On Super Vertex-Magic Total Labeling of the Disjoint Union of k Copies of K_n

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Abstract

Let $G = (V, E)$ be a finite non-empty graph. A vertex-magic total labeling (VMTL) is a bijection λ from $V \cup E$ to the set of consecutive integers $\{1, 2, \ldots, |V| + |E|\}$ with the property that for every $v \in V$, $\lambda(v) + \sum_{w \in N(v)} \lambda(vw) = h$, for some constant h. Such a labeling is called super if the vertex labels are $1, 2, \ldots, |V|$.

There are some results known about super VMTL of kG only when the graph G has a super VMTL. In this paper we focus on the case when G is the complete graph K_n. It was shown that a super VMTL of kK_n exists for n odd and any k, for $4 < n \equiv 0 \pmod 4$ and any k, and for $n = 4$ and k even. We continue the study and examine the graph kK_n for $n \equiv 2 \pmod 4$. Let $n = 4l + 2$ for a positive integer l. The graph kK_{4l+2} does not admit a super VMTL for k odd. We give a large number of super VMTLs of kK_{4l+2} for any even k based on super VMTL of $4K_{2l+1}$.

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1 Definitions and basic properties

Throughout this paper $G = (V, E)$ will be an undirected graph with the vertex set V and edge set E, where $|V| = n$. The set of neighbors of v we denote by $N(v)$. The maximum degree of G will be denoted by $\Delta(G)$ or just Δ.

Though magic labelings were introduced in 1968 by Sedláček [11], the concept of vertex-magic total labeling first appeared in 2002 [7]. A total labeling of G is a bijection $\lambda : V \cup E \to \{1, 2, \ldots, n + |E|\}$ and the associated *weight* of a vertex $v_i \in V$ in the labeling λ is

$$w_\lambda(v_i) = \lambda(v_i) + \sum_{v_j \in N(v_i)} \lambda(v_i v_j).$$

If each vertex of G in a particular total labeling λ has the same weight, the labeling is called *vertex-magic* (denoted by VMTL for short). In this case we denote the weight by h or $h(G)$. It is clear that the set of possible values of h has an upper and lower bound, see [8]. Any integer h between these bounds is called a feasible value for h.

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By $\lambda(X)$ we denote the set of images of elements in $X \subseteq (V \cup E)$. A vertex-magic total labeling λ is called *super* if $\lambda(V) = \{1, 2, \ldots, n\}$. A graph which admits a super vertex-magic total labeling is called super vertex-magic. The value of the magic constant h of a super vertex-magic graph is the largest feasible value, see [6] and [8].

Various properties of vertex-magic total labelings and super vertex-magic total labelings have been studied in [1,6–8]. The value of the magic constant was given in several works:

$$h = 2|E| + \frac{|E|(|E|+1)}{n} + \frac{n+1}{2}. \tag{1}$$

In [1] the degrees of super vertex-magic graphs have been examined. For a dynamic survey of various graph labelings see [2]. A good survey on vertex-magic graphs is in [14]. So far only a few methods for constructing VMTL of a graph have been given. There is no universal method known for constructing a super VMTL of a general graph and none such method is expected to exist. Therefore, usually only particular graphs or families of graphs are studied. E.g., two constructions of a super VMTL of K_n for odd n were given in [7,8]. Fig. 1 gives a super VMTL of K_5 with $h = 45$ described in [8]. In [3] the results on super VMTL K_n were completed. Other super VMTL of graphs

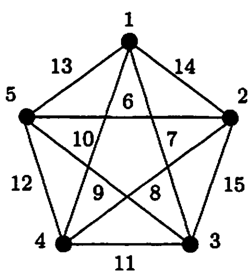

Fig. 1. A super vertex-magic of K_5 with $h = 45$.

were given in [1,4,13]. E.g., a super VMTL of rC_s (the disjoint union of r cycles of length s), for odd integers r and s was given in [1]. In [13] it was proved that the Knödel graph $W_{3,n}$ has a super VMTL for $n \equiv 0 \pmod 4$. Finally, several super VMTL of kK_n (the disjoint union of k copies of K_n) were given in [4] for n, k odd and for $8 \leq n \equiv 0 \pmod 4$ and k arbitrary. In this paper we continue the study of super VMTL of kK_n (the disjoint union of k copies of K_n) for $n \equiv 2 \pmod 4$. It shall be pointed out that if kK_n admits a super VMTL often it can be used to obtain different super VMTL by applying methods presented in [4,10].

In some papers general methods to obtain super VMTL of graphs based on super VMTL of their subgraphs are presented. Suppose $G = (V, E_1 \cup E_2)$ can be decomposed into edge-disjoint factors $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$. In [10] it was shown that if G_1 admits a super VMTL and G_2 is a regular graph of even degree, then G admits a super VMTL. In [4,9] methods to construct super VMTL of graphs, obtained from graphs that admit a super VMTL, were presented. Yet for non-regular graphs or for regular graphs not satisfying the necessary conditions these methods cannot be used.

2 Known results on super VMTL of kK_n

It was shown in [6] that if G is a graph of even order having a super VMTL then either $n \equiv 0 \pmod 8$ and $e \equiv 0, 3 \pmod 4$ or $n \equiv 4 \pmod 8$ and $e \equiv 1, 2 \pmod 4$. Hence we have the following corollary.

Lemma 2.1 *No graph G of order $n \equiv 2 \pmod 4$ admits a super VMTL.*

Lemma 2.2 *[7] [1] The magic constant of a super VMTL of*

K_n *is*

$$h(K_n) = \frac{n(n+1)^2}{4}. \tag{2}$$

It is easy to observe that K_{4l+2} does not admit a super VMTL also by Lemma 2.2. Indeed, the magic constant of K_{4l+2} would have to be $h(K_{4l+2}) = \frac{(2l+1)(4l+3)^2}{2}$, which is not an integer.

In [7] it was shown that K_4 does not admit a super VMTL. But for odd or doubly even order $n > 4$, K_n has a super VMTL:

Theorem 2.3 *[7,8] If n is odd then K_n admits a super VMTL.*

Theorem 2.4 *[3] If $n \equiv 0 \pmod 4$, $n > 4$, then K_n has a super VMTL.*

Next we examine the disjoint union of k copies of K_n. Suppose G admits a super VMTL and is Δ-regular. The following lemma gives $h(kG)$ as a function of $h(G)$.

Lemma 2.5 *[4] Let $h(G)$ be the magic constant of a Δ-regular graph G of order n. The magic constant of kG is*

$$h(kG) = kh(G) - \frac{(k-1)(\Delta+1)}{2}. \tag{3}$$

In [4] it was shown that for certain regular graphs kG it is enough to focus on finding super VMTL of a single copy. Notice that this approach is *not* true for all regular graphs G, e.g. when the single copy G does not admit a super VMTL.

Theorem 2.6 *[4] Let k be a positive integer. Suppose G is a Δ-regular graph of order n, such that $(k-1)(\Delta+1)/2$ is an integer. If G admits a super VMTL λ_G with $w_{\lambda_G}(v_i) = h(G)$, then kG admits a super VMTL.*

Thus an even number of copies of K_4 admits a super VMTL based on Theorem 2.6 and the following theorem.

Theorem 2.7 *[6] The graph* $2K_4$ *admits a super VMTL.*

Moreover, from Theorems 2.3, 2.4, and 2.6 we can conclude the following:

Theorem 2.8 *[4] Let* n*,* k*, and* $l \geq 2$ *be positive integers. If* n *and* k *are odd or* $n = 4l$ *and* k *is arbitrary then the graph* kK_n *has a super VMTL.*

Now by the following two lemmas super VMTL of certain kK_n cannot exist.

Lemma 2.9 *If* k *is even and* n *is odd then* kK_n *does not admit a super VMTL.*

PROOF. By Lemma 2.2 the magic constant for a super VMTL of K_n is the integer $h(K_n) = n(n+1)^2/4$. Also $kh(G)$ is an integer, however, $(k-1)(\Delta+1)/2$ is not an integer. Hence, according to (3), if k is even and n is odd then kK_n does not admit a super VMTL. □

Lemma 2.10 *Let* l *be a nonnegative integer. If* k *is odd then the graph* kK_{4l+2} *does not admit a super VMTL.*

PROOF. By Lemma 2.1, the graph kK_{4l+2} does not admit a super VMTL for k odd. □

By a result from [1] no vertices of small degree can exist in a super VMTL graph.

Theorem 2.11 *[1] The minimum degree of a super vertex-magic graph G is at least two.*

Table 1 summarizes known results on super VMTL of kK_n. The symbol $\exists$ in the table stands for a result that such a labeling is known to exists. If a particular super VMTL does not exist, this is denoted by the symbol $\nexists$. The reference points to the relevant paper or to a proposition in this section. The question mark stands for a conjectured result. For both unsolved cases it is conjectured that they admit a super VMTL.

Table 1
Summary of results and conjectures on super VMTL of kK_n.

	$k = 1$	odd $k > 1$	even k
$n = 4$	$\nexists$, see [7]	$\exists$?	$\exists$ Thms. 2.7, 2.6
$4 < n \equiv 0 \pmod 4$	$\exists$, see [3]	$\exists$ Thm. 2.8	$\exists$ Thm. 2.8
$n \equiv 1 \pmod 4$	$\exists$, see [7,8]	$\exists$ Thm. 2.8	$\nexists$ Lem. 2.9
$n \equiv 2 \pmod 4$	$\nexists$ Lem. 2.1	$\nexists$ Lem. 2.1	$\exists$?
$n \equiv 3 \pmod 4$	$\exists$, see [7,8]	$\exists$ Thm. 2.8	$\nexists$ Lem. 2.9

The main result of this paper is that in Sec. 4 we prove the conjecture for k even and $n \equiv 2 \pmod 4$.

3 Related magic labelings

In this section we recall results on magic-type labelings used in Sec. 4. Having a graph $G = (V, E)$ a one-to-one mapping $\lambda : E \to \{1, 2, \ldots, |E|\}$ is called a *supermagic* labeling of G if there exists a constant k such that for every vertex x of G

$$w_\lambda(x) = \sum_{y \in N(x)} \lambda(xy) = k.$$

In [14] Wallis calls this a *vertex-magic edge labeling.* The constant k is the *magic constant* for the supermagic labeling λ.

There are a number of results known for supermagic labeling of graphs. Among these we pick only those relevant for our constructions below. For further reference one should consult [2].

In [15] it was shown that any complete bipartite graph $K_{n,n}$ for $n \geq 3$ admits a supermagic labeling. An example is given in Fig. 2.

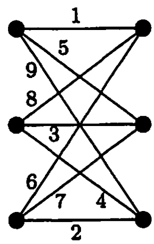

Fig. 2. A supermagic labeling σ of $K_{3,3}$ with $k = 15$.

Theorem 3.1 *[15] There exists a supermagic labeling of $K_{n,n}$ for every $n \geq 3$ with the magic constant $k = n(n^2+1)/2$.*

Ivančo in [5] showed the following theorem. Independently was this result shown in [9] using a technique similar to the one used in Sec. 4.

Theorem 3.2 *[5] Let r be an integer at least 3. Let G be an r-regular graph with a proper edge r coloring, which has an supermagic labeling λ.*

(1) If r is odd then nG has a supermagic labeling whenever n is an odd positive integer.
(2) If r is even then nG has a supermagic labeling for every positive integer n.

In fact a supermagic labeling of $K_{n,n}$ corresponds to the concept known as *magic squares*, see e.g. [12,15].

It is easy to verify that one can add an integer s to all labels in an r-regular supermagic graph G with the magic constant k and obtain another supermagic labeling of G with the magic constant $k' = k + rs$.

4 Super VMTL of $2mK_{4l+2}$

Let m, l be positive integers. In this section we prove that $2mK_{4l+2}$ admits a super VMTL. The proof is constructive and has several steps. First we find an injective total labeling, called *pre magic* labeling, of $4K_{2l+1}$ which will *not* be magic, there will be 2 pairs of K_{2l+1} having the same weight at every vertex but for each pair this sum will differ by 1. We shall point out that by Lemma 2.9 there exists no super VMTL of $4K_{2l+1}$. Next we find a supermagic labeling of the 2 copies of the complete bipartite graph $K_{2l+1,2l+1}$, each having a different magic sum that also will differ by 1. Notice again, that there is no supermagic labeling of $2K_{2l+1,2l+1}$ guaranteed by Theorem 3.2. In fact it is easy to show that no such labeling has to exists, see [5]. Finally, we glue the graphs together so that the resulting 2 copies of K_{4l+2} will have a super VMTL.

4.1 Pre magic labeling of $4K_{2l+1}$

Let l be a positive integer. By Theorem 2.3 there exists a super VMTL λ of K_{2l+1}:

$$\lambda : V(K_{2l+1}) \cup E(K_{2l+1}) \to \{1, 2, \dots, (2l+1)(l+1)\}.$$

Next we use an idea similar as in [9] and [15]. By Vizing's Theorem there exists a proper edge coloring of K_{2l+1} by $2l+1$ colors.

Now every vertex can be colored by the color not appearing on the $2l$ adjacent edges. As a result we obtain a proper *total* coloring η of K_{2l+1} by $2l+1$ colors (in a proper coloring no two incident graph elements have the same color):

$$\eta : V(K_{2l+1}) \cup E(K_{2l+1}) \rightarrow \{1, 2, \ldots, 2l+1\}.$$

Now we take four copies G_1, G_2, G_3, G_4 of $G = K_{2l+1}$. In each G_i we denote the copy of each vertex $v \in V(G)$ by v_i and the copy of each edge $e \in E(G)$ by e_i for $i = 1, 2, 3, 4$.

Based on the coloring η and the labeling λ we construct a pre magic labeling g of $4K_{2l+1}$ according to Table 2. We multiply each label $\lambda(x)$ by 4, where $x \in V(G) \cup E(G)$, and subtract 0, 1, 2, or 3 as described in the table.

Table 2
Labeling of $4K_{2l+1}$ based on a super VMTL λ and a total coloring η of K_{2l+1}.

color \ copy	$i=1$	$i=2$	$i=3$	$i=4$
$\eta(x) = 1, \ldots, l-1$	$4\lambda(x) - 0$	$4\lambda(x) - 1$	$4\lambda(x) - 2$	$4\lambda(x) - 3$
$\eta(x) = l, \ldots, 2l-1$	$4\lambda(x) - 3$	$4\lambda(x) - 2$	$4\lambda(x) - 1$	$4\lambda(x) - 0$
$\eta(x) = 2l$	$4\lambda(x) - 1$	$4\lambda(x) - 0$	$4\lambda(x) - 3$	$4\lambda(x) - 2$
$\eta(x) = 2l+1$	$4\lambda(x) - 0$	$4\lambda(x) - 2$	$4\lambda(x) - 1$	$4\lambda(x) - 3$
$w_g(v_i), \forall v_i \in V(G_i)$	$4h - 3l - 1$	$4h - 3l - 1$	$4h - 3l - 2$	$4h - 3l - 2$

First we show that the labeling

$$g : \bigcup_{i=1} (V(G_i) \cup E(G_i)) \rightarrow \{1, 2, \ldots, 4|V(G) \cup E(G)|\}$$

described above is injective. Suppose two elements x, y have the same label $4\lambda(x) - a = 4\lambda(y) - b$, where $a, b \in 0, 1, 2, 3$. Counting modulo 4 immediately follows $a = b \pmod 4$, thus $\lambda(x) = \lambda(y) \pmod 4$. We have $x = y$ since λ is injective and in each copy always a different integer is subtracted. This concludes the proof that g is injective.

Next we evaluate the vertex weight for every vertex v_1 in G_1. By $S_v(H)$ we denote the set $\{v\} \cup \{e : e \text{ adjacent to } v \text{ in } H\}$ for a vertex v in a graph H. Since λ is a super VMTL we have $w_\lambda(v) = \sum_{x \in S_v} \lambda(x) = h$. Now for every $v_1 \in V(G_1)$ we have

$$\begin{aligned} w_g(v_1) = & \sum_{x \in S_{v_1}(G_1)} g(x) \\ = & 4 \sum_{x \in S_v(G)} \lambda(x) - (l-1) \cdot 0 - l \cdot 3 - 1 - 0 \\ = & 4h - 3l - 1. \end{aligned}$$

Similarly, for every vertex v_2 in G_2 we get $w_g(v_2) = 4h - 3l - 1$ and for vertices v_3 in G_3 and v_4 in G_4 we get $w_g(v_3) = w_g(v_4) = 4h - 3l - 2$. The last row of Table 2 gives the weight of every vertex in each particular copy.

An example for $l = 1$ is in the figures below. In Fig. 3 there is a super VMTL λ of K_3, in Fig. 4 is a proper total 3-coloring η, and in Fig. 5 is the pre magic labeling g of $4K_3$ based on the super VMTL λ of K_3 from Fig. 3 and the total coloring η of K_3 from Fig. 4.

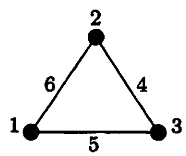

Fig. 3. A super VMTL λ of K_3 with $h = 12$.

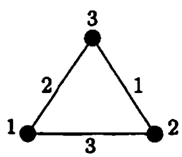

Fig. 4. A proper total coloring η of K_3 with 3 colors.

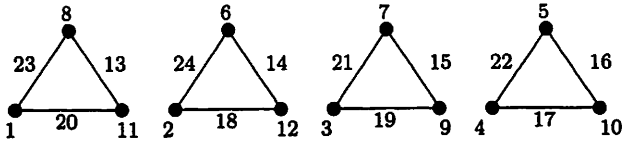

Fig. 5. Pre magic labeling g of $4K_3$.

4.2 *Complete bipartite graphs* $2K_{2l+1,2l+1}$

Now we find a supermagic labeling, for each of the 2 copies of the complete bipartite graph $K_{2l+1,2l+1}$. The magic constant will differ by 1 for each copy. The idea of the construction is similar as in the previous section.

By Theorem 3.1 there exists a supermagic labeling σ of $K_{2l+1,2l+1}$

$$\sigma : E(K_{2l+1,2l+1}) \to \{1, 2, \ldots, (2l+1)^2\}.$$

It is well known by Hall's Theorem that there exists a proper *edge* coloring χ of $K_{2l+1,2l+1}$ by $2l+1$ colors:

$$\chi : E(K_{2l+1,2l+1}) \to \{1, 2, \ldots, 2l+1\}.$$

Now we take two copies F_1, F_2 of $F = K_{2l+1,2l+1}$. In F_i we denote the copy of each edge $e \in E(F)$ by e_i for $i = 1, 2$.

Based on the coloring χ and the labeling σ we construct a supermagic labeling f of each $K_{2l+1,2l+1}$ according to Table 3. We multiply each edge label $\sigma(e)$ by 2, where $e \in E(F)$, add a constant s (since F is regular), and subtract 0 or 1 as described in the table. We set $s = 4|V(G) \cup E(G)|$, where $G = K_{2l+1}$.

We show that the labeling

$$f : E(F_1) \cup E(F_2) \to \{s+1, s+2, \ldots, s+2(2l+1)^2\},$$

is injective, where $s = 4|V(G) \cup E(G)|$ and $G = K_{2l+1}$. This is easy to observe. Since σ is injective, the smallest label assigned

Table 3
Labeling f of $2K_{2l+1,2l+1}$ based on a supermagic labeling σ and an edge coloring χ.

color \ copy	$i = 1$	$i = 2$
$\chi(e) = 1, \ldots, l$	$s + 2\sigma(e) - 1$	$s + 2\sigma(e) - 0$
$\chi(e) = l + 1, \ldots, 2l + 1$	$s + 2\sigma(e) - 0$	$s + 2\sigma(e) - 1$
$w_f(v_i)$, $\forall v_i \in V(F_i)$	$(2l + 1)s + 2k - l$	$(2l + 1)s + 2k - l - 1$

is $s + 1$, and we assign always two consecutive integers (one odd and one even) to two corresponding edges in F_1 and F_2.

Again we evaluate the vertex weight for every vertex v_1 in F_1. By $T_v(H)$ we denote the set $\{e : e \text{ adjacent to } v \text{ in } H\}$ for some vertex v in a graph H. Since σ is a super VMTL we have $w_\sigma(v) = \sum_{x \in T_v(F)} \sigma(x) = k$. Now for every $v_1 \in V(F_1)$ we have

$$\begin{aligned} w_f(v_1) &= \sum_{x \in T_{v_1}(F_1)} f(x) \\ &= (2l + 1)s + 2 \sum_{x \in T_v(F)} \sigma(x) - l \cdot 1 - (l + 1) \cdot 0 \\ &= (2l + 1)s + 2k - l. \end{aligned}$$

Similarly, for every vertex v_2 in F_2 we get $w_f(v_2) = (2l + 1)s + 2k - l - 1$. The last row of Table 3 gives the weight of every vertex in each particular copy.

Fig. 6 shows a proper edge 3-coloring χ of $K_{3,3}$. From the supermagic labeling σ of $K_{3,3}$ given in Fig. 2 and the edge coloring χ we construct two supermagic labelings of $K_{3,3}$, see Fig. 7. In this example we take $s = 0$.

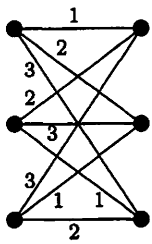

Fig. 6. An edge coloring χ of $K_{3,3}$.

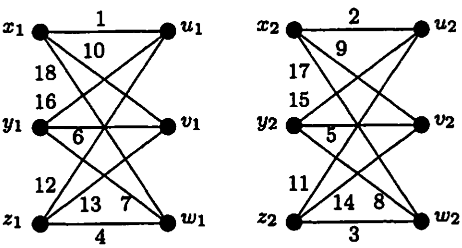

Fig. 7. Labeling f of $2K_{3,3}$.

4.3 Main result

Now we glue together the four labeled copies of K_{2l+1} and the two labeled copies of $K_{2l+1,2l+1}$ to obtain a super VMTL of $2K_{4l+2}$.

Theorem 4.1 *The graph $2K_{4l+2}$ admits a super VMTL for any positive integer l.*

PROOF. Obviously each K_{4l+2} can be decomposed into a complete bipartite graph $K_{2l+1,2l+1}$ and two copies of K_{2l+1}. We label the complete bipartite graph $F_1 = K_{2l+1,2l+1}$ by f (from the previous section) and edges of the complete graph K_{2l+1} in one partite set by g as in G_3 and in the second partite set as in G_4. Similarly, we label the second copy of K_{4l+2} using labels from $F_2 = K_{2l+1,2l+1}$ and G_1, G_2. We obtain a labeled graph $2K_{4l+2}$. We denote the resulting labeling ω.

Both the labeling g of $4K_{2l+1}$ and the labeling f of $2K_{2l+1,2l+1}$ are injective. Recall that $s = 4|V(K_{2l+1}) \cup E(K_{2l+1})|$. The set of labels for g is $\{1, 2, \dots, s\}$ and the set of labels of f is $\{s + 1, s + 2, \dots, s + 2(2l + 1)^2\}$, thus ω is an injective labeling

$$\omega : V(2K_{4l+2}) \cup E(2K_{4l+2}) \to \{1, \dots, |V(2K_{4l+2})| + |E(2K_{4l+2})|\},$$

since $s + 2(2l + 1)^2 = 4\left((2l + 1) + \binom{2l+1}{2}\right) + 2(2l + 1)^2 = (4l + 2)(4l+3) = 2(4l+2)+(4l+2)(4l+1) = |V(2K_{4l+2})|+|E(2K_{4l+2})|$.

The weight of every vertex $v \in 2K_{4l+2}$ with respect to ω is $\omega(v) = g(v) + f(v)$. For all vertices v in the graph K_{4l+2} consisting of F_1, G_3, and G_4, we get the same sum

$$w_\omega(v) = g(v) + f(v) = 4h - 3l - 1 + (2l + 1)s + 2k - 1.$$

Taking $h = (2l + 1)(2l + 2)^2/4$ by Lemma 2.2 and $k = (2l + 1)\left((2l + 1)^2 + 1)\right)/2$ by Theorem 3.1, we obtain

$$w_\omega(v) = 8(2l + 1)(2l^2 + 3l + 1).$$

Similarly, all vertices v in the second copy of K_{4l+2} consisting of F_2, G_1, G_2 will have the same weight $w_\omega(v) = 8(2l + 1)(2l^2 + 3l + 1)$.

Finally, to show that ω is a super VMTL of $2K_{4l+2}$ it is enough to observe that the smallest labels $1, 2, \dots, 8l + 4$ are at the vertices which immediately follows from Table 2 describing the labeling g. This completes the proof. □

As a consequence of Theorems 4.1 and 2.6, we have the following theorem.

Theorem 4.2 *The graph $2mK_{4l+2}$ admits a super VMTL for any positive integers m and l.*

In Fig. 8 there is a super VMTL of $2K_6$ obtained from the the labeling g of $4K_3$ in Fig. 5 and the labeling f of $2K_{3,3}$ in Fig. 7. We take $x_1 = 3$, $y_1 = 9$, $z_1 = 7$, $u_1 = 4$, $v_1 = 10$, $w_1 = 5$, $x_2 = 1$, $y_2 = 11$, $z_2 = 8$, $u_2 = 2$, $v_2 = 12$, $w_2 = 6$, and $s = 24$.

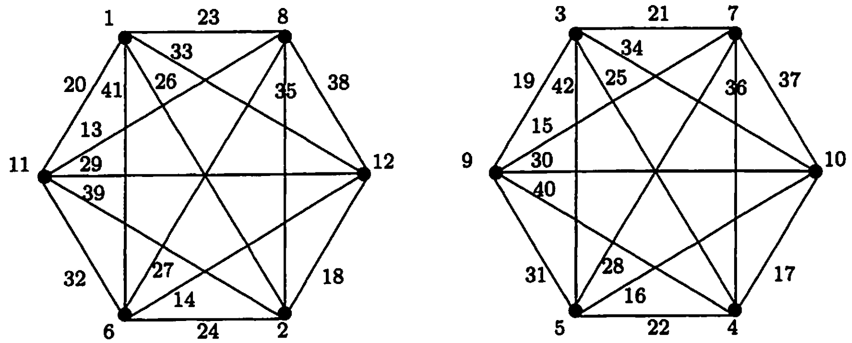

Fig. 8. A super VMTL of $2K_6$.

4.4 Final remarks

We shall point out that by the construction in the proof of Theorem 4.1 we did not find just one, but several different super VMTL of $2K_{4l+2}$. In each copy we can pick $l-1$ colors for the first row of Table 2, two colors for the last row and take any permutation of vertices in G_3, G_4. Thus we have at least $\binom{2l+1}{l-1}\binom{l+2}{2}((2l+1)!)^2$ different super VMTL of $2K_{4l+2}$.

There exists an alternate approach which yields a different labeling of $2K_{4l+2}$. First we find an injective total labeling (*pre magic* labeling) of $4K_{2l+1}$ which again will *not* be magic. There will be 2 pairs of K_{2l+1} having the same weight at every vertex but for each pair this sum will differ by $2l+1$. Next we find a supermagic labeling of $2K_{2l+1,2l+1}$, each having a different magic sum that also will differ by $2l+1$. These can be then combined to obtain a super VMTL of $2K_{4l+2}$.

There remains one unsolved case, namely finding a super VMTL of kK_n for $n = 4$ and odd $k \geq 3$. We believe there exists such labeling for every $k \geq 2$. A super VMTL of the smallest example $3K_4$ of the unsolved case is in Fig. 9.

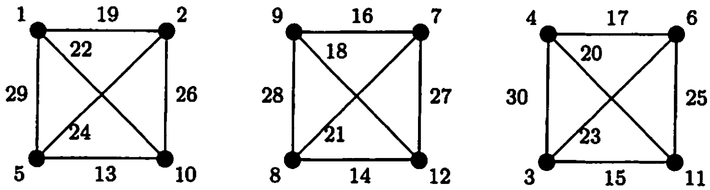

Fig. 9. A super VMTL of $3K_4$ with $h = 71$.

Conjecture 1 *If k is an odd integer, $k > 1$, then kK_4 admits a super VMTL.*

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References

[1] C. Balbuena, K.C. Das, Y. Lin, M. Miller, J. Ryan, Slamin, K. Sugeng, M. Tkáč, On the degrees of a strongly vertex-magic graph, Discrete Mathematics **306** (2006) 539–551.

[2] J.A. Gallian, A dynamic survey of graph labeling, Electronic Journal Combinatorics 5 (2008), #DS6.

[3] J. Gómez, Solution of the conjecture: If $n \equiv 0(\text{mod}4)$, $n > 4$, then K_n has a super vertex-magic total labeling, Discrete mathematics, **307** (2007) 2525–2534.

[4] J. Gómez, Two new methodos to obtain super vertex-magic total labelings of graphs, Discrete Mathematics, **308** (2008) 3361–3372.

[5] J. Ivančo, On supermagic regular graphs, Mathematica Bohemica, **125** (2000) 99–114.

[6] J.A. MacDougall, M. Miller, and K.A. Sugeng, Super vertex-magic total labelings of graphs, Proc. 15th Australian Workshop on Combinatorial Algorithms (2004) 222–229.

[7] J.A. MacDougall, M. Miller, Slamin and W.D. Wallis, Vertex-magic total labelings of graphs, Utilitas Mathematica **61** (2002) 3–21.

[8] D. McQuillan and K. Smith, Vertex-magic total labeling of odd complete graphs, Discrete Mathematics **305** (2005) 240–249.

[9] P. Kovář, Vertex magic total labeling of products of regular VMT graphs and regular supermagic graphs, JCMCC, **54** (2005) 21–31.

[10] P. Kovář, Magic labelings of regular graphs, AKCE International Journal of Graphs and Combinatorics **4** (2007) 261–275.

[11] J. Sedláček, Problem 27, Theory of graphs and its applications, Proc. symposium Smolenice (1963) 163–167.

[12] M.M. Postnikov, Magic squares, Nauka, Moskva, (1964).

[13] X. Yue, Y. Yuansheng, Mominul, W. Liping, Super Vertex-Magic Total Labeling of $W_{3,n}$, submitted to Ars Combinatoria (2007).

[14] W.D. Wallis, Vertex Magic Labelings of Multiple Graphs, Congressus Numerantium **152** (2001) 81–83.

[15] W.D. Wallis, Magic graphs, Birkhauser, (2001).