG is (super) edge-magic.*

It means, that if G is a super edge-magic tripartite graph and m is odd, then mG is super $(a, 2)$-EAT. We are able to extend this result. The main theorem of our paper is

Theorem 1. *If G is a (super) $(a,2)$-EAT tripartite graph and m is odd, then mG is (super) $(a',2)$-EAT.*

Proof. Let G be (super) $(a,2)$-EAT tripartite (p,q)-graph with the partite sets V_1, V_2 and V_3. Then $E(G) = V_1V_2 \cup V_2V_3 \cup V_1V_3$, where the juxtaposition of two partite sets denotes the set of edges between those two sets. Let $f : V(G) \cup E(G) \longrightarrow \{1, 2, \ldots, p+q\}$ be a (super) $(a,2)$-EAT labeling of G.

By the symbol x_i we denote the element (a vertex or an edge) in the ith copy of mG corresponding to the element $x \in V(G) \cup E(G)$.

We define a new labeling g of mG, for m odd, in the following way.

$$g(x_i) = \begin{cases} m[f(x)-1]+i & \text{if } x \in V_1 \cup V_2V_3, \\ m[f(x)-1]+i+\frac{m+1}{2} & \text{if } x \in V_2 \cup V_1V_3 \text{ and } i < \frac{m}{2}, \\ m[f(x)-1]+i-\frac{m+1}{2}+1 & \text{if } x \in V_2 \cup V_1V_3 \text{ and } i > \frac{m}{2}, \\ m[f(x)-1]+2i & \text{if } x \in V_3 \cup V_1V_2 \text{ and } i < \frac{m}{2}, \\ m[f(x)-1]+2i-m & \text{if } x \in V_3 \cup V_1V_2 \text{ and } i > \frac{m}{2}. \end{cases}$$

Let $t \in \{1, 2, \ldots, p+q\}$. We consider the following three cases:

Case 1. If the number t is assigned by the labeling f to some element of $V_1 \cup V_2V_3$, then the corresponding elements in the copies of G in mG have labels

$m(t-1)+1,$	$m(t-1)+2,$	$\ldots$	$m(t-1)+i,$	$\ldots$	mt
in G_1	in G_2	$\ldots$	in G_i	$\ldots$	in G_m

i.e. the numbers $m(t-1)+1, m(t-1)+2, \ldots, mt$.

Case 2. If the number t is assigned by the labeling f to some element of $V_2 \cup V_1V_3$, then the corresponding elements in the copies of G in mG receive labels

$mt + \frac{3-m}{2}$	in G_1
$mt + \frac{5-m}{2}$	in G_2
$\vdots$	$\vdots$
mt	in $G_{\frac{m-1}{2}}$
$m(t-1)+1$	in $G_{\frac{m+1}{2}}$
$m(t-1)+2$	in $G_{\frac{m+3}{2}}$
$\vdots$	$\vdots$
$mt + \frac{1-m}{2}$	in G_m

thus the numbers $m(t-1)+1, m(t-1)+2, \ldots, mt$.

Case 3. If the number t is assigned by the labeling f to some element of $V_3 \cup V_1V_2$, then the corresponding elements in the copies of G in mG receive labels

$$
\begin{array}{ll}
m(t-1)+2 & \text{in } G_1 \\
m(t-1)+4 & \text{in } G_2 \\
\vdots & \vdots \\
mt-1 & \text{in } G_{\frac{m-1}{2}} \\
m(t-1)+1 & \text{in } G_{\frac{m+1}{2}} \\
m(t-1)+3 & \text{in } G_{\frac{m+3}{2}} \\
\vdots & \vdots \\
mt & \text{in } G_m
\end{array}
$$

hence the corresponding labels are $m(t-1)+1, m(t-1)+2, \ldots, mt$.

Thus the set of the labels in mG corresponding to the value t is independent on the labeled element. It means the labeling g is evidently total and assigns the number $1, 2, \ldots, m(p+q)$ to the elements of mG. Moreover, if the labeling f is super, then also the smallest possible labels are used to label the vertices in mG and thus g is also super.

In the next part we will calculate the edge-weight of an edge $uv \in E(G_i)$. We again distinguish three cases. If $u \in V_1^i$ and $v \in V_2^i$, if $u \in V_1^i$ and $v \in V_3^i$ and if $u \in V_2^i$ and $v \in V_3^i$. By the symbol V_j^i, $j = 1, 2, 3$ and $i = 1, 2, \ldots, m$, we denote the vertex set corresponding to the vertex set V_j in the ith copy of G.

It is easy to verify that in all cases we obtain for the edge-weights

$$
g(u_i) + g(v_i) + g(u_i v_i) =
$$

$$
\begin{cases}
m[f(u) + f(v) + f(uv) - 3] + \frac{m+1}{2} + 4i & \text{if } i < \frac{m}{2}, \\
m[f(u) + f(v) + f(uv) - 3] + \frac{m+1}{2} - 2m + 4i & \text{if } i > \frac{m}{2}.
\end{cases}
$$

Thus to the edge-weight A of some edge uv in G, $A = f(u) + f(v) + f(uv)$, corresponds to the following edge-weights in mG

$$
\begin{array}{ll}
m(A-3) + \frac{m+1}{2} + 4 & \text{in } G_1 \\
m(A-3) + \frac{m+1}{2} + 8 & \text{in } G_2 \\
\vdots & \vdots \\
m(A-3) + \frac{m+1}{2} + 2m - 2 & \text{in } G_{\frac{m-1}{2}} \\
m(A-3) + \frac{m+1}{2} + 2 & \text{in } G_{\frac{m+1}{2}} \\
m(A-3) + \frac{m+1}{2} + 6 & \text{in } G_{\frac{m+3}{2}} \\
\vdots & \vdots \\
m(A-3) + \frac{m+1}{2} + 2m & \text{in } G_m.
\end{array}
$$

It means that the edge-weights are

$$
m(A-3) + \frac{m+1}{2} + 2, m(A-3) + \frac{m+1}{2} + 4, \ldots, m(A-1) + \frac{m+1}{2}.
$$

As f is $(a, 2)$-EAT labeling then the edge-weights in G are

$$a, a+2, a+4, \dots, a+2(q-1).$$

Thus to the edge-weight $A+2$ in g the corresponding edge-weights in mG are

$$m(A-1)+\frac{m+1}{2}+2, m(A-1)+\frac{m+1}{2}+4, \dots, m(A+1)+\frac{m+1}{2},$$

hence the edge-weights in mG again form an arithmetic sequence with the difference 2 and the initial term $m(a-3)+\frac{m+1}{2}+2$. This concludes the proof. □

Immediately from the previous theorem we get the following result

Corollary 1. *If G is a (super) $(a, 2)$-EAT bipartite graph and m is odd, then mG is (super) $(a', 2)$-EAT.*

In [6] Ivančo and Lučkaničová proved a more general result than the one in the Proposition 2 for disjoint union of edge-magic graphs. A mapping $c : V(G) \cup E(G) \longrightarrow \{1, 2, 3\}$ is called an *e-m-coloring* of a graph G if $\{c(u), c(v), c(uv)\} = \{1, 2, 3\}$ for any edge uv of G. They proved

Proposition 3. *Let m be an odd positive integer. For $i = 1, 2, \dots, m$, let G_i, g_i and c_i be an edge-magic graph with p_i vertices and q_i edges, an edge-magic total labeling of G_i with its magic number σ_i and an e-m-coloring of G_i, respectively. Suppose that the following conditions are satisfied*

1. *there is an integer σ such that $\sigma_i = \sigma$ for all $i = 1, 2, \dots, m$,*
2. *if $g_i(x) = g_j(y)$, then $c_i(x) = c_j(y)$ for all $i, j = 1, 2, \dots, m$, $x \in V(G_i) \cup E(G_i)$ and $y \in V(G_j) \cup E(G_j)$,*
3. *there is an integer r such that $r = p_1 + q_1 \geq \dots \geq p_m + q_m \geq r - 1$.*

Then the disjoint union $\cup_{i=1}^{m} G_i$ is an edge-magic graph.

Moreover, if all g_i are super edge-magic labelings and $p_1 = p_2 = \dots = p_m$, then $\cup_{i=1}^{m} G_i$ is a super edge-magic graph.

It is not difficult to prove a similar result for (super) $(a, 2)$-EAT graphs.

Theorem 2. *Let m be an odd positive integer. For $i = 1, 2, \dots, m$, let G_i, f_i and c_i be an $(a, 2)$-EAT graph with p_i vertices and q_i edges, an $(a, 2)$-EAT labeling of G_i and an e-m-coloring of G_i, respectively. Suppose that the following conditions are satisfied*

1. *if $f_i(x) = f_j(y)$, then $c_i(x) = c_j(y)$ for all $i, j = 1, 2, \dots, m$, $x \in V(G_i) \cup E(G_i)$ and $y \in V(G_j) \cup E(G_j)$,*

2. there is an integer r such that $r = p_1 + q_1 \geq \cdots \geq p_m + q_m \geq r - 1$.

Then the disjoint union $\cup_{i=1}^{m} G_i$ is an $(a', 2)$-EAT graph.

Moreover, if all f_i are super $(a, 2)$-EAT labelings and $p_1 = p_2 = \cdots = p_m$, then $\cup_{i=1}^{m} G_i$ is a super $(a', 2)$-EAT graph.

Proof. For $i = 1, 2, \ldots, m$, let G_i, f_i and c_i be an $(a, 2)$-EAT graph with p_i vertices and q_i edges, an $(a, 2)$-EAT labeling of G_i and an e-m-coloring of G_i, respectively.

We define a new labeling g of $\cup_{i=1}^{m} G_i$, for m odd, in the following way.

$$g(x_i) = \begin{cases} m[f_i(x) - 1] + i & \text{if } c_i(x) = 1, \\ m[f_i(x) - 1] + i + \frac{m+1}{2} & \text{if } c_i(x) = 2 \text{ and } i < \frac{m}{2}, \\ m[f_i(x) - 1] + i - \frac{m+1}{2} + 1 & \text{if } c_i(x) = 2 \text{ and } i > \frac{m}{2}, \\ m[f_i(x) - 1] + 2i & \text{if } c_i(x) = 3 \text{ and } i < \frac{m}{2}, \\ m[f_i(x) - 1] + 2i - m & \text{if } c_i(x) = 3 \text{ and } i > \frac{m}{2}. \end{cases}$$

According to condition (1) it is not difficult to check that the labeling g uses each integer $1, 2, \ldots, |V(\cup_{i=1}^{m} G_i) \cup E(\cup_{i=1}^{m} G_i)|$ exactly once. As f_i is an $(a, 2)$-EAT labeling of G_i and c_i is an e-m-coloring of G_i, then analogously as in the proof of Theorem 1 we show, that $\cup_{i=1}^{m} G_i$ is an $(a', 2)$-EAT graph. Moreover, if all f_i are super $(a, 2)$-EAT labelings and $p_1 = p_2 = \cdots = p_m$ then $1 \leq g(u) \leq (p_i - 1)m + m = |V(\cup_{i=1}^{m} G_i)|$. Thus, g is a super $(a', 2)$-EAT labeling, too. □

In the literature there are some known conditions for the non-existence of the (a, d)-EAT labelings for some graphs depending on the order and the size of a graph. In [9] it is proved

Proposition 4. *Let G be a graph with all vertices of odd degrees. If $|E(G)| \equiv 0 \pmod 4$ and $|V(G)| \equiv 2 \pmod 4$ then G has no (a, d)-EAT labeling.*

Moreover, for $(a, 0)$-EAT graphs, the following is proved in [8]

Proposition 5. *Let G be a graph with all vertices of odd degrees. If $|E(G)| \equiv 0 \pmod 2$ and $|V(G)| + |E(G)| \equiv 2 \pmod 4$ then G has no $(a, 0)$-EAT labeling.*

These results are based on the arguments using divisibility. More precisely, we get the following lemma

Lemma 1. *Let G be a graph with all vertices of odd degrees and let d be an even integer. If one of the following conditions holds*

i) $|E(G)| \equiv 0 \pmod 4$ and $|V(G)| \equiv 1 \pmod 4$ or $|V(G)| \equiv 2 \pmod 4$, *ii)* $|E(G)| \equiv 2 \pmod 4$ *and* $|V(G)| \equiv 0 \pmod 4$ *or* $|V(G)| \equiv 3 \pmod 4$

then G has no (a,d)*-EAT labeling.*

Proof. If a graph G has an (a,d)-EAT labeling f then it is known that the following connection between the order, the size, the degrees of a graph and the parameters a and d must hold

$$\sum_{uv \in E(G)} w_f(uv) = \sum_{uv \in E(G)} f(uv) + \sum_{v \in V(G)} deg(v) f(v).$$

Thus we get

$$a|E(G)| + \frac{|E(G)|(|E(G)|-1)d}{2}$$
$$= \frac{(|V(G)|+|E(G)|)(|V(G)|+|E(G)|+1)}{2} + \sum_{v \in V(G)} (deg(v)-1) f(v).$$

If G is a graph with all vertices of odd degree, d is even and if one of the conditions in the lemma holds then using the parity considerations on the left hand and on the right hand side of the formula we get a contradiction. □

Thus for an even number of copies of a graph we have

Theorem 3. *Let* d, k *be positive integers,* $d \equiv 0 \pmod 2$, $k \equiv 1 \pmod 2$. *Let G be a graph with all vertices of odd degrees. If the size and the order of G have a different parity then the graph* $2kG$ *has no* (a,d)*-EAT labeling.*

Proof. Consider a graph $2kG$, where G is a graph with all vertices of odd degrees. Let $k \equiv 1 \pmod 2$.

If the size is odd and the order is even then

$$|E(2kG)| = 2k|E(G)| \equiv 2 \pmod 4, \ |V(2kG)| = 2k|V(G)| \equiv 0 \pmod 4.$$

If the size is even and the order is odd then

$$|E(2kG)| = 2k|E(G)| \equiv 0 \pmod 4, \ |V(2kG)| = 2k|V(G)| \equiv 2 \pmod 4.$$

Thus according to Lemma 1 the graph $2kG$ is not (a,d)-EAT for d even. □

For example, let us consider a star $K_{1,n}$. In [10] it is proved that every star is super $(a,0)$-EAT and super $(a,2)$-EAT. As the star is a bipartite graph then the odd number of copies of a star $K_{1,n}$ is super $(a,0)$-EAT and super $(a,2)$-EAT according to Proposition 2 and Theorem 1. However, for $n \equiv 1 \pmod 2$ using Theorem 3 we get that $(4k+2)$ copies of $K_{1,n}$ is neither $(a,0)$-EAT nor $(a,2)$-EAT. Thus also the graph G is (a,d)-EAT for $d \equiv 0 \pmod 2$, in many cases there exist no such a labeling of even number of copies of G. This indicates that there exists no general construction of (a,d)-EAT labeling for even number of copies of a graph for d even.

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