A Characterization On Potentially $K_6 - C_4$-graphic Sequences *

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Abstract

For given a graph H, a graphic sequence $\pi = (d_1, d_2, \cdots, d_n)$ is said to be potentially H-graphic if there exists a realization of π containing H as a subgraph. Let $K_m - H$ be the graph obtained from K_m by removing the edges set $E(H)$ where H is a subgraph of K_m. In this paper, we characterize the potentially $K_6 - C_4$-graphic sequences. This characterization implies a theorem due to Hu and Lai [7].
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1 Introduction

We consider finite simple graphs. Any undefined notation follows that of Bondy and Murty [1]. The set of all non-increasing nonnegative integer sequence $\pi = (d_1, d_2, \cdots, d_n)$ is denoted by NS_n. A sequence $\pi \in NS_n$ is said to be graphic if it is the degree sequence of a simple graph G of order n; such a graph G is referred as a realization of π. The set of all graphic sequences in NS_n is denoted by GS_n. Let C_k and P_k denote a cycle on k vertices and a path on $k+1$ vertices, respectively. Let $\sigma(\pi)$ be the sum of all the terms of π, and let [x] be the largest integer less than or equal

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to x. A graphic sequence π is said to be potentially H-graphic if it has a realization G containing H as a subgraph. Let $G-H$ denote the graph obtained from G by removing the edges set $E(H)$ where H is a subgraph of G. In the degree sequence, r^t means r repeats t times, that is, in the realization of the sequence there are t vertices of degree r.

Gould et al.[6] considered an extremal problem on potentially H-graphic sequences as follows: determine the smallest even integer $\sigma(H,n)$ such that every n-term positive graphic sequence π with $\sigma(\pi) \geq \sigma(H,n)$ has a realization G containing H as a subgraph. A harder question is to characterize the potentially H-graphic sequences without zero terms. Yin and Li [18] gave two sufficient conditions for $\pi \in GS_n$ to be potentially $K_r - e$-graphic. Luo [15] characterized the potentially C_k-graphic sequences for each $k = 3,4,5$. Chen [2] characterized the potentially C_6-graphic sequences. Chen et al.[3] characterized the potentially ${}_kC_l$-graphic sequences for each $3 \leq k \leq 5$, $l = 6$. Recently, Luo and Warner [16] characterized the potentially K_4-graphic sequences. Eschen and Niu [5] characterized the potentially $K_4 - e$-graphic sequences. Yin et al.[19] characterized the potentially ${}_3C_4$, ${}_3C_5$ and ${}_4C_5$-graphic sequences. Yin and Chen [20] characterized the potentially $K_{r,s}$-graphic sequences for $r = 2, s = 3$ and $r = 2, s = 4$. Yin et al.[21] characterized the potentially $K_5 - e$ and K_6-graphic sequences. Besides, Yin [22] characterized the potentially $K_6 - K_3$-graphic sequences. Chen and Li [4] characterized the potentially $K_{1,t} + e$-graphic sequences. Xu and Lai[17] characterized the potentially $K_6 - C_5$-graphic sequences. Hu and Lai [8,11] characterized the potentially $K_5 - C_4$ and $K_5 - E_3$-graphic sequences where E_3 denotes graphs with 5 vertices and 3 edges. In [12], they characterized the potentially $K_{3,3}$ and $K_6 - C_6$-graphic sequences. Moreover, Hu, Lai and Wang [10] characterized the potentially $K_5 - P_4$ and $K_5 - Y_4$-graphic sequences where Y_4 is a tree on 5 vertices and 3 leaves.

In this paper, we characterize the potentially $K_6 - C_4$-graphic sequences. This characterization implies a theorem due to Hu and Lai [7]. Up to now, the problem of characterizing the potentially $K_6 - C_k (3 \leq k \leq 6)$-graphic sequences has been completely solved.

2 Preparations

Let $\pi = (d_1, \cdots, d_n) \in NS_n, 1 \le k \le n$. Let

$$\pi''_k = \begin{cases} (d_1 - 1, \cdots, d_{k-1} - 1, d_{k+1} - 1, \cdots, d_{d_k+1} - 1, d_{d_k+2}, \cdots, d_n), \\ \text{if } d_k \ge k, \\ (d_1 - 1, \cdots, d_{d_k} - 1, d_{d_k+1}, \cdots, d_{k-1}, d_{k+1}, \cdots, d_n), \\ \text{if } d_k < k. \end{cases}$$

Denote $\pi'_k = (d'_1, d'_2, \cdots, d'_{n-1})$, where $d'_1 \ge d'_2 \ge \cdots \ge d'_{n-1}$ is a rearrangement of the $n-1$ terms of π''_k. Then π'_k is called the residual sequence obtained by laying off d_k from π. For simplicity, we denote π'_n by π' in this paper. We need the following results.

Theorem 2.1 [6] If $\pi = (d_1, d_2, \cdots, d_n)$ is a graphic sequence with a realization G containing H as a subgraph, then there exists a realization G' of π containing H as a subgraph so that the vertices of H have the largest degrees of π.

Theorem 2.2 [8] Let $\pi = (d_1, d_2, \cdots, d_n) \in NS_n$ be a graphic sequence with $n \ge 5$. Then π is potentially $K_5 - C_4$-graphic if and only if the following conditions hold:

(1) $d_1 \ge 4$, $d_5 \ge 2$;

(2)$\pi \ne (4, 2^5), (4, 2^6), ((n-2)^2, 2^{n-2}), (n-k, k+i, 2^i, 1^{n-i-2})$ where $i = 3, 4, \cdots, n-2k$ and $k = 1, 2, \cdots, [\frac{n-1}{2}] - 1$.

Lemma 2.3 [9] If $\pi = (d_1, d_2, \cdots, d_n)$ is a nonincreasing sequence of positive integers with even $\sigma(\pi)$, $n \ge 4$, $d_1 \le 3$ and $\pi \ne (3^3, 1), (3^2, 1^2)$, then π is graphic.

Lemma 2.4 [22] Let $\pi = (4^x, 3^y, 2^z, 1^m)$ with even $\sigma(\pi)$, $x+y+z+m = n \ge 5$ and $x \ge 1$. Then $\pi \in GS_n$ if and only if $\pi \notin A$, where $A = \{(4, 3^2, 1^2), (4, 3, 1^3), (4^2, 2, 1^2), (4^2, 3, 2, 1), (4^3, 1^2), (4^3, 2^2), (4^3, 3, 1), (4^4, 2), (4^2, 3, 1^3), (4^2, 1^4), (4^3, 2, 1^2), (4^4, 1^2), (4^3, 1^4)\}$.

Lemma 2.5 (Kleitman and Wang [13]) π is graphic if and only if π'_k is graphic.

The following corollary is obvious.

Corollary 2.6 Let H be a simple graph. If π' is potentially H-graphic, then π is potentially H-graphic.

3 Main Theorems

Theorem 3.1 Let $\pi = (d_1, d_2, \cdots, d_n) \in NS_n$ be a graphic sequence with $n \geq 6$. Then π is potentially $K_6 - C_4$-graphic if and only if the following conditions hold:

(1) $d_2 \geq 5$, $d_6 \geq 3$;

(2) $\pi = (d_1, d_2, d_3, 3^k, 2^t, 1^{n-3-k-t})$ implies $d_1 + d_2 + d_3 \leq n + 2k + t + 1$;

(3) $\pi \neq (5^2, 4^6)$, $(5^2, 4^7)$, $(6^2, 3^6)$, $(6, 5, 4, 3^5)$, $(6, 5, 3^7)$, $(5^3, 4, 3^3)$, $(5^3, 3^5)$, $(5^2, 4^2, 3^4)$, $(5^2, 4, 3^6)$, $(5^2, 4, 3^4)$, $(5^2, 3^6)$, $(6, 5, 3^5, 2)$, $(5^3, 3^3, 2)$, $(5^2, 4, 3^4, 2)$, $(5^2, 3^6, 2)$, $(5^2, 3^4, 2)$, $(5^2, 3^4, 2^2)$, $(6, 5, 3^6, 1)$, $(5^3, 3^4, 1)$, $(5^2, 4, 3^5, 1)$, $(5^2, 3^7, 1)$, $(5^2, 3^6, 1^2)$, $(5^2, 3^5, 1)$, $(n-1, 5, 3^5, 1^{n-7})$, $(n-1, 5, 3^6, 1^{n-8})$.

Proof: First we show the conditions (1)-(3) are necessary conditions for π to be potentially $K_6 - C_4$-graphic. Assume that π is potentially $K_6 - C_4$-graphic. (1) is obvious. If $\pi = (d_1, d_2, d_3, 3^k, 2^t, 1^{n-3-k-t})$ is potentially $K_6 - C_4$-graphic, then according to Theorem 2.1, there exists a realization G of π containing $K_6 - C_4$ as a subgraph so that the vertices of $K_6 - C_4$ have the largest degrees of π. Therefore, the sequence $\pi_1 = (d_1 - 5, d_2 - 5, d_3 - 3, 3^{k-3}, 2^t, 1^{n-3-k-t})$ obtained from $G - (K_6 - C_4)$ is graphic and there exists no edge among three vertices with degree $d_1 - 5$, $d_2 - 5$ and $d_3 - 3$ in the realization of π_1. It follows $d_1 - 5 + d_2 - 5 + d_3 - 3 \leq 3(k-3) + 2t + n - 3 - k - t$, i.e., $d_1 + d_2 + d_3 \leq n + 2k + t + 1$. Hence, (2) holds. Now it is easy to check that $(5^2, 4^6)$, $(5^2, 4^7)$, $(6^2, 3^6)$, $(6, 5, 4, 3^5)$, $(6, 5, 3^7)$, $(5^3, 4, 3^3)$, $(5^3, 3^5)$, $(5^2, 4^2, 3^4)$, $(5^2, 4, 3^6)$, $(5^2, 4, 3^4)$, $(5^2, 3^6)$, $(6, 5, 3^5, 2)$, $(5^3, 3^3, 2)$, $(5^2, 4, 3^4, 2)$, $(5^2, 3^6, 2)$, $(5^2, 3^4, 2)$, $(5^2, 3^4, 2^2)$, $(6, 5, 3^6, 1)$, $(5^3, 3^4, 1)$, $(5^2, 4, 3^5, 1)$, $(5^2, 3^7, 1)$, $(5^2, 3^6, 1^2)$ and $(5^2, 3^5, 1)$ are not potentially $K_6 - C_4$-graphic. Since $(4, 2^5)$ and $(4, 2^6)$ are not potentially $K_5 - C_4$-graphic by Theorem 2.2, we have $\pi \neq (n-1, 5, 3^5, 1^{n-7})$ and $(n-1, 5, 3^6, 1^{n-8})$. Hence, (3) holds.

To prove the sufficiency, we use induction on n. Suppose the graphic sequence π satisfies the conditions (1)-(3). We first prove the base case where $n = 6$. Since $\pi \neq (5^3, 3^3)$, then π is one of the following: (5^6), $(5^4, 4^2)$, $(5^3, 4^2, 3)$, $(5^2, 4^4)$, $(5^2, 4^2, 3^2)$, $(5^2, 3^4)$. It is easy to check that all of these are potentially $K_6 - C_4$-graphic. Now suppose that the sufficiency holds for $n - 1 (n \geq 7)$, we will show that π is potentially $K_6 - C_4$-graphic in terms of the following cases:

Case 1: $d_n \geq 4$. Consider $\pi' = (d'_1, d'_2, \cdots, d'_{n-1})$ where $d'_{n-3} \geq 4$ and $d'_{n-1} \geq 3$. If π' satisfies (1) and (3), then by the induction hypothesis, π'

is potentially $K_6 - C_4$-graphic, and hence so is π.

If π' does not satisfy (1), i.e., $d_2' = 4$, then $d_2 = 5$. We will proceed with the following two cases $d_1 = 5$ and $d_1 \geq 6$.

Subcase 1: $d_1 = 5$. Then $\pi = (5^k, 4^{n-k})$ where $1 \leq k \leq 5$. Since $\sigma(\pi)$ is even, we have $k = 2$ or $k = 4$. If $k = 2$, then $\pi = (5^2, 4^{n-2})$. Since $\pi \neq (5^2, 4^6)$ and $(5^2, 4^7)$, we have $n = 7$ or $n \geq 10$. It is easy to check that $(5^2, 4^5)$ and $(5^2, 4^8)$ are potentially $K_6 - C_4$-graphic. If $n \geq 11$, let $\pi_1 = (5^2, 4^4)$, $\pi_2 = (4^{n-6})$. Then by lemma 2.4, π_2 is graphic. Let G_1 be a realization of π_2, then $(K_6 - 2K_2) \cup G_1$ is a realization of π. Thus, $\pi = (5^2, 4^{n-2})(n \neq 8, 9)$ is potentially $K_6 - C_4$-graphic since $K_6 - C_4 \subseteq K_6 - 2K_2$. Similarly, one can show that $\pi = (5^4, 4^{n-4})$ is also potentially $K_6 - C_4$-graphic.

Subcase 2: $d_1 \geq 6$. Then $\pi = (d_1, 5^k, 4^{n-1-k})$ where $1 \leq k \leq 3$, d_1 and k have the same parity. We will show that π is potentially $K_6 - C_4$-graphic.

If $k = 1$, then $\pi = (d_1, 5, 4^{n-2})$ where d_1 is odd. If $n \leq 10$, then π is one of the following: $(7, 5, 4^6)$, $(7, 5, 4^7)$, $(7, 5, 4^8)$, $(9, 5, 4^8)$. It is easy to check that all of these are potentially $K_6 - C_4$-graphic. If $n \geq 11$, let $\pi_1 = (5^2, 4^4)$, $\pi_2 = (d_1 - 5, 4^{n-6})$. Then the residual sequence $\pi_2' = (4^{n-1-d_1}, 3^{d_1-5})$ obtained by laying off $d_1 - 5$ from π_2 is graphic by lemma 2.3 and lemma 2.4. Hence, π_2 is graphic. Let G_1 be a realization of π_2, and $x \in V(G_1)$ with $d_{G_1}(x) = d_1 - 5$. Denote $G = (K_{1,2,2} \cup G_1) \cup \{xx_1, xx_2, xx_3, xx_4, xx_5\}$ where $x_i \in V(K_{1,2,2}), i = 1, \cdots, 5$. i.e., G is the graph obtained from $K_{1,2,2} \cup G_1$ by adding new edges $xx_1, xx_2, xx_3, xx_4, xx_5$ to $K_{1,2,2} \cup G_1$. Clearly, G is a realization of π and contains $K_6 - C_4$.

If $k = 2$, then $\pi = (d_1, 5^2, 4^{n-3})$ where d_1 is even. If $n \leq 11$, then π is one of the following: $(6, 5^2, 4^4)$, $(6, 5^2, 4^5)$, $(6, 5^2, 4^6)$, $(6, 5^2, 4^7)$, $(6, 5^2, 4^8)$, $(8, 5^2, 4^6)$, $(8, 5^2, 4^7)$, $(8, 5^2, 4^8)$, $(10, 5^2, 4^8)$. It is easy to check that all of these are potentially $K_6 - C_4$-graphic. If $n \geq 12$, let $\pi_1 = (6, 5^2, 4^4)$, $\pi_2 = (d_1 - 6, 4^{n-7})$. Then the residual sequence $\pi_2' = (4^{n-1-d_1}, 3^{d_1-6})$ obtained by laying off $d_1 - 6$ from π_2 is graphic by lemma 2.3 and lemma 2.4. Hence, π_2 is graphic. Let G_1 be a realization of π_2 and $x \in V(G_1)$ with $d_{G_1}(x) = d_1 - 6$. Denote $G = (K_6 - P_5 \cup G_1) \cup \{xx_1, xx_2, xx_3, xx_4, xx_5, xx_6\}$ where $x_i \in V(K_6 - P_5), i = 1, \cdots, 6$. i.e., G is the graph obtained from $K_6 - P_5 \cup G_1$ by adding new edges $xx_1, xx_2, xx_3, xx_4, xx_5, xx_6$ to $K_6 - P_5 \cup G_1$. Clearly, G is a realization of π and contains $K_6 - C_4$.

Similarly, one can show that $\pi = (d_1, 5^3, 4^{n-4})$ is also potentially $K_6 - C_4$-graphic.

If π' does not satisfy (3), then $\pi' = (5^2, 4^6)$ or $(5^2, 4^7)$. Hence, $\pi = (6^2, 5^2, 4^5)$, $(6, 5^4, 4^4)$, $(5^6, 4^3)$, $(6^2, 5^2, 4^6)$, $(6, 5^4, 4^5)$, $(5^6, 4^4)$. It is easy to check that all of these are potentially $K_6 - C_4$-graphic.

Case 2: $d_n = 3$. Consider $\pi' = (d'_1, d'_2, \cdots, d'_{n-1})$ where $d'_2 \geq 4$ and $d'_{n-2} \geq 3$. If π' satisfies (1)-(3), then by the induction hypothesis, π' is potentially $K_6 - C_4$-graphic, and hence so is π.

If π' does not satisfy (1), there are three subcases:

Subcase 1: $d'_2 \geq 5$ and $d'_6 = 2$. Then $\pi = (d_1, d_2, 3^5)$ where $d_1 \geq d_2 \geq 6$, which is impossible.

Subcase 2: $d'_2 = 4$ and $d'_6 = 2$. Then $\pi = (6, 5, 3^5)$, which contradicts condition (3).

Subcase 3: $d'_2 = 4$ and $d'_6 \geq 3$. Then $d_2 = 5$, we will proceed with the following two cases $d_1 = 5$ and $d_1 \geq 6$.

Subcase 3.1: $d_1 = 5$. Then $\pi = (5^k, 4^t, 3^{n-k-t})$ where $2 \leq k \leq 4$, $n - k - t \geq 1$ and $n - t$ is even.

If $k = 2$, then $\pi = (5^2, 4^t, 3^{n-2-t})$. We will show that π is potentially $K_6 - C_4$-graphic. If $t = 0$, then $\pi = (5^2, 3^{n-2})$. Since $\pi \neq (5^2, 3^6)$, we have $n \geq 10$. It is enough to show $\pi_1 = (3^{n-6})$ is graphic. It follows by lemma 2.3. If $t = 1$, then $\pi = (5^2, 4, 3^{n-3})$. Since $\pi \neq (5^2, 4, 3^4)$ and $(5^2, 4, 3^6)$, we have $n \geq 11$. We only need to show that $\pi_1 = (3^{n-6}, 1)$ is graphic. It also follows by lemma 2.3. If $t = 2$, then $\pi = (5^2, 4^2, 3^{n-4})$. Since $\pi \neq (5^2, 4^2, 3^4)$, we have $n \geq 10$. It is enough to show $\pi_1 = (3^{n-6}, 1^2)$ is graphic. It follows by lemma 2.3. If $t = 3$, then $\pi = (5^2, 4^3, 3^{n-5})$. Since $\pi_1 = (3^{n-6}, 1^3)$ is graphic by lemma 2.3, $\pi = (5^2, 4^3, 3^{n-5})$ is potentially $K_6 - C_4$-graphic. If $t \geq 4$, let $\pi_1 = (5^2, 4^4)$, $\pi_2 = (4^{t-4}, 3^{n-2-t})$. If $n \geq 11$, then by lemma 2.3 and lemma 2.4, π_2 is graphic. Let G_1 be a realization of π_2, then $(K_6 - 2K_2) \cup G_1$ is a realization of π. Since $K_6 - C_4 \subseteq K_6 - 2K_2$, π is potentially $K_6 - C_4$-graphic. If $n \leq 10$, then $\pi = (5^2, 4^4, 3^2)$, $(5^2, 4^4, 3^4)$, $(5^2, 4^5, 3^2)$ or $(5^2, 4^6, 3^2)$. It is easy to check that all of these are potentially $K_6 - C_4$-graphic.

If $k = 3$, then $\pi = (5^3, 4^t, 3^{n-3-t})$. We will show that π is potentially $K_6 - C_4$-graphic. If $t = 0$, then $\pi = (5^3, 3^{n-3})$. Since $\pi \neq (5^3, 3^5)$, we have $n \geq 10$. It is easy to check that $(5^3, 3^7)$ and $(5^3, 3^9)$ are potentially $K_6 - C_4$-graphic. If $n \geq 14$, let $\pi_1 = (5^3, 3^7)$, $\pi_2 = (3^{n-10})$ and G_1 be a realization of π_1 which contains $K_6 - C_4$. Then by lemma 2.3, π_2 is graphic. Let G_2 be a realization of π_2, then $G_1 \cup G_2$ is a realization of $\pi = (5^3, 3^{n-3})$. Similarly, one can show that $\pi = (5^3, 4^t, 3^{n-3-t})$ is potentially $K_6 - C_4$-graphic for the cases $t = 1$ and $t = 2$. If $t \geq 3$, let $\pi_1 = (5^3, 4^2, 3)$, $\pi_2 = (4^{t-2}, 3^{n-4-t})$. If $n \geq 11$, then π_2 is graphic by lemma 2.4. Let G_1 be a realization of π_2, then $(K_6 - P_2) \cup G_1$ is a realization of π. Since $K_6 - C_4 \subseteq K_6 - P_2$, π is potentially $K_6 - C_4$-graphic. If $n \leq 10$, then $\pi = (5^3, 4^3, 3)$, $(5^3, 4^3, 3^3)$, $(5^3, 4^4, 3)$, $(5^3, 4^4, 3^3)$, $(5^3, 4^5, 3)$ or $(5^3, 4^6, 3)$. It is easy to check that all of these are potentially $K_6 - C_4$-graphic.

If $k = 4$, then $\pi = (5^4, 4^t, 3^{n-4-t})$. We will show that π is potentially $K_6 - C_4$-graphic. If $t = 0$, then $\pi = (5^4, 3^{n-4})$. It is easy to check that $(5^4, 3^4)$ and $(5^4, 3^6)$ are potentially $K_6 - C_4$-graphic. If $n \geq 12$, let $\pi_1 = (5^4, 3^4)$, $\pi_2 = (3^{n-8})$ and G_1 be a realization of π_1 which contains $K_6 - C_4$. Then by lemma 2.3, π_2 is graphic. Let G_2 be a realization of π_2, then $G_1 \cup G_2$ is a realization of $\pi = (5^4, 3^{n-4})$. Similarly, one can show that $\pi = (5^4, 4^t, 3^{n-4-t})$ is potentially $K_6 - C_4$-graphic for the cases $t = 1$ and $t = 2$. If $t \geq 3$, let $\pi_1 = (5^4, 4^2)$, $\pi_2 = (4^{t-2}, 3^{n-4-t})$. If $n \geq 11$, then π_2 is graphic by lemma 2.4. Let G_1 be a realization of π_2, then $(K_6 - e) \cup G_1$ is a realization of π. Since $K_6 - C_4 \subseteq K_6 - e$, π is potentially $K_6 - C_4$-graphic. If $n \leq 10$, then $\pi = (5^4, 4^3, 3^2)$ or $(5^4, 4^4, 3^2)$. It is easy to check that both of them are potentially $K_6 - C_4$-graphic.

Subcase 3.2: $d_1 \geq 6$. Then $\pi = (d_1, 5, 4^k, 3^{n-2-k})$ where $n-2-k \geq 1$, d_1 and $n-1-k$ have the same parity. We will show that π is potentially $K_6 - C_4$-graphic.

If $k = 0$, then $\pi = (d_1, 5, 3^{n-2})$. Since $\pi \neq (6, 5, 3^5)$ and $(7, 5, 3^6)$, we have $n \geq 9$. If $n = 9$, since $\pi \neq (6, 5, 3^7)$, then $\pi = (8, 5, 3^7)$ which is potentially $K_6 - C_4$-graphic. If $n \geq 10$, we only need to show that $\pi_1 = (d_1 - 5, 3^{n-6})$ is graphic. Since the residual sequence $\pi_1' = (3^{n-1-d_1}, 2^{d_1-5})$ obtained by laying off $d_1 - 5$ from π_1 is graphic by lemma 2.3, π_1 is graphic.

If $k = 1$, then $\pi = (d_1, 5, 4, 3^{n-3})$. Since $\pi \neq (6, 5, 4, 3^5)$, we have $n \geq 9$. It is enough to show $\pi_1 = (d_1 - 5, 3^{n-6}, 1)$ is graphic and there exists no edge between two vertices with degree $d_1 - 5$ and 1 in the realization of π_1. Hence, it suffices to show $\pi_2 = (3^{n-1-d_1}, 2^{d_1-5}, 1)$ is graphic. It follows by lemma 2.3. With the same argument as above, one can show that $\pi = (d_1, 5, 4^k, 3^{n-2-k})$ is potentially $K_6 - C_4$-graphic for the cases $k = 2$ and $k = 3$.

Now we consider the case where $k \geq 4$. If $n \leq 10$, then π is one of the following: $(6, 5, 4^4, 3)$, $(6, 5, 4^4, 3^3)$, $(6, 5, 4^5, 3)$, $(6, 5, 4^5, 3^3)$, $(6, 5, 4^6, 3)$, $(6, 5, 4^7, 3)$, $(7, 5, 4^4, 3^2)$, $(7, 5, 4^4, 3^4)$, $(7, 5, 4^5, 3^2)$, $(7, 5, 4^6, 3^2)$, $(8, 5, 4^4, 3^3)$, $(8, 5, 4^5, 3^3)$, $(8, 5, 4^6, 3)$, $(8, 5, 4^7, 3)$, $(9, 5, 4^4, 3^4)$, $(9, 5, 4^6, 3^2)$. It is easy to check that all of these are potentially $K_6 - C_4$-graphic. If $n \geq 11$, let $\pi_1 = (5^2, 4^4)$, $\pi_2 = (d_1 - 5, 4^{k-4}, 3^{n-2-k})$. Then the residual sequence π_2' obtained by laying off $d_1 - 5$ from π_2 is graphic by lemma 2.3 and lemma 2.4, and hence π_2 is also graphic. Let G_1 be a realization of π_2 and $x \in V(G_1)$ with $d_{G_1}(x) = d_1 - 5$. Denote $G = (K_{1,2,2} \cup G_1) \cup \{xx_1, xx_2, xx_3, xx_4, xx_5\}$ where $x_i \in V(K_{1,2,2}), i = 1, \cdots, 5$. i.e., G is the graph obtained from $K_{1,2,2} \cup G_1$ by adding new edges $xx_1, xx_2, xx_3, xx_4, xx_5$ to $K_{1,2,2} \cup G_1$. Clearly, G is a realization of π and contains $K_6 - C_4$.

If π' does not satisfy (2), there are two subcases:

Subcase 1: $\pi' = (d_1', d_2', d_3', 3^{n-4})$ and $d_1' + d_2' + d_3' > n - 1 + 2(n-4) + 1$, i.e., $d_1' + d_2' + d_3' > 3n - 8$. If $d_3' \leq 4$, then $d_1' + d_2' > 3n - 12$. It follows $3n - 10 \leq d_1' + d_2' \leq 2(n-2)$, i.e., $n \leq 6$, a contradiction. Thus, $d_3' \geq 5$. Therefore, $\pi = (d_1, d_2, d_3, 3^{n-3})$ and $d_1 + d_2 + d_3 > 3n - 5$, a contradiction.

Subcase 2: $\pi' = (d_1', d_2', 3^{n-4}, 2)$ and $d_1' + d_2' + 3 > n - 1 + 2(n-5) + 1 + 1$, i.e., $d_1' + d_2' > 3n - 12$. Hence, $3n - 10 \leq d_1' + d_2' \leq 2(n-2)$, i.e., $n \leq 6$, a contradiction.

If π' does not satisfy (3), since $\pi \neq (6^2, 3^6)$, then π' is one of the following: $(5^2, 4^6)$, $(5^2, 4^7)$, $(6^2, 3^6)$, $(6, 5, 4, 3^5)$, $(6, 5, 3^7)$, $(5^3, 4, 3^3)$, $(5^3, 3^5)$, $(5^2, 4^2, 3^4)$, $(5^2, 4, 3^6)$, $(5^2, 4, 3^4)$, $(5^2, 3^6)$, $(6, 5, 3^5, 2)$, $(5^2, 3^6, 2)$, $(6, 5, 3^5)$, $(7, 5, 3^6)$. Hence, π is one of the following: $(6^2, 5, 4^5, 3)$, $(6, 5^3, 4^4, 3)$, $(5^5, 4^3, 3)$, $(6^2, 5, 4^6, 3)$, $(6, 5^3, 4^5, 3)$, $(5^5, 4^4, 3)$, $(7^2, 4, 3^6)$, $(7, 6, 5, 3^6)$, $(7, 6, 4^2, 3^5)$, $(7, 6, 4, 3^7)$, $(6^3, 4, 3^4)$, $(6^2, 5^2, 3^4)$, $(6^3, 3^6)$, $(6^2, 5, 4, 3^5)$, $(6^2, 4^3, 3^4)$, $(6, 5^3, 3^5)$, $(6^2, 5, 3^7)$, $(6^2, 4^2, 3^6)$, $(6^2, 5, 3^5)$, $(6^2, 4^2, 3^4)$, $(6^2, 4, 3^6)$, $(7, 6, 3^7)$, $(6^2, 3^8)$, $(7, 6, 4, 3^5)$, $(8, 6, 4, 3^6)$. It is easy to check that all of these are potentially $K_6 - C_4$-graphic.

Case 3: $d_n = 2$. Consider $\pi' = (d_1', d_2', \cdots, d_{n-1}')$ where $d_2' \geq 4$, $d_6' \geq 3$ and $d_{n-1}' \geq 2$. If π' satisfies (1)-(3), then by the induction hypothesis, π' is potentially $K_6 - C_4$-graphic, and hence so is π.

If π' does not satisfy (1), i.e., $d_2' = 4$, then $d_2 = 5$. There are two subcases:

Subcase 1: $d_1 \geq 6$. Then $\pi = (d_1, 5, 4^k, 3^t, 2^{n-2-k-t})$ where $k + t \geq 4$, $n - 2 - k - t \geq 1$, and, d_1 and t have different parities. We will show that π is potentially $K_6 - C_4$-graphic.

If $k = 0$, then $\pi = (d_1, 5, 3^t, 2^{n-2-t})$. If $n \geq 10$, we only need to show that $\pi_1 = (d_1 - 5, 3^{t-4}, 2^{n-2-t})$ is graphic. The residual sequence π_1' obtained by laying off $d_1 - 5$ from π_1 clearly satisfies the hypothesis of lemma 2.3, and so π_1' is graphic and hence so is π_1. If $n \leq 9$, since $\pi \neq (6, 5, 3^5, 2)$, then $\pi = (6,5,3^5,2^2)$, $(7,5,3^4,2^2)$, $(7,5,3^4,2^3)$, $(7,5,3^6,2)$, $(8,5,3^5,2^2)$. It is easy to check that all of these are potentially $K_6 - C_4$-graphic.

If $k = 1$, then $\pi = (d_1,5,4,3^t,2^{n-3-t})$. If $n \geq 9$, we only need to show that $\pi_1 = (d_1 - 5, 3^{t-3}, 2^{n-3-t}, 1)$ is graphic and there exists no edge between two vertices with degree $d_1 - 5$ and 1 in the realization of π_1. The residual sequence π_1' obtained by laying off $d_1 - 5$ from π_1 clearly satisfies the hypothesis of lemma 2.3, and so π_1' is graphic and hence so is π_1. If $n \leq 8$, then $\pi = (6,5,4,3^3,2)$, $(6,5,4,3^3,2^2)$, $(7,5,4,3^4,2)$. It is easy to check that all of these are potentially $K_6 - C_4$-graphic. With the same argument as above, one can show that $\pi = (d_1,5,4^k,3^t,2^{n-2-k-t})$ is potentially $K_6 - C_4$-graphic for the cases $k = 2$ and $k = 3$.

Now we consider the case where $k \geq 4$. If $n \geq 11$, let $\pi_1 = (5^2,4^4)$, $\pi_2 = (d_1-5,4^{k-4},3^t,2^{n-2-k-t})$. Then the residual sequence π_2' obtained by laying off $d_1 - 5$ from π_2 is graphic by lemma 2.3 and lemma 2.4, and hence π_2 is also graphic. Let G_1 be a realization of π_2 and and $x \in V(G_1)$ with $d_{G_1}(x) = d_1 - 5$. Denote $G = (K_{1,2,2} \cup G_1) \cup \{xx_1, xx_2, xx_3, xx_4, xx_5\}$ where $x_i \in V(K_{1,2,2}), i = 1, \cdots, 5$. i.e., G is the graph obtained from $K_{1,2,2} \cup G_1$ by adding new edges $xx_1, xx_2, xx_3, xx_4, xx_5$ to $K_{1,2,2} \cup G_1$. Clearly, G is a realization of π and contains $K_6 - C_4$. If $n \leq 10$, then π is one of the following: $(6,5,4^4,3,2)$, $(6,5,4^4,3,2^2)$, $(6,5,4^4,3,2^3)$, $(6,5,4^4,3^3,2)$, $(6,5,4^5,3,2)$, $(6,5,4^5,3,2^2)$, $(6,5,4^6,3,2)$, $(7,5,4^4,2^2)$, $(7,5,4^4,2^3)$, $(7,5,4^4,2^4)$, $(7,5,4^4,3^2,2)$, $(7,5,4^4,3^2,2^2)$, $(7,5,4^5,2)$, $(7,5,4^5,2^2)$, $(7,5,4^5,2^3)$, $(7,5,4^5,3^2,2)$, $(7,5,4^6,2)$, $(7,5,4^6,2^2)$, $(7,5,4^7,2)$, $(8,5,4^4,3,2^2)$, $(8,5,4^4,3,2^3)$, $(8,5,4^4,3^3,2)$, $(8,5,4^5,3,2)$, $(8,5,4^5,3,2^2)$, $(8,5,4^6,3,2)$, $(9,5,4^4,2^4)$, $(9,5,4^4,3^2,2^2)$, $(9,5,4^5,2^3)$, $(9,5,4^5,3^2,2)$, $(9,5,4^6,2^2)$, $(9,5,4^7,2)$. It is easy to check that all of these are potentially $K_6 - C_4$-graphic.

Subcase 2: $d_1 = 5$. Then $\pi = (5^i,4^k,3^t,2^{n-i-k-t})$ where $2 \leq i \leq 3$, $i+k+t \geq 6$, $n-i-k-t \geq 1$ and $i+t$ is even. We will show that π is potentially $K_6 - C_4$-graphic.

Subcase 2.1: $i = 2$. Then $\pi = (5^2,4^k,3^t,2^{n-2-k-t})$. If $k = 0$, then $\pi = (5^2,3^t,2^{n-2-t})$. If $n \geq 10$, it is enough to show $\pi_1 = (3^{t-4},2^{n-2-t})$ is graphic. It follows by lemma 2.3. If $n \leq 9$, since $\pi \neq (5^2,3^4,2)$, $(5^2,3^4,2^2)$ and $(5^2,3^6,2)$, we have $\pi = (5^2,3^4,2^3)$ which is potentially $K_6 - C_4$-graphic.

If $k = 1$, then $\pi = (5^2,4,3^t,2^{n-3-t})$. Since $\pi \neq (5^2,4,3^4,2)$, we have $n \geq 9$. It is enough to show $\pi_1 = (3^{t-3},2^{n-3-t},1)$ is graphic. It follows by lemma 2.3. With the same argument as above, one can show that

$\pi = (5^2, 4^k, 3^t, 2^{n-2-k-t})$ is potentially $K_6 - C_4$-graphic for the cases $k = 2$ and $k = 3$.

Now we consider the case where $k \geq 4$. If $n \geq 11$, let $\pi_1 = (5^2, 4^4)$, $\pi_2 = (4^{k-4}, 3^t, 2^{n-2-k-t})$. If $\pi_2 \neq (4^3, 2^2)$ and $(4^4, 2)$, then π_2 is graphic by lemma 2.3 and lemma 2.4. Let G_1 be a realization of π_2, then $(K_6 - 2K_2) \cup G_1$ is a realization of $\pi = (5^2, 4^k, 3^t, 2^{n-2-k-t})$. Since $K_6 - C_4 \subseteq K_6 - 2K_2$, π is potentially $K_6 - C_4$-graphic. If $n = 11$ and $\pi_2 = (4^3, 2^2)$ or $(4^4, 2)$, then $\pi = (5^2, 4^7, 2^2)$ or $(5^2, 4^8, 2)$. If $n \leq 10$, then π is one of the following: $(5^2, 4^4, 2)$, $(5^2, 4^4, 2^2)$, $(5^2, 4^4, 2^3)$, $(5^2, 4^4, 2^4)$, $(5^2, 4^4, 3^2, 2)$, $(5^2, 4^4, 3^2, 2^2)$, $(5^2, 4^5, 2)$, $(5^2, 4^5, 2^2)$, $(5^2, 4^5, 2^3)$, $(5^2, 4^5, 3^2, 2)$, $(5^2, 4^6, 2)$, $(5^2, 4^6, 2^2)$, $(5^2, 4^7, 2)$. It is easy to check that all of these are potentially $K_6 - C_4$-graphic.

Subcase 2.2: $i = 3$. Then $\pi = (5^3, 4^k, 3^t, 2^{n-3-k-t})$. If $k = 0$, then $\pi = (5^3, 3^t, 2^{n-3-t})$. Since $\pi \neq (5^3, 3^3, 2)$, we have $n \geq 8$. If $n \geq 9$, it is enough to show $\pi_1 = (3^{t-3}, 2^{n-2-t})$ is graphic. It follows by lemma 2.3. If $n = 8$, then $\pi = (5^3, 3^3, 2^2)$ which is potentially $K_6 - C_4$-graphic.

If $k = 1$, then $\pi = (5^3, 4, 3^t, 2^{n-4-t})$. Let $\pi_1 = (5^3, 4, 3^3, 2)$, $\pi_2 = (3^{t-3}, 2^{n-5-t})$. It is easy to see that π_1 is potentially $K_6 - C_4$-graphic. Let G_1 be a realization of π_1 with $K_6 - C_4 \subseteq G_1$. If $n \geq 12$, then π_2 is graphic by lemma 2.3. Let G_2 be a realization of π_2, then $G_1 \cup G_2$ is a realization of π. If $n \leq 11$, then π is one of the following: $(5^3, 4, 3^3, 2)$, $(5^3, 4, 3^3, 2^2)$, $(5^3, 4, 3^3, 2^3)$, $(5^3, 4, 3^3, 2^4)$, $(5^3, 4, 3^5, 2)$, $(5^3, 4, 3^5, 2^2)$. It is easy to check that all of these are potentially $K_6 - C_4$-graphic. Similarly, one can show that $\pi = (5^3, 4^k, 3^t, 2^{n-3-k-t})$ is potentially $K_6 - C_4$-graphic for the cases $k = 2$ and $k = 3$.

Now we consider the case where $k \geq 4$. If $n \geq 12$, let $\pi_1 = (5^3, 4^3, 3)$, $\pi_2 = (4^{k-3}, 3^{t-1}, 2^{n-3-k-t})$. It is easy to see that π_1 is potentially $K_6 - C_4$-graphic. Let G_1 be a realization of π_1 with $K_6 - C_4 \subseteq G_1$. If $\pi_2 \neq (4^3, 2^2)$ and $(4^4, 2)$, then π_2 is graphic by lemma 2.3 and lemma 2.4. Let G_2 be a realization of π_2, then $G_1 \cup G_2$ is a realization of $\pi = (5^3, 4^k, 3^t, 2^{n-3-k-t})$. If $\pi_2 = (4^3, 2^2)$ or $(4^4, 2)$, then $\pi = (5^3, 4^6, 3, 2^2)$ or $(5^3, 4^7, 3, 2)$. If $n \leq 11$, then π is one of the following: $(5^3, 4^4, 3, 2)$, $(5^3, 4^4, 3, 2^2)$, $(5^3, 4^4, 3, 2^3)$, $(5^3, 4^4, 3^3, 2)$, $(5^3, 4^5, 3, 2)$, $(5^3, 4^5, 3, 2^2)$, $(5^3, 4^6, 3, 2)$. It is easy to check that all of these are potentially $K_6 - C_4$-graphic.

If π' does not satisfy (2), then $\pi' = (d_1', d_2', d_3', 3^k, 2^{n-4-k})$ and $d_1' + d_2' + d_3' > n - 1 + 2k + n - 4 - k + 1$, i.e., $d_1' + d_2' + d_3' > 2n + k - 4$. Hence, $\pi = (d_1, d_2, d_3, 3^k, 2^{n-3-k})$ and $d_1 + d_2 + d_3 > 2n + k - 2$, a contradiction.

If π' does not satisfy (3), then π' is one of the following: $(5^2,4^6)$, $(5^2,4^7)$, $(6^2,3^6)$, $(6,5,4,3^5)$, $(6,5,3^7)$, $(5^3,4,3^3)$, $(5^3,3^5)$, $(5^2,4^2,3^4)$, $(5^2,4,3^6)$, $(5^2,4,3^4)$, $(5^2,3^6)$, $(6,5,3^5,2)$, $(5^3,3^3,2)$, $(5^2,4,3^4,2)$, $(5^2,3^6,2)$, $(5^2,3^4,2)$, $(5^2,3^4,2^2)$, $(6,5,3^5)$, $(7,5,3^6)$. Hence, π is one of the following: $(6^2,4^6,2)$, $(6,5^2,4^5,2)$, $(5^4,4^4,2)$, $(6^2,4^7,2)$, $(6,5^2,4^6,2)$, $(5^4,4^5,2)$, $(7^2,3^6,2)$, $(7,6,4,3^5,2)$,$(7,5^2,3^5,2)$,$(7,6,3^7,2)$, $(6^2,5,4,3^3,2)$,$(6,5^3,3^3,2)$,$(6^2,5,3^5,2)$, $(6^2,4^2,3^4,2)$,$(6,5^2,4,3^4,2)$,$(5^4,3^4,2)$, $(6^2,4,3^6,2)$,$(6,5^2,3^6,2)$,$(6^2,4,3^4,2)$, $(6,5^2,3^4,2)$, $(6^2,3^6,2)$, $(7,6,3^5,2^2)$, $(6^2,5,3^3,2^2)$, $(6^2,4,3^4,2^2)$, $(6,5^2,3^4,2^2)$, $(6^2,3^6,2^2)$, $(6^2,3^4,2^2)$, $(6^2,3^4,2^3)$, $(7,6,3^5,2)$, $(8,6,3^6,2)$. It is easy to check that all of these are potentially K_6-C_4-graphic.

Case 4: $d_n=1$. Consider $\pi'=(d'_1,d'_2,\cdots,d'_{n-1})$ where $d'_1\geq 5$, $d'_2\geq 4$ and $d'_6\geq 3$. If π' satisfies (1)-(3), then by the induction hypothesis, π' is potentially K_6-C_4-graphic, and hence so is π.

If π' does not satisfy (1), i.e., $d'_2=4$, then $\pi=(5^2,4^k,3^t,2^i,1^{n-2-k-t-i})$ where $k+t\geq 4$, $n-2-k-t-i\geq 1$ and $n-k-i$ is even. We will show that π is potentially K_6-C_4-graphic.

If $k=0$, then $\pi=(5^2,3^t,2^i,1^{n-2-t-i})$. If $n\geq 10$, we only need to show that $\pi_1=(3^{t-4},2^i,1^{n-2-t-i})$ is graphic. Since $\pi\neq(5^2,3^7,1)$ and $(5^2,3^6,1^2)$, then $\pi_1\neq(3^3,1)$, $(3^2,1^2)$. By lemma 2.3, π_1 is graphic. If $n\leq 9$, since $\pi\neq(5^2,3^5,1)$ and $(5^2,3^5,2,1)$, then $\pi=(5^2,3^4,1^2)$ or $(5^2,3^4,2,1^2)$. It is easy to check that both of them are potentially K_6-C_4-graphic.

If $k=1$, then $\pi=(5^2,4,3^t,2^i,1^{n-3-t-i})$. If $n\geq 9$, it is enough to show $\pi_1=(3^{t-3},2^i,1^{n-2-t-i})$ is graphic. Since $\pi\neq(5^2,4,3^6)$ and $(5^2,4,3^5,1)$, then $\pi_1\neq(3^3,1)$, $(3^2,1^2)$. By lemma 2.3, π_1 is graphic. If $n\leq 8$, then $\pi=(5^2,4,3^3,1)$ or $(5^2,4,3^3,2,1)$. It is easy to check that both of them are potentially K_6-C_4-graphic. With the same argument as above, one can show that $\pi=(5^2,4^k,3^t,2^i,1^{n-2-k-t-i})$ is potentially K_6-C_4-graphic for the cases $k=2$ and $k=3$.

Now we consider the case where $k\geq 4$. If $n\geq 11$, let $\pi_1=(5^2,4^4)$, $\pi_2=(4^{k-4},3^t,2^i,1^{n-2-k-t-i})$. If $\pi_2\neq(4,3^2,1^2)$, $(4,3,1^3)$, $(4^2,2,1^2)$, $(4^2,3,2,1)$, $(4^3,1^2)$, $(4^3,3,1)$, $(4^2,3,1^3)$, $(4^2,1^4)$, $(4^3,2,1^2)$, $(4^4,1^2)$ and $(4^3,1^4)$, then π_2 is graphic by lemma 2.3 and lemma 2.4. Let G_1 be a realization of π_2, then $(K_6-2K_2)\cup G_1$ is a realization of $\pi=(5^2,4^k,3^t,2^i,1^{n-2-k-t-i})$ Since $K_6-C_4\subseteq K_6-2K_2$, π is potentially K_6-C_4-graphic. If π_2 is one of the following: $(4,3^2,1^2)$, $(4,3,1^3)$, $(4^2,2,1^2)$, $(4^2,3,2,1)$, $(4^3,1^2)$, $(4^3,3,1)$, $(4^2,3,1^3)$, $(4^2,1^4)$, $(4^3,2,1^2)$, $(4^4,1^2)$, $(4^3,1^4)$, then π is one of the following: $(5^2,4^5,3^2,1^2)$, $(5^2,4^5,3,1^3)$, $(5^2,4^6,2,1^2)$, $(5^2,4^6,3,2,1)$, $(5^2,4^7,1^2)$, $(5^2,4^7,3,1)$, $(5^2,4^6,3,1^3)$, $(5^2,4^6,1^4)$, $(5^2,4^7,2,1^2)$, $(5^2,4^8,1^2)$, $(5^2,4^7,1^4)$. If $n \le 10$, then π is one of the following: $(5^2,4^4,1^2)$, $(5^2,4^4,2,1^2)$, $(5^2,4^4,2^2,1^2)$, $(5^2,4^4,3^2,1^2)$, $(5^2,4^4,3,1)$, $(5^2,4^4,3,2,1)$, $(5^2,4^4,3,2^2,1)$, $(5^2,4^4,3^3,1)$, $(5^2,4^4,3,1^3)$, $(5^2,4^4,1^4)$, $(5^2,4^5,1^2)$, $(5^2,4^5,2,1^2)$, $(5^2,4^5,3,1)$, $(5^2,4^5,3,2,1)$, $(5^2,4^6,1^2)$, $(5^2,4^6,3,1)$. It is easy to check that all of these are potentially K_6-C_4-graphic.

If π' does not satisfy (2), then $\pi' = (d'_1,d'_2,d'_3,3^k,2^t,1^{n-4-k-t})$ and $d'_1+d'_2+d'_3 > n-1+2k+t+1$, i.e., $d'_1+d'_2+d'_3 > n+2k+t$. Hence, $\pi = (d_1,d_2,d_3,3^k,2^t,1^{n-3-k-t})$ and $d_1+d_2+d_3 > n+2k+t+1$, a contradiction.

If π' does not satisfy (3), since $\pi \ne (6,5,3^5,1)$ and $(n-1,5,3^6,1^{n-8})$, then π' is one of the following: $(5^2,4^6)$, $(5^2,4^7)$, $(6^2,3^6)$, $(6,5,4,3^5)$, $(6,5,3^7)$, $(5^3,4,3^3)$, $(5^3,3^5)$, $(5^2,4^2,3^4)$, $(5^2,4,3^6)$, $(5^2,4,3^4)$, $(6,5,3^5,2)$, $(5^3,3^3,2)$, $(5^2,4,3^4,2)$, $(5^2,3^6,2)$, $(5^2,3^4,2)$, $(5^2,3^4,2^2)$, $(6,5,3^6,1)$, $(5^3,3^4,1)$, $(5^2,4,3^5,1)$, $(5^2,3^7,1)$, $(5^2,3^6,1^2)$, $(5^2,3^5,1)$, $(6,5,3^5)$. Since $\pi \ne (5^3,3^4,1)$ and $(n-1,5,3^5,1^{n-7})$, then π is one of the following: $(6,5,4^6,1)$, $(5^3,4^5,1)$, $(6,5,4^7,1)$, $(5^3,4^6,1)$, $(7,6,3^6,1)$, $(7,5,4,3^5,1)$, $(6^2,4,3^5,1)$, $(7,5,3^7,1)$, $(6^2,3^7,1)$, $(6,5^2,4,3^3,1)$, $(5^4,3^3,1)$,$(6,5^2,3^5,1)$,$(6,5,4^2,3^4,1)$, $(5^3,4,3^4,1)$, $(6,5,4,3^6,1)$,$(5^3,3^6,1)$, $(6,5,4,3^4,1)$,$(7,5,3^5,2,1)$,$(6^2,3^5,2,1)$, $(6,5^2,3^3,2,1)$, $(6,5,4,3^4,2,1)$, $(5^3,3^4,2,1)$, $(6,5,3^6,2,1)$, $(6,5,3^4,2,1)$, $(6,5,3^4,2^2,1)$, $(7,5,3^6,1^2)$, $(6^2,3^6,1^2)$, $(6,5^2,3^4,1^2)$, $(6,5,4,3^5,1^2)$, $(5^3,3^5,1^2)$, $(6,5,3^7,1^2)$, $(6,5,3^6,1^3)$, $(6,5,3^5,1^2)$, $(6^2,3^5,1)$. It is easy to check that all of these are potentially K_6-C_4-graphic.

4 Application

In the remaining of this section, we will use theorem 3.1 to find exact values of $\sigma(K_6-C_4,n)$. Note that the value of $\sigma(K_6-C_4,n)$ was determined by Hu and Lai in [7] so a much simpler proof is given here.

Theorem (Hu and Lai [7]) If $n \ge 6$, then $\sigma(K_6-C_4,n) = 6n-10$.

Proof: First we claim that for $n \ge 6$, $\sigma(K_6-C_4,n) \ge 6n-10$. Take $\pi_1 = ((n-1)^3,3^{n-3})$, then $\sigma(\pi_1) = 6n-12$, and it is easy to see that π_1 is not potentially K_6-C_4-graphic by condition (2) in Theorem 3.1.

Now we show that if π is an n-term ($n \ge 6$) graphic sequence with $\sigma(\pi) \ge 6n-10$, then there exists a realization of π containing a K_6-C_4.

If $d_2 \le 4$, then $\sigma(\pi) \le d_1+4(n-1) \le n-1+4(n-1) = 5n-5 < 6n-10$, a contradiction. Hence, $d_2 \ge 5$.

If $d_6 \le 2$, then $\sigma(\pi) \le d_1 + d_2 + d_3 + d_4 + d_5 + 2(n-5) \le 20 + 2(n-5) + 2(n-5) = 4n < 6n-10$, a contradiction. Hence, $d_6 \ge 3$.

Since $\sigma(\pi) \ge 6n-10$, then π is not one of the following: $(d_1, d_2, d_3, 3^k, 2^t, 1^{n-3-k-t})$, $(5^2, 4^6)$, $(5^2, 4^7)$, $(6^2, 3^6)$, $(6, 5, 4, 3^5)$, $(6, 5, 3^7)$, $(5^3, 4, 3^3)$, $(5^3, 3^5)$, $(5^2, 4^2, 3^4)$, $(5^2, 4, 3^6)$, $(5^2, 4, 3^4)$, $(5^2, 3^6)$, $(6, 5, 3^5, 2)$, $(5^3, 3^3, 2)$, $(5^2, 4, 3^4, 2)$, $(5^2, 3^6, 2)$,$(5^2, 3^4, 2)$, $(5^2, 3^4, 2^2)$,$(6, 5, 3^6, 1)$,$(5^3, 3^4, 1)$, $(5^2, 4, 3^5, 1)$, $(5^2, 3^7, 1)$, $(5^2, 3^6, 1^2)$, $(5^2, 3^5, 1)$, $(n-1, 5, 3^5, 1^{n-7})$, $(n-1, 5, 3^6, 1^{n-8})$. Thus, π satisfies the conditions (1)-(3) in Theorem 3.1. Therefore, π is potentially $K_6 - C_4$-graphic.

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