# FIBONACCI MODULES AND MULTIPLE FIBONACCI SEQUENCES

ABDUL RAUF NIZAMI
*Abdus Salam School of Mathematical Sciences,*
*GC University, Lahore-Pakistan.*
*e-mail: arnizami@yahoo.com*

ABSTRACT. Double Fibonacci sequences $(x_{n,k})$ are introduced and they are related to operations with Fibonacci modules. Generalizations and examples are also discussed.

## 1. Introduction

Let us fix a commutative ring $\mathcal{R}$; $\mathcal{R}^2$ will denote the rank 2 free $\mathcal{R}$-module and also the product ring $\mathcal{R} \times \mathcal{R}$. The main object of study is the Fibonacci module of type $(a, b) \in \mathcal{R}^2$ associated to the $\mathcal{R}$-module M:

**Definition 1.1.** $\mathcal{F}_{\mathrm{M}}(a,b)$ is the set of sequences $\{(x_n)_{n\geq 0} : x_n \in \mathrm{M},\ x_{n+2} = ax_{n+1} + bx_n, \forall n \geq 0\}$. If $\mathrm{M} = \mathcal{R}$, we use the shorter notation $\mathcal{F}(a, b)$.

**Remark 1.2.** Using the $\mathcal{R}[T]$ structure of the $\mathcal{R}$-module of all sequences in M: $\mathcal{S}_{\mathrm{M}} = \{(x_n)_{n\geq 0} : x_n \in \mathrm{M}\}$, where the action $T$ is given by the shift $T(x_0, x_1, x_2, \ldots) = (x_1, x_2, x_3, \ldots)$, one can describe $\mathcal{F}_{\mathrm{M}}(a, b)$ as the sub $\mathcal{R}[T]$-module $\ker(T^2 - aT - b)$. We also consider $\tilde{\mathcal{F}}_{\mathrm{M}}(a,b) = \{(x_n)_{n\in \mathrm{Z}} : x_n \in \mathrm{M}, x_{n+2} = ax_{n+1} + bx_n, \forall n\}$.

It is well known (at least in the vector space case) that $\mathcal{F}(a, b)$ is a free $\mathcal{R}$-module of rank 2; more generally:

**Proposition 1.3.**

$$\mathcal{F}_{\mathrm{M}}(a,b) \cong \mathrm{M} \oplus \mathrm{M} \cong \mathcal{F}(a,b) \otimes \mathrm{M}.$$

An explicit basis can be found for $\mathcal{F}_{\mathrm{M}}(a, b)$ (see, for example, [2] in which Lucas functions are used):

**Proposition 1.4.** *The sequences* $\left(P_0^{[n]}(a,b)\right)_{n\geq 0}$ *and* $\left(P_1^{[n]}(a,b)\right)_{n\geq 0}$ *in* $\mathcal{F}(a,b)$ *defined by* $P_0^{[0]}(a,b) = 1$, $P_0^{[1]}(a,b) = 0$, *respectively by* $P_1^{[0]}(a,b) = 0$, $P_1^{[1]}(a,b) = 1$, *and by* $P_i^{[n+2]}(a,b) = aP_i^{[n+1]}(a,b) + bP_i^{[n]}(a,b)$ $(i = 0,1)$ *give a canonical basis of the* $\mathcal{R}$*-module* $\mathcal{F}(a,b)$.

---

This research is partially supported by Higher Education Commission, Pakistan.

ARS COMBINATORIA 113(2014), pp. 151-159

Standard operations with modules give the following:

**Proposition 1.5.** *a) There is a natural $\mathcal{R}[T]$-module isomorphism:*

$$\mathcal{F}_{\mathbf{M}}(a,b) \oplus \mathcal{F}_{\mathbf{N}}(a,b) \cong \mathcal{F}_{\mathbf{M}\oplus\mathbf{N}}(a,b)\,.$$

*b) There is a natural $\mathcal{R}^2$-module isomorphism:*

$$\mathcal{F}_{\mathbf{M}}(a,b) \oplus \mathcal{F}_{\mathbf{N}}(c,d) \cong \mathcal{F}_{\mathbf{M}\oplus\mathbf{N}}\big((a,c),(b,d)\big)\,.$$

In order to describe multiplicative operations (tensor product, symmetric power, exterior power), we introduce double Fibonacci sequences.

**Definition 1.6.** The double sequence $(x_{n,k})_{n,k\geq 0}$, $x_{n,k} \in \mathbf{M}$ is a double Fibonacci sequence of type $(a,b)\otimes(c,d) \in \mathcal{R}^2 \otimes \mathcal{R}^2$ if for any $n,k \geq 0$ we have:

$$x_{n+2,k} = a x_{n+1,k} + b x_{n,k}\,,$$
$$x_{n,k+2} = c x_{n,k+1} + d x_{n,k}\,.$$

As an example, let us consider the element in $\mathcal{F}^{[2]}_{\mathbf{Z}}\big((1,1)\otimes(1,3)\big)$ with $x_{0,0} = x_{1,0} = x_{1,1} = 1$ and $x_{0,1} = 0$ (we locate the terms in the first quadrant):

$$\begin{array}{ccccc}
\vdots & \vdots & \vdots & \vdots & \\
3 & 7 & 10 & 17 & \cdots \\
3 & 4 & 7 & 11 & \cdots \\
0 & 1 & 1 & 2 & \cdots \\
1 & 1 & 2 & 3 & \cdots
\end{array}$$

The set of double Fibonacci sequences is denoted by $\mathcal{F}^{[2]}_{\mathbf{M}}\big((a,b)\otimes(c,d)\big)$ and it is naturally an $\mathcal{R}[H,V]$-module ($H$, $V$ are horizontal and vertical shifts: $H(x_{n,k}) = (x_{n+1,k})$, respectively $V(x_{n,k}) = (x_{n,k+1})$). If $(a,b) = (c,d)$ we use the simplified notation $\mathcal{F}^{[2]}_{\mathbf{M}}(a,b)$. In [3] double sequences $(x_{n,k})$ given by a different recurrency are considered: $x_{n,k}$ depends linearly on the terms $\{x_{i,j}\}_{i+j<n+k}$. In our definition, $x_{n,k}$ depends on $x_{n-1,k}$ and $x_{n-2,k}$ and also depends on $x_{n,k-1}$ and $x_{n,k-2}$, using two different relations. Even the existence of a sequence with prescribed initial four terms $x_{i,j}$, $(i,j) \in \{0,1\}^2$, is not an obvious fact. Now we present some properties and operations with these sequences.

In Section 2 the proofs of the previous results are given. In Section 3 we generalize these results in two directions: we consider higher order linear recurrency:

$$x_{n+d} = a_1 x_{n+d-1} + \cdots + a_d x_n\,,$$

and also we consider multiple sequences: $(x_{n_1,n_2,\ldots,n_d})_{n_i\geq 0}$.

In the last section examples of double Fibonacci sequences are given and also an interesting property of their diagonals is presented.

**Proposition 1.7.** *There is a natural isomorphism of $\mathcal{R}[H,V]$-modules:*

$$\mathcal{F}_{\mathrm{M}}(a,b) \otimes_{\mathcal{R}} \mathcal{F}_{\mathrm{N}}(c,d) \cong \mathcal{F}^{[2]}_{\mathrm{M}\otimes\mathrm{N}}\big((a,b)\otimes(c,d)\big)\,.$$

**Corollary 1.8.** *The module $\mathcal{F}^{[2]}\big((a,b)\otimes(c,d)\big)$ is a free $\mathcal{R}$-module of rank 4. In general, $\mathcal{F}^{[2]}_{\mathrm{M}\otimes\mathrm{N}}\big((a,b)\otimes(c,d)\big)$ is isomorphic to $(\mathrm{M}\otimes\mathrm{N})^4$.*

An explicit basis of $\mathcal{F}^{[2]}\big((a,b)\otimes(c,d)\big)$ is given by the four sequences $\big(P^{[n,k]}_{i,j}(a,b)\otimes(c,d)\big)_{n,k\geq 0} = \big(P^{[n]}_i(a,b)P^{[k]}_j(c,d)\big)_{n,k\geq 0}$, where $(i,j)\in\{0,1\}^2$.

The generating function of a double sequence $(x_{n,k})_{n,k}\geq 0$ is the formal series in $\mathcal{R}[[t,s]]\otimes\mathrm{M}\cong\mathrm{M}[[t,s]]$ :

$$G(t,s) = x_{0,0} + x_{1,0}t + x_{0,1}s + \cdots + x_{n,k}t^n s^k + \cdots\,.$$

**Proposition 1.9.** *A Fibonacci sequence $(x_{n,k})$ of type $(a,b)\otimes(c,d)$ has a rational generating function given by*

$$G(t,s) = q(t)^{-1}r(s)^{-1}\left[x_{0,0}(1-at)(1-cs)+x_{1,0}t(1-cs)+x_{0,1}(1-at)s+x_{1,1}ts\right]$$

*where $q(t) = 1-at-bt^2$, $r(s) = 1-cs-ds^2$.*

## 2. Proofs

We can write well-known results on Fibonacci sequences in the following form:

**Lemma 2.1.** *There are polynomials $P^{[n]}_0, P^{[n]}_1 \in \mathcal{R}[T,U]$ such that for any $(x_n)_{n\geq 0}\in\mathcal{F}_{\mathrm{M}}(a,b)$ :*

$$x_n = P^{[n]}_0(a,b)x_0 + P^{[n]}_1(a,b)x_1 \tag{2.1}$$

*for every $n\geq 0$.*

*Proof.* We define $P^{[0]}_0 = 1$, $P^{[1]}_0 = 0$ and $P^{[0]}_1 = 0$, $P^{[1]}_1 = 1$, and $P^{[n+2]}_i = aP^{[n+1]}_i + bP^{[n]}_i$ $(i = 0,1)$. These satisfy the equation (2.1) by definition for $n = 0,1$ and by induction for $n\geq 2$. ☐

**Remark 2.2.** The Lemma 2.1 shows that the $\mathcal{R}$-module $\mathcal{F}(a,b)$ is free of rank 2 with basis $\big(P^{[n]}_0(a,b)\big)_{n\geq 0}$, $\big(P^{[n]}_1(a,b)\big)_{n\geq 0}$.

**Remark 2.3.** If $a = r_1 + r_2$, $b = -r_1r_2$ then one can describe $P^{[n]}_0$ and $P^{[n]}_1$ in the classical way as polynomials in $r_1, r_2$:

$$P^{[n]}_0(r_1+r_2,-r_1r_2) = R^{[n]}_0(r_1,r_2) = -r_1^{n-1}r_2 - r_1^{n-2}r_2^2 - \cdots - r_1r_2^{n-1},$$
$$P^{[n]}_1(r_1+r_2,-r_1r_2) = R^{[n]}_1(r_1,r_2) = r_1^{n-1} + r_1^{n-2}r_2^1 + \cdots + r_2^{n-1},$$

or as rational functions in $r_1, r_2$:

$$R^{[n]}_0(r_1,r_2) = \frac{r_1^n r_2 - r_1 r_2^n}{r_2 - r_1}\,,\quad R^{[n]}_1(r_1,r_2) = \frac{r_2^n - r_1^n}{r_2 - r_1}\,. \tag{2.2}$$

**Remark 2.4.** The previous formulae are also correct in $\widetilde{\mathcal{F}}_{\mathbf{M}}(a,b)$, *i.e.* for negative $n$, if we extend the scalars to a suitable ring of fractions.

For an arbitrary sequence $(x_n)_{n\geq 0}$ in $\mathcal{S}_{\mathbf{M}}$ we define its *generating function* $G(t)$ as a formal series in $\mathcal{R}[[t]] \otimes \mathbf{M} \cong \mathbf{M}[[t]]$ :

$$G(t) = x_0 + x_1 t + x_2 t^2 + \cdots .$$

Another classical result is (see, for example, [1]):

**Lemma 2.5.** *The generating function of the Fibonacci sequence* $(x_n)_{n\geq 0} \in \mathcal{F}_{\mathbf{M}}(a,b)$ *is the rational function*

$$G(t) = \frac{(1-at)x_0 + tx_1}{1-at-bt^2} = q(t)^{-1}\left[x_0 + (x_1 - ax_0)t\right],$$

*where* $q(t) = 1 - at - bt^2$.

*Proof.* [Proposition 1.4] From Lemma 2.1, an arbitrary sequence $(x_n)_{n\geq 0} \in \mathcal{F}(a,b)$ can be written as $(x_n)_{n\geq 0} = \left(P_0^{[n]}(a,b)\right)_{n\geq 0} x_0 + \left(P_1^{[n]}(a,b)\right)_{n\geq 0} x_1$. □

*Proof.* [Proposition 1.3] Define the morphisms

$$\mathcal{F}_{\mathbf{M}}(a,b) \xrightarrow{\varphi} \mathbf{M} \oplus \mathbf{M} \xrightarrow{\psi} \mathcal{F}(a,b) \otimes \mathbf{M} \xrightarrow{\eta} \mathcal{F}_{\mathbf{M}}(a,b)$$

by

$$\varphi\big((x_n)_{n\geq 0}\big) = (x_0, x_1),$$

$$\psi(x_0, x_1) = \left(P_0^{[n]}(a,b)\right)_{n\geq 0} \otimes x_0 + \left(P_1^{[n]}(a,b)\right)_{n\geq 0} \otimes x_1,$$

and

$$\eta\big((c_n)_{n\geq 0} \otimes x\big) = (c_n x)_{n\geq 0}.$$

It is easy to check that $\eta\psi\varphi$, $\varphi\eta\psi$ and $\psi\varphi\eta$ are identities, so $\varphi$, $\psi$, $\eta$ are $\mathcal{R}$-module isomorphisms. It is also obvious that $\eta$ and $\psi\varphi$ are $\mathcal{R}[T]$-linear. □

*Proof.* [Proposition 1.5] There are canonical maps:

$$\Phi : \mathcal{F}_{\mathbf{M}}(a,b) \oplus \mathcal{F}_{\mathbf{N}}(a,b) \longrightarrow \mathcal{F}_{\mathbf{M}\oplus\mathbf{N}}(a,b)$$

defined by

$$\Phi\big((x_n)_{n\geq 0}, (y_n)_{n\geq 0}\big) = (x_n, y_n)_{n\geq 0}$$

and

$$\Psi : \mathcal{F}_{\mathbf{M}}(a,b) \oplus \mathcal{F}_{\mathbf{N}}(c,d) \longrightarrow \mathcal{F}_{\mathbf{M}\oplus\mathbf{N}}\big((a,c),(b,d)\big)$$

defined by

$$\Psi\big((x_n)_{n\geq 0}, (y_n)_{n\geq 0}\big) = (x_n, y_n)_{n\geq 0}.$$

Both are compatible with the shift. □

*Proof.* [Proposition 1.7] Define the morphism of $\mathcal{R}[H,V]$-modules:

$$\Phi : \mathcal{F}_{\mathbf{M}}(a,b) \otimes \mathcal{F}_{\mathbf{N}}(c,d) \longrightarrow \mathcal{F}^{[2]}_{\mathbf{M}\otimes\mathbf{N}}\big((a,b)\otimes(c,d)\big)$$

by

$$\Phi\big((x_n)_{n\geq 0} \otimes (y_k)_{k\geq 0}\big) = (x_n \otimes y_k)_{n,k\geq 0}\,.$$

The inverse morphism $\Psi$ can be constructed using canonical bases $P_0^{[n]}(a,b)$, $P_1^{[n]}(a,b)$ of $\mathcal{F}_{\mathbf{M}}(a,b)$, respectively $P_0^{[k]}(c,d)$, $P_1^{[k]}(c,d)$ of $\mathcal{F}_{\mathbf{N}}(c,d)$ and the corresponding basis $P_i^{[n]}(a,b) \otimes P_j^{[k]}(c,d)$, $i,j \in \{0,1\}^2$ of $\mathcal{F}_{\mathbf{M}}(a,b) \otimes \mathcal{F}_{\mathbf{N}}(c,d)$: if the first four terms are given by $Z_{0,0} = \sum_{i\in I} m_i \otimes n_i$, $Z_{1,0} = \sum_{j\in J} m_j' \otimes n_j'$, $Z_{0,1} = \sum_{h\in H} m_h'' \otimes n_h''$, $Z_{1,1} = \sum_{l\in L} m_l''' \otimes n_l'''$, then $\Psi$ is defined by:

$$\begin{aligned}\Psi\big((Z_{n,k})_{n,k\geq 0}\big) = \;& \textstyle\sum_{i\in I} \big(P_0^{[n]}(a,b)m_i\big)_{n\geq 0} \otimes \big(P_0^{[k]}(c,d)n_i\big)_{k\geq 0}\\ &+ \textstyle\sum_{j\in J} \big(P_1^{[n]}(a,b)m_j'\big)_{n\geq 0} \otimes \big(P_0^{[k]}(c,d)n_j'\big)_{k\geq 0}\\ &+ \textstyle\sum_{h\in H} \big(P_0^{[n]}(a,b)m_h''\big)_{n\geq 0} \otimes \big(P_1^{[k]}(c,d)n_h''\big)_{k\geq 0}\\ &+ \textstyle\sum_{l\in L} \big(P_1^{[n]}(a,b)m_l'''\big)_{n\geq 0} \otimes \big(P_1^{[k]}(c,d)n_l'''\big)_{k\geq 0}\,.\end{aligned}$$

□

*Proof.* [Corollary 1.8] The proof is clear as $\mathcal{F}^{[2]}\big((a,b)\otimes(c,d)\big) \cong \mathcal{F}(a,b) \otimes \mathcal{F}(c,d) \cong (\mathcal{R}\oplus\mathcal{R})\otimes(\mathcal{R}\oplus\mathcal{R}) \cong \mathcal{R}^4$. In general, $\mathcal{F}^{[2]}_{\mathbf{M}\otimes\mathbf{N}}\big((a,b)\otimes(c,d)\big) \cong \mathcal{F}_{\mathbf{M}}(a,b)\otimes\mathcal{F}_{\mathbf{N}}(c,d) \cong (\mathbf{M}\oplus\mathbf{M})\otimes(\mathbf{N}\oplus\mathbf{N}) \cong (\mathbf{M}\otimes\mathbf{N})^4$. □

**Corollary 2.6.** *Using $a = r_1 + r_2$, $b = -r_1 r_2$, the general term $x_{n,k}$ of a sequence in $\mathcal{F}^{[2]}_{\mathbf{M}\otimes\mathbf{N}}(a,b)$ is given by*

$$\begin{aligned}x_{n,k} = \Delta^{-2}\big[&(r_1^n r_2 - r_1 r_2^n)(r_1^k r_2 - r_1 r_2^k)x_{0,0} + (r_2^n - r_1^n)(r_1^k r_2 - r_1 r_2^k)x_{1,0}\\ &+ (r_1^n r_2 - r_1 r_2^n)(r_2^k - r_1^k)x_{0,1} + (r_2^n - r_1^n)(r_2^k - r_1^k)x_{1,1}\big],\end{aligned}$$

*where $\Delta = r_2 - r_1$. This formula is correct for arbitrary integers $n,k$ (as an equality in the ring $\mathcal{R}(r_1,r_2)$ of rational functions).*

*Proof.* [Proposition 1.9] Apply two times Lemma 2.5:

$$\begin{aligned}G(t,s) =\;& \textstyle\sum_{n\geq 0}\big(\sum_{k\geq 0} x_{n,k}s^k\big)t^n\\ =\;& \textstyle\sum_{n\geq 0}\big[r(s)^{-1}x_{n,0}(1-cs) + r(s)^{-1}x_{n,1}s\big]t^n\\ =\;& r(s)^{-1}\big[(1-cs)\textstyle\sum_{n\geq 0} x_{n,0}t^n + s\sum_{n\geq 0} x_{n,1}t^n\big]\\ =\;& q(t)^{-1}r(s)^{-1}\big\{(1-cs)[x_{0,0}(1-at) + x_{1,0}t]\\ &+ s[x_{0,1}(1-at) + x_{1,1}t]\big\}\,.\end{aligned}$$

□

We consider also other operations with Fibonacci modules, for example symmetric powers and exterior products (we suppose that 2 is a unit in $\mathcal{R}$):

**Proposition 2.7.** *There are natural isomorphisms:*

$$\mathrm{Symm}^{(2)}\,\mathcal{F}_{\mathbf{M}}(a,b) \cong \left\{(x_{n,k}) \in \mathcal{F}^{[2]}_{\mathbf{M}\otimes\mathbf{N}}(a,b) :\ x_{n,k} = x_{k,n}\ \ \forall\ k,n \geq 0\right\},$$

$$\wedge^{(2)}\,\mathcal{F}_{\mathbf{M}}(a,b) \cong \left\{(x_{n,k}) \in \mathcal{F}^{[2]}_{\mathbf{M}\otimes\mathbf{N}}(a,b) :\ x_{n,k} = -x_{k,n}\ \ \forall\ k,n \geq 0\right\}.$$

## 3. Generalizations

First we introduce recurrency of order $d$:

**Definition 3.1.** Let $\mathbf{a} = (a_1, \dots, a_d)$ be an element in $\mathcal{R}^d$. The Fibonacci module of type $\mathbf{a}$ associated to the module $\mathbf{M}$ is the $\mathcal{R}[T]$-module:

$$\mathcal{F}_{\mathbf{M}}(\mathbf{a}) = \left\{(x_n)_{n\geq 0} \in \mathcal{S}_{\mathbf{M}} :\ x_{n+d} = a_1 x_{n+d-1} + \cdots + a_d x_n,\ \forall\ n \geq 0\right\}.$$

Next we consider multiple Fibonacci sequences $(x_{n_1,\dots,n_p})_{n_i\geq 0}$ in $\mathbf{M}$:

**Definition 3.2.** Let $\mathbf{a}^{(1)} \in \mathcal{R}^{d_1}, \dots, \mathbf{a}^{(p)} \in \mathcal{R}^{d_p}$. The Fibonacci module of type $(\mathbf{a}^{(1)}, \dots, \mathbf{a}^{(p)})$ associated to the module $\mathbf{M}$ is the $\mathcal{R}[T_1, \dots, T_p]$-module:

$$\mathcal{F}^{[p]}_{\mathbf{M}}(\mathbf{a}^{(1)}, \dots, \mathbf{a}^{(p)}) = \left\{(x_{n_1,\dots,n_p})_{n_i\geq 0} : x_{n_1,\dots,n_p} \in \mathbf{M},\ x_{n_1,\dots,n_i+d_i,\dots,n_p} = \sum_{j=1}^{d_i} a^{(i)}_j x_{n_1,\dots,n_i+d_i-j,\dots,n_p} \text{ for } i = 1,2,\dots,p\right\}.$$

If $\mathbf{a}^{(1)} = \cdots = \mathbf{a}^{(p)} = \mathbf{a} = (a_1, \dots, a_d)$, we denote simply $\mathcal{F}^{[p]}_{\mathbf{M}}(\mathbf{a}) = \mathcal{F}^{[p]}_{\mathbf{M}}(a_1, \dots, a_d)$.

The previous results have obvious generalizations. For example:

**Proposition 3.3.**

$$\mathcal{F}_{\mathbf{M}}(a_1, \dots, a_d) \cong \mathbf{M}^d \cong \mathcal{F}(a_1, \dots, a_d) \otimes \mathbf{M}.$$

**Proposition 3.4.** *Fix* $\mathbf{a} = (a_1, \dots, a_d) \in \mathcal{R}^d$. *The sequences* $\left(P_i^{[n]}(\mathbf{a})\right)_{n\geq 0}$, $i = 0, \dots, d-1$ *in* $\mathcal{F}(\mathbf{a})$ *defined by* $P_i^{[j]}(\mathbf{a}) = \delta_{ij}$ *(for* $j = 0, \dots, d-1$*) give a canonical basis of* $\mathcal{F}(\mathbf{a})$.

**Lemma 3.5.** *The generating function of* $(x_n)_{n\geq 0}$ *in* $\mathcal{F}(\mathbf{a})$ *is*

$$G(t) = q(t)^{-1}\left[Q_0(t)x_0 + Q_1(t)x_1 + \cdots + Q_{d-1}(t)x_d\right],$$

*where*

$$Q_i(t) = t^i\left(1 - a_1 t - a_2 t^2 - \cdots - a_{d-i-1} t^{d-i-1}\right),\ i \in \{0, \dots, d-1\},$$

*and* $q(t) = 1 - a_1 t - a_2 t^2 - \cdots - a_d t^d$.

**Proposition 3.6.**

$$\mathcal{F}_{\mathbf{M}_1}(\mathbf{a}^{(1)}) \otimes \cdots \otimes \mathcal{F}_{\mathbf{M}_p}(\mathbf{a}^{(p)}) \cong \mathcal{F}^{[p]}_{\mathbf{M}_1\otimes\cdots\otimes\mathbf{M}_p}(\mathbf{a}^{(1)}, \dots, \mathbf{a}^{(p)}).$$

*In particular,* $\mathcal{F}^{[p]}(\mathbf{a}^{(1)}, \dots, \mathbf{a}^{(p)})$ *is free of rank* $D = d_1 d_2 \cdots d_p$.

**Proposition 3.7.** *A multiple Fibonacci sequence* $(x_{n_1,\dots,n_p})$ *of type* $(\mathbf{a}^{(1)},\dots,\mathbf{a}^{(p)})$ *has a rational generating function:*

$$G(t_1,\dots,t_p)=q_1(t_1)^{-1}\cdots q_p(t_p)^{-1}\Big[\sum_{0\leq j_i\leq d_i-1} Q^{(1)}_{j_1}(t_1)\cdots Q^{(p)}_{j_p}(t_p)x_{j_1,\dots,j_p}\Big],$$

*where* $q_i(t)=1-a^{(i)}_1 t-\dots-a^{(i)}_{d_i}t^{d_i}$ *and* $Q^{(i)}_0,\dots,Q^{(i)}_{d_i-1}$ *are like in Lemma 3.5.*

For further applications in knot theory, we will use the next specializations:

**Theorem 3.8.** *Let* $(x_{n_1,\dots,n_p})_{\geq 0}$ *be an element in* $\mathcal{F}^{[p]}_{\mathrm{M}}(r_1+r_2,-r_1r_2)$.
*a) The general term is given by*

$$x_{n_1,\dots,n_p}=\Delta^{-p}\sum_{0\leq j_1,\dots,j_p\leq 1} S^{[n_1]}_{j_1}(r_1,r_2)\cdots S^{[n_p]}_{j_p}(r_1,r_2)x_{j_1,\dots,j_p}\,,$$

*where* $\Delta=r_2-r_1$, $S^{[n]}_0(r_1,r_2)=r_1^n r_2-r_1r_2^n$, $S^{[n]}_1(r_1,r_2)=r_2^n-r_1^n$;
*b) the generating function of* $(x_{n_1,\dots,n_p})$ *is given by*

$$G(t_1,\dots,t_p)=q(t_1)^{-1}\cdots q(t_p)^{-1}\sum_{0\leq j_1,\dots,j_p\leq 1} Q_{j_1}(t_1)\cdots Q_{j_p}(t_p)x_{j_1,\dots,j_p}\,,$$

*where* $q(t)=(1-r_1t)(1-r_2t)$, $Q_0(t)=1-(r_1+r_2)t$ *and* $Q_1(t)=t$.

## 4. Examples

**Example 4.1.** Fibonacci module $\mathcal{F}^{[2]}_{\mathbb{Z}}(1,1)$: let us analyze sequences with the first four entries $(c_{i,j})_{(i,j)\in\{0,1\}^2}$ equal to 0 or 1. From the sixteen possible choices there are 5 primitive sequences:

$B_0=\boxed{\begin{matrix}0&0\\0&0\end{matrix}}$, $B_1=\boxed{\begin{matrix}0&0\\1&0\end{matrix}}$, $B_2=\boxed{\begin{matrix}1&0\\0&1\end{matrix}}$, $B_3=\boxed{\begin{matrix}0&1\\1&0\end{matrix}}$, $B_4=\boxed{\begin{matrix}1&0\\1&1\end{matrix}}$.

The others are shifts of these primitive sequences (see figure below):

$H(B_1)=\boxed{\begin{matrix}0&0\\0&1\end{matrix}}$, $H^2(B_1)=\boxed{\begin{matrix}0&0\\1&1\end{matrix}}$, $V(B_1)=\boxed{\begin{matrix}1&0\\0&0\end{matrix}}$, $V^2(B_1)=\boxed{\begin{matrix}1&0\\1&0\end{matrix}}$,

$HV(B_1)=\boxed{\begin{matrix}0&1\\0&0\end{matrix}}$, $H^2V(B_1)=\boxed{\begin{matrix}1&1\\0&0\end{matrix}}$, $HV^2(B_1)=\boxed{\begin{matrix}0&1\\0&1\end{matrix}}$, $H^2V^2(B_1)=\boxed{\begin{matrix}1&1\\1&1\end{matrix}}$.

$$H(B_2) = \boxed{\begin{matrix} 0 & 1 \\ 1 & 1 \end{matrix}},\ V(B_2) = \boxed{\begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix}} \text{ and } H(B_3) = V(B_3) = \boxed{\begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix}}.$$

In fact, using the structure of $\mathbb{Z}[H, V]$-module, $\mathcal{F}_{\mathbb{Z}}^{[2]}(1,1)$ is generated by $B_1$.

It is obvious that an element $(x_n)_{n\geq 0} \in \mathcal{F}_{\mathbb{Q}}(1,1)$ can be defined by any two terms $\{x_p, x_q\}$; in the case of a double sequence $(x_{n,k})_{n,k\geq 0} \in \mathcal{F}_{\mathbb{Q}}^{[2]}(1,1)$, not any four terms $\{x_{l,m}, x_{p,q}, x_{r,s}, x_{u,v}\}$ can define the sequence.

$$\begin{array}{lllllllll}
13 \\
8 & 0 \\
5 & 0 & 5 \\
3 & 0 & 3 & 3 \\
2 & 0 & 2 & 2 & 4 \\
1 & 0 & 1 & 1 & 2 & 3 \\
1 & 0 & 1 & 1 & 2 & 3 & 5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 2 & 3 & 5 & 8 & 13
\end{array}
\qquad\qquad
\begin{array}{lllllllll}
21 \\
13 & 8 \\
8 & 5 & 13 \\
5 & 3 & 8 & 11 \\
3 & 2 & 5 & 7 & 12 \\
2 & 1 & 3 & 4 & 7 & 11 \\
1 & 1 & 2 & 3 & 5 & 8 & 13 \\
1 & 0 & 1 & 1 & 2 & 3 & 5 & 8 \\
0 & 1 & 1 & 2 & 3 & 5 & 8 & 13 & 21
\end{array}$$

A curious property of these sequences is the alternating monotonicity along the lines parallel to the secondary diagonal:

$$x_{n+2,k} \geq x_{n+1,k+1} \leq x_{n,k+2}$$

or

$$x_{n+2,k} \leq x_{n+1,k+1} \geq x_{n,k+2}\,.$$

In general we do not have this strong alternating property (look at the sequence given by $x_{0,0} = x_{1,0} = 3, x_{0,1} = 2, x_{1,1} = 0$: the 4th diagonal is $(7,3,2,9)$ ). In general we have only a "weak alternating property":

$$x_{n+2,k+1} \geq x_{n+1,k+2} \text{ if and only if } x_{n+3,k} \leq x_{n,k+3}$$

(see the next corollary).

The general statement explaining these two facts is given by:

**Proposition 4.2.** (diagonal property) *If $a^2d = bc^2$, any four diagonal consecutive terms of the sequence $(x_{n,k})_{n,k\geq 0} \in \mathcal{F}_{\mathbb{M}}^{[2]}\big((a,b)\otimes(c,d)\big)$ satisfy the relation:*

$$abx_{n,k+3} + (a^2+b)cx_{n+1,k+2} = a(c^2+d)x_{n+2,k+1} + cdx_{n+3,k}\,.$$

*Proof.* Express the terms as combinations of $x_{n,k}$, $x_{n+1,k}$, $x_{n,k+1}$ and $x_{n+1,k+1}$.
□

**Corollary 4.3.** *Four diagonal consecutive terms in* $(x_{n,k})_{n,k\geq 0} \in \mathcal{F}_{\mathbb{Z}}^{[2]}(1,1)$ *satisfy*

$$x_{n,k+3} - x_{n+3,k} = 2(x_{n+2,k+1} - x_{n+1,k+2})\,.$$

**Acknowledgment.** I would like to thank the referee for many comments and improvements of the first version of the paper.

## REFERENCES

[1] M. Aigner, *A Course in Enumeration*, Springer 2007.

[2] I. Niven, H. Montgomery, *An Introduction to the Theory of Numbers*, John Wiley and Sons, 2006.

[3] A. M. Odlyzko, *Asymptotic Enumeration Methods*, in R. L. Graham, M. Grötschel, L. Lovász: Handbook of Combinatorics, Vol.II, pp.1063-1230, Elsevier 1995.

**2000 Mathematics Subject Classification: 05A15, 11B39**