$\vec{P}_{2k+1}$-factorization of symmetric complete bipartite multi-digraphs*

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Abstract

Let $\vec{P}_l$ be the directed path on l vertices and $\lambda K^*_{m,n}$ be the symmetric complete bipartite multi-digraph with two partite sets having m and n vertices. A $\vec{P}_l$-factorization of $\lambda K^*_{m,n}$ is a set of arc-disjoint $\vec{P}_l$-factors of $\lambda K^*_{m,n}$ which is a partition of the set of arcs of $\lambda K^*_{m,n}$. In this paper, it is shown that a necessary and sufficient condition for the existence of a $\vec{P}_{2k+1}$-factorization of $\lambda K^*_{m,n}$ for any positive integer k.

1 Introduction

Let $\vec{P}_l$ be the directed path on l vertices and $K^*_{m,n}$ be the symmetric complete bipartite digraph with partite sets X and Y, where $|X| = m$ and $|Y| = n$. symmetric complete bipartite multi-digraph $\lambda K^*_{m,n}$ is the arc-disjoint union of λ digraphs each isomorphic to $K^*_{m,n}$. A subgraph F of $\lambda K^*_{m,n}$ is called a spanning subgraph of $\lambda K^*_{m,n}$ if F contains all the vertices of $\lambda K^*_{m,n}$. A $\vec{P}_l$-factor of $\lambda K^*_{m,n}$ is a spanning subgraph F of $\lambda K^*_{m,n}$ such that every component of F is a $\vec{P}_l$ and every pair of $\vec{P}_l$'s have no vertex in

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common. A $\vec{P}_l$-factorization of $\lambda K^*_{m,n}$ is a set of arc-disjoint $\vec{P}_l$-factors of $\lambda K^*_{m,n}$ which is a partition of the set of arcs of $\lambda K^*_{m,n}$. The graph $\lambda K^*_{m,n}$ is called $\vec{P}_l$-factorizable whenever it has a $\vec{P}_l$-factorization. For graph theoretical terms, see [1] and [4].

For the directed path on even vertices $\vec{P}_{2k}$, the spectrum problem for a $\vec{P}_{2k}$-factorization of $\lambda K^*_{m,n}$ has been completely solved by Wang [5] and Du [3].

Theorem 1.1 Let m, n and k be positive integers, there exists a $\vec{P}_{2k}$-factorization of $\lambda K^*_{m,n}$ if and only if $m = n \equiv 0 \ (mod\ k(2k-1)/d)$, where $d = gcd(\lambda, 2k-1)$.

For the directed path on odd vertices $\vec{P}_{2k+1}$, the spectrum problem for a $\vec{P}_{2k+1}$-factorization of $\lambda K^*_{m,n}$ seems to be much less tractable. The second author of this paper, in paper [2], gave a necessary and sufficient condition for the existence of a $\vec{P}_3$-factorization of $K^*_{m,n}$. In newly paper [6], we gave a necessary and sufficient condition for $\vec{P}_3$-factorization of the symmetric complete bipartite multi-digraph $\lambda K^*_{m,n}$. In this paper a necessary and sufficient condition for the existence of a $\vec{P}_{2k+1}$-factorization of the symmetric complete bipartite digraph $\lambda K^*_{m,n}$ will be given.

Theorem 1.2 Let m, n and k be positive integers, there exists a $\vec{P}_{2k+1}$-factorization of $\lambda K^*_{m,n}$ if and only if (1) $(k+1)n \geq km$, (2) $(k+1)m \geq kn$, (3) $m + n \equiv 0 \ (mod\ (2k+1))$, and (4) $\lambda(2k+1)mn/[k(m+n)]$ is an integer.

2 Main result

First, assume that a $\vec{P}_{2k+1}$-factorization of $\lambda K^*_{m,n}$ is given. Certain integers are defined as follows:

$e =$ the number of copies of $\vec{P}_{2k+1}$ in any factor,

$r =$ the number of $\vec{P}_{2k+1}$-factors in the factorization,

$a =$ the number of copies of $\vec{P}_{2k+1}$ with its endpoints in Y in a particular $\vec{P}_{2k+1}$-factor,

$b =$ the number of copies of $\vec{P}_{2k+1}$ with its endpoints in X in a particular $\vec{P}_{2k+1}$-factor,

$c =$ the total number of copies of $\vec{P}_{2k+1}$ in the whole factorization.

Since any $\vec{P}_{2k+1}$-factor spans $\lambda K^*_{m,n}$,

$$e = \tfrac{m+n}{2k+1}. \tag{2.1}$$

Every $\vec{P}_{2k+1}$-factor has $2ke$ arcs so that in a factorization $2\lambda mn = 2kre = 2kc$. Thus

$$r = \tfrac{(2k+1)\lambda mn}{k(m+n)}. \tag{2.2}$$

By definition of a and b, we get $2ka+(2k+1)b = m$ and $(2k+1)a+2kb = n$. Hence

$$a = \tfrac{(k+1)n-km}{2k+1}, \tag{2.3}$$

$$b = \tfrac{(k+1)m-kn}{2k+1}. \tag{2.4}$$

Since expressions (2.1)-(2.4) must be integers, we have the following necessary condition for the existence of a $\vec{P}_{2k+1}$-factorization of the symmetric complete bipartite digraph $\lambda K^*_{m,n}$

Theorem 2.1 If $\lambda K^*_{m,n}$ has a $\vec{P}_{2k+1}$-factorization, then (1) $(k+1)n \geq km$, (2) $(k+1)m \geq kn$, (3) $m+n \equiv 0$ $(mod\ (2k+1))$, and (4) $\lambda(2k+1)mn/[k(m+n)]$ is an integer.

The proof of the sufficiency of Theorem 1.2 consists of the following lemmas. We use xy to denote the arc from x to y in this paper. For any two integers x and y, we use $gcd(x,y)$ to denote the greatest common divisor of x and y. The following lemma is obvious.

Lemma 2.2 Let g, p and q be positive integers, if $gcd(p,q) = 1$, then

$$gcd(pq, p+gq) = gcd(p,g).$$

We first prove the following two results, which are used later in this paper.

Theorem 2.3 If $\lambda K^*_{m,n}$ has a $\vec{P}_{2k+1}$-factorization, then $\lambda s K^*_{m,n}$ has a $\vec{P}_{2k+1}$-factorization for every positive integer s.

Proof Construct a $\vec{P}_{2k+1}$-factorization of $\lambda K^*_{m,n}$ repeatly s times. Then we have a $\vec{P}_{2k+1}$-factorization of $\lambda s K^*_{m,n}$.

Theorem 2.4 If $\lambda K^*_{m,n}$ has a $\vec{P}_{2k+1}$-factorization, then $\lambda K^*_{sm,sn}$ has a $\vec{P}_{2k+1}$-factorization for every positive integer s.

Proof Let $\{F_i : 1 \leq i \leq s\}$ be a 1-factorization of $K_{s,s}$ (whose existence, see [1, 4]). For each $i \in \{1, 2, \cdots, s\}$, replace every edge of F_i by a $\lambda K^*_{m,n}$ to get a factor G_i of $\lambda K^*_{sm,sn}$ such that the graph G_i are pairwise arc-disjoint and their union is $\lambda K^*_{sm,sn}$. Since $\lambda K^*_{m,n}$ has a $\vec{P}_{2k+1}$-factorization, it is clear that the graph G_i, too, has a $\vec{P}_{2k+1}$-factorization. Consequently, $\lambda K^*_{sm,sn}$ has a $\vec{P}_{2k+1}$-factorization. This proves the theorem.

Now we start to prove our main result Theorem 1.2. There are three cases to consider.

Case $(k+1)m = kn$: In this case, for $1 \leq j \leq k+1$, let $F_j = \{x_i y_{i+j}, y_{i+j+1} x_i : 1 \leq i \leq k\}$, which is a $\vec{P}_{2k+1}$-factor of $K^*_{k,k+1}$, and then $\cup_{1 \leq j \leq k+1} F_j$ is a $\vec{P}_{2k+1}$-factorization of $K^*_{k,k+1}$. From Theorem 2.3 and Theorem 2.4 $\lambda K^*_{m,n}$ has a $\vec{P}_{2k+1}$-factorization.

Case $km = (k+1)n$: Obviously, $\lambda K^*_{m,n}$ has a $\vec{P}_{2k+1}$-factorization.

Case $(k+1)m > kn$ and $(k+1)n > km$: In this case, let $a = [(k+1)n - km]/(2k+1)$, $b = [(k+1)m - kn]/(2k+1)$, $e = (m+n)/(2k+1)$, and $r = \lambda(2k+1)mn/[k(m+n)]$. Then from Conditions (1)-(4) in theorem 1.2, a, b, e, r are integers and $0 < a < m$ and $0 < b < n$. We have $ka + (k+1)b = m$ and $(k+1)a + kb = n$. Hence $r = \lambda(k+1)(a+b) + \lambda ab/[k(a+b)]$. Let $z = \lambda ab/[k(a+b)]$, which is a positive integer. And let $gcd(ka, (k+1)b) = d$, $ka = dp$, $(k+1)b = dq$, where $gcd(p,q) = 1$. Then $z = \lambda dpq/\{k[(k+1)p + kq]\}$. These equalities imply the following equalities:

$$d = \frac{k[(k+1)p+kq]z}{\lambda pq},$$

$$m = \frac{k(p+q)[(k+1)p+kq]z}{\lambda pq},$$

$$n = \frac{[(k+1)^2 p + k^2 q][(k+1)p+kq]z}{\lambda(k+1)pq},$$

$$r = \frac{(p+q)[(k+1)^2 p + k^2 q]z}{pq},$$

$$a = \frac{p[(k+1)p+kq]z}{\lambda pq},$$

$$b = \frac{kq[(k+1)p+kq]z}{\lambda(k+1)pq}.$$

Let $k = {p_1}^{k_1} {p_2}^{k_2} \cdots {p_\gamma}^{k_\gamma}$, where p_1, p_2, $\cdots$, p_γ are distinct prime numbers, $k_1, k_2, \cdots, k_\gamma$ are positive integers, and $k+1 = q_1^{h_1} q_2^{h_2} \cdots q_\omega^{h_\omega}$, where $q_1, q_2, \cdots, q_\omega$ are distinct prime numbers, $h_1, h_2, \cdots, h_\omega$ are positive integers. In the following lemma we classify m and n by use of the prime factorization of $gcd(p, k^2)$ and $gcd(q, (k+1)^2)$, such that the proof of our main result is reduced to the construction of one case.

Lemma 2.5 If $gcd(p, k^2) = p_1^{i_1} p_2^{i_2} \cdots p_\alpha^{i_\alpha} p_{\alpha+1}^{2k_{\alpha+1}-i_{\alpha+1}} p_{\alpha+2}^{2k_{\alpha+2}-i_{\alpha+2}} \cdots p_\beta^{2k_\beta - i_\beta} p_{\beta+1}^{2k_{\beta+1}} p_{\beta+2}^{2k_{\beta+2}} \cdots p_\gamma^{2k_\gamma}$, where $1 \le \alpha \le \beta \le \gamma$, $0 \le i_j \le k_j$ (when $1 \le j \le \alpha$) or $0 < i_j < k_j$ (when $\alpha + 1 \le j \le \beta$). And $gcd(q, (k+1)^2) = q_1^{j_1} q_2^{j_2} \cdots q_\mu^{j_\mu} q_{\mu+1}^{2h_{\mu+1}-j_{\mu+1}} q_{\mu+2}^{2h_{\mu+2}-j_{\mu+2}} \cdots q_\nu^{2h_\nu - j_\nu} q_{\nu+1}^{2h_{\nu+1}} q_{\nu+2}^{2h_{\nu+2}} \cdots q_\omega^{2h_\omega}$, where $1 \le \mu \le \nu \le \omega$, $0 \le j_i \le h_i$ (when $1 \le i \le \mu$) or $0 < j_i < h_i$ (when $\mu + 1 \le i \le \nu$). Let

$$s = p_1^{i_1} p_2^{i_2} \cdots p_\alpha^{i_\alpha}, \quad t = p_1^{k_1 - i_1} p_2^{k_2 - i_2} \cdots p_\alpha^{k_\alpha - i_\alpha}, \quad u = p_{\alpha+1}^{i_{\alpha+1}} p_{\alpha+2}^{i_{\alpha+2}} \cdots p_\beta^{i_\beta},$$

$$v = p_{\alpha+1}^{k_{\alpha+1} - i_{\alpha+1}} p_{\alpha+2}^{k_{\alpha+2} - i_{\alpha+2}} \cdots p_\beta^{k_\beta - i_\beta}, \quad w = p_{\beta+1}^{k_{\beta+1}} p_{\beta+2}^{k_{\beta+2}} \cdots p_\gamma^{k_\gamma},$$

$$s' = q_1^{j_1} q_2^{j_2} \cdots q_\mu^{j_\mu}, \quad t' = q_1^{h_1 - j_1} q_2^{h_2 - j_2} \cdots q_\mu^{h_\mu - j_\mu}, \quad u' = q_{\mu+1}^{j_{\mu+1}} q_{\mu+2}^{j_{\mu+2}} \cdots q_\nu^{j_\nu},$$

$$v' = q_{\mu+1}^{h_{\mu+1} - j_{\mu+1}} q_{\mu+2}^{h_{\mu+2} - j_{\mu+2}} \cdots q_\nu^{h_\nu - j_\nu}, \quad w' = q_{\nu+1}^{h_{\nu+1}} q_{\nu+2}^{h_{\nu+2}} \cdots q_\omega^{h_\omega},$$

and let $p = suv^2w^2p'$, $q = s'u'v'^2w'^2q'$ and $gcd(su(vwt'p' + tv'w'q'), \lambda) = \eta$. Then

$$d = stut'(vwt'p' + tv'w'q')z'/\eta,,$$

$$m = stut'(suv^2w^2p' + s'u'v'^2w'^2q')(vwt'p' + tv'w'q')z'/\eta,$$

$$n = suvwv'w'(s't'^2u'p' + st^2uq')(vwt'p' + tv'w'q')z'/\eta,$$

$$r = \lambda t'v'w'(suv^2w^2p' + s'u'v'^2w'^2q')(s't'^2u'p' + st^2uq')z'/\eta,$$

$$a = suvwt'p'(vwt'p' + tv'w'q')z'/\eta, \quad b = stuv'w'q'(vwt'p' + tv'w'q')z'/\eta,$$

for some positive integer z'.

Proof We assume the $gcd(p,k^2) = suv^2w^2$, $gcd(q,(k+1)^2) = s'u'v'^2w'^2$, $gcd(p,q) = 1$ and $p = suv^2w^2p'$, $q = s'u'v'^2w'^2q'$ holds. Then $gcd(vwt'p', tv'w'q') = gcd(p',q') = 1$, $gcd(stu, p'q') = 1$ and $gcd(stu, v'w') = gcd(vwt'p' + tv'w'q', v'w') = 1$. It is easy to see that

$$d = \frac{stu(vwt'p'+tv'w'q')z}{\lambda v'w'p'q'}.$$

By Lemma 2.2, we see that $gcd(vwt'p' + tv'w'q', p'q') = 1$. therefore

$$\frac{z}{v'w'p'q'}$$

must be an integer. Let

$$z_1 = \frac{z}{v'w'p'q'},$$

then we have

$$b = \frac{stuv'w'q'(vwt'p'+tv'w'q')z_1}{\lambda t'}.$$

Since $gcd(stuv'w'q', t') = gcd(vwt'p' + tv'w'q', t') = 1$, we see that

$$\frac{z_1}{t'}$$

must be an integer. Let

$$z_2 = \frac{z_1}{t'}.$$

Let $gcd(\lambda, suvwt'p'(vwt'p'+tv'w'q')) = \eta_1$ and $gcd(\lambda, stuv'w'q'(vwt'p'+tv'w'q')) = \eta_2$. By $a = suvwt'p'(vwt'p' + tv'w'q')z_2/\lambda$ and $b = stuv'w'q'(vwt'p' + tv'w'q')z_2/\lambda$, we see that $z_2\eta_1/\lambda$ and $z_2\eta_2/\lambda$ must be integers. Since $gcd(vwt'p', tv'w'q') = 1$, so we have $z_2\eta/\lambda$ must be an integer, where $gcd(\lambda, su(vwt'p' + tv'w'q')) = \eta$. Let $z' = z_2\eta/\lambda$. Then the equalities hold.

For the proof of Theorem 1.2, we only need the following direct construction. In this construction, we first give a set of arcs, which form some copies of $\vec{P}_{2k+1}$. And, it is shown that these copies of $\vec{P}_{2k+1}$ are vertex-disjoint and cover all the vertices of $\lambda K^*_{m,n}$, and piece together to a $\vec{P}_{2k+1}$-factor.

Finally, we get r $\vec{P}_{2k+1}$-factors by rotational variants of this $\vec{P}_{2k+1}$-factor, and prove that r distinct $\vec{P}_{2k+1}$-factors are piece together to $\lambda K^*_{m,n}$, then construct a $\vec{P}_{2k+1}$-factorization of $\lambda K^*_{m,n}$.

Lemma 2.6 For any positive integers s, t, u, v, w, s', t', u', v', w', p' and q', let

$$m = stut'(suv^2w^2p' + s'u'v'^2w'^2q')(vwt'p' + tv'w'q')/\lambda,$$

$$n = suvwv'w'(s't'^2u'p' + st^2uq')(vwt'p' + tv'w'q')/\lambda.$$

Then $\lambda K^*_{m,n}$ has a $\vec{P}_{stuvw+s't'u'v'w'}$-factorization if $stuvw + 1 = s't'u'v'w'$ and $su(vwt'p' + tv'w'q')/\lambda$ is an integer.

Proof Let $\xi = su$, $\xi' = s'u'$, $\zeta = vw$, $\zeta' = v'w'$ and $f = su(vwt'p' + tv'w'q')/\lambda = \xi(\zeta t'p' + t\zeta'q')/\lambda$. Expressions (2.1)-(2.4) imply that $e = su(vwt'p'+tv'w'q')^2/\lambda = (\zeta t'p'+t\zeta'q')f$, $r = t'v'w'(suv^2w^2p'+s'u'v'^2w'^2q')(s't'^2u'p'+st^2uq') = t'\zeta'(\xi\zeta^2p'+\xi'\zeta'^2q')(\xi't'^2p'+\xi t^2q')$, $a = suvwt'p'(vwt'p' +tv'w'q')/\lambda = \zeta t'p'f$, $b = stuv'w'q'(vwt'p' + tv'w'q')/\lambda = t\zeta'q'f$. Let $r_1 = t'(suv^2w^2p' + s'u'v'^2w'^2q') = t'(\xi\zeta^2p' + \xi'\zeta'^2q')$, $r_2 = v'w'(s't'^2u'p' + st^2uq') = \zeta'(\xi't'^2p' + \xi t^2q')$. Let X and Y be the two partite sets of $\lambda K^*_{m,n}$ and set

$$X = \{x_{i,j} :\ 1 \le i \le r_1;\ 1 \le j \le tf\},$$

$$Y = \{y_{i,j} :\ 1 \le i \le r_2;\ 1 \le j \le \zeta f\}.$$

We remark in advance that the additions in the first subscripts of $x_{i,j}$'s and $y_{i,j}$'s are taken modulo r_1 and r_2 in $\{1, 2, \cdots, r_1\}$ and $\{1, 2, \cdots, r_2\}$, respectively, and the additions in the second subscripts of $x_{i,j}$'s and $y_{i,j}$'s are taken modulo tf and ζf in $\{1, 2, \cdots, tf\}$ and $\{1, 2, \cdots, \zeta f\}$, respectively.

In the remainder of the proof, we first construct a model of a $\vec{P}_{2stuvw+1}$-factor of $\lambda K^*_{m,n}$, and then get r arc-disjoint $\vec{P}_{2stuvw+1}$-factors by rotational variants of this model, finally piece these factors together to form the required factorization.

For making a $\vec{P}_{2stuvw+1}$-factor of $\lambda K^*_{m,n}$, we need $e = su(vwt'p' + tv'w'q')^2/\lambda = (\zeta t'p' + t\zeta'q')f$ vertex-disjoint copies of $\vec{P}_{2stuvw+1}$. Among these copies, there consist of $a = suvwt'p'(vwt'p' + tv'w'q')/\lambda = \zeta t'p'f$ Type I copies and $b = stuv'w'q'(vwt'p' + tv'w'q')/\lambda = t\zeta'q'f$ Type II copies, where Type I denote the $\vec{P}_{2stuvw+1}$ with its start vertex and end vertex in Y and Type II with its start vertex and end vertex in X.

Type I: copies of $\vec{P}_{2stuvw+1}$ with their start vertices and end vertices in Y

Let

$$I_0 = \{x_{y',1+fz}y_{y'+\xi\zeta z,1+y'} : 1 \le y' \le \xi\zeta,\ 0 \le z \le t-1\}$$

$$\cup\{y_{y'+\xi\zeta z+1,2+y'}x_{y',1+fz} : 1 \le y' \le \xi\zeta - 1,\ 0 \le z \le t-1\}$$

$$\cup\{y_{\xi\zeta(z+1)+1,2}x_{\xi\zeta,1+fz} : 0 \le z \le t-1\},$$

which is a Type I copy of $\vec{P}_{2stuvw+1}$ where $y_{\xi\zeta t+1,2}$ is the start vertex and $y_{1,2}$ is end. Let "1" be changed to variable "j" ($1 \le j \le f$) in the second subscripts of $x_{i,j}$'s and $y_{i,j}$'s in I_0, we have set I which consist of f vertex-disjoint Type I copies.

$$I = \{x_{y',j+fz}y_{y'+\xi\zeta z,j+y'} : 1 \le y' \le \xi\zeta,\ 0 \le z \le t-1,\ 1 \le j \le f\}$$

$$\cup\{y_{y'+\xi\zeta z+1,j+y'+1}x_{y',j+fz} : 1 \le y' \le \xi\zeta - 1,\ 0 \le z \le t-1,\ 1 \le j \le f\}$$

$$\cup\{y_{\xi\zeta(z+1)+1,j+1}x_{\xi\zeta,j+fz} : 0 \le z \le t-1,\ 1 \le j \le f\},$$

For $1 \le x' \le \zeta$, Let the first subscripts of $x_{i,j}$'s, and the second subscripts of $y_{i,j}$'s in I add expression "$\xi\zeta(x'-1)$" and "$f(x'-1)$", respectively. We get E_0 as follows.

$$E_0 = \{x_{y'+\xi\zeta(x'-1),j+fz}y_{y'+\xi\zeta z,j+y'+f(x'-1)} :$$

$$1 \le y' \le \xi\zeta,\ 0 \le z \le t-1,\ 1 \le j \le f,\ 1 \le x' \le \zeta\}$$

$$\cup\{y_{y'+\xi\zeta z+1,j+y'+1+f(x'-1)}x_{y'+\xi\zeta(x'-1),j+fz} :$$

$$1 \le y' \le \xi\zeta - 1,\ 0 \le z \le t-1,\ 1 \le j \le f,\ 1 \le x' \le \zeta\}$$

$$\cup\{y_{\xi\zeta(z+1)+1,j+1+f(x'-1)}x_{\xi\zeta x',j+fz} :$$

$$0 \le z \le t-1,\ 1 \le j \le f,\ 1 \le x' \le \zeta\},$$

E_0 contains ζf vertex-disjoint Type I copies. For $1 \le i \le t'p'$, let the first subscripts of $x_{i,j}$'s and $y_{i,j}$'s in E_0 add expression "$\xi\zeta^2(i-1)$" and "$t'\xi'\zeta'(i-1)$", respectively, the second subscripts of $y_{i,j}$'s add expression "$\xi\zeta(i-1)$". We get all Type I copies.

For each $1 \le i \le t'p'$, let

$$E_i = \{x_{y'+\xi\zeta(x'-1)+\xi\zeta^2(i-1),j+fz}y_{y'+\xi\zeta z+t'\xi'\zeta'(i-1),j+y'+f(x'-1)+\xi\zeta(i-1)}:$$

$$1 \le y' \le \xi\zeta,\ 0 \le z \le t-1,\ 1 \le j \le f,\ 1 \le x' \le \zeta\}$$

$$\cup\{y_{y'+\xi\zeta z+1+t'\xi'\zeta'(i-1),j+y'+1+f(x'-1)+\xi\zeta(i-1)}x_{y'+\xi\zeta(x'-1)+\xi\zeta^2(i-1),j+fz}:$$

$$1 \le y' \le \xi\zeta-1,\ 0 \le z \le t-1,\ 1 \le j \le f,\ 1 \le x' \le \zeta\}$$

$$\cup\{y_{\xi\zeta(z+1)+1+t'\xi'\zeta'(i-1),j+1+f(x'-1)+\xi\zeta(i-1)}x_{\xi\zeta x'+\xi\zeta^2(i-1),j+fz}:$$

$$0 \le z \le t-1,\ 1 \le j \le f,\ 1 \le x' \le \zeta\},$$

Notice $\xi\zeta t+1 = t'\xi'\zeta'$ ($stuvw+1 = s't'u'v'w'$), for each i ($1 \le i \le t'p'$), E_i contains ζf vertex-disjoint Type I copies of $\vec{P}_{2stuvw+1}$. And $\cup_{1\le i\le t'p'}E_i$ contains $a = \zeta t'p'f$ vertex-disjoint Type I copies of $\vec{P}_{2stuvw+1}$. The first subscripts of $x_{i,j}$'s and $y_{i,j}$'s cover $\{1,2,\cdots,\xi\zeta^2 t'p'\}$ and $\{1,2,\cdots,t'^2\xi'\zeta'p'\}$, respectively. And, for each of the first subscripts of $x_{i,j}$'s and $y_{i,j}$'s, the second subscripts of $x_{i,j}$'s and $y_{i,j}$'s cover $\{1,2,\cdots,tf\}$ and $\{1,2,\cdots,\zeta f\}$, respectively. And there are $2\xi\zeta^2 tft'p'$ arcs in $\cup_{1\le i\le t'p'}E_i$.

Type II: copies of $\vec{P}_{2stuvw+1}$ with their start vertices and end vertices in X

Let

$$R_0 = \{x_{\zeta(x'-1)+y',1}y_{x',1+f(y'-1)+x'},\ y_{x',1+f(y'-1)+x'}x_{\zeta(x'-1)+y'+1,1}:$$

$$1 \le x' \le t\xi,\ 1 \le y' \le \zeta\},$$

which is a Type II copy of $\vec{P}_{2stuvw+1}$ where $x_{1,1}$ is the start and $x_{t\xi\zeta+1,1}$ is end vertex. Let "1" be changed to the variable "j" ($1 \le j \le f$) in the second subscripts of $x_{i,j}$'s and $y_{i,j}$'s in R_0, we have set R which consist of f vertice-disjoint Type II copies.

$$R = \{x_{\zeta(x'-1)+y',j}y_{x',j+f(y'-1)+x'},\ y_{x',j+f(y'-1)+x'}x_{\zeta(x'-1)+y'+1,j} :$$

$$1 \le x' \le t\xi,\ 1 \le y' \le \zeta,\ 1 \le j \le f\}.$$

For $0 \le z \le t-1$, Let the second subscripts of $x_{i,j}$'s, and the first subscripts of $y_{i,j}$'s in R add expression "fz" and "ξtz", respectively. We get C_0 as follows.

$$C_0 =$$

$$\{x_{\zeta(x'-1)+y',j+fz}y_{x'+\xi tz,j+f(y'-1)+x'},\ y_{x'+\xi tz,j+f(y'-1)+x'}x_{\zeta(x'-1)+y'+1,j+fz} :$$

$$1 \le x' \le t\xi,\ 1 \le y' \le \zeta,\ 1 \le j \le f,\ 0 \le z \le t-1\}.$$

C_0 contains tf vertex-disjoint Type II copies. For $1 \le i \le \zeta' q$, let the first subscripts of $x_{i,j}$'s and $y_{i,j}$'s in C_0 add expression "$t'\xi'\zeta'(i-1)$" and "$t^2\xi(i-1)$", respectively, the second subscripts of $y_{i,j}$'s add expression "$t\xi(i-1)$". We have C_i $(1 \le i \le \zeta' q')$

$$C_i = \{x_{\zeta(x'-1)+y'+t'\xi'\zeta'(i-1),j+fz}y_{x'+\xi tz+t^2\xi(i-1),j+f(y'-1)+x'+t\xi(i-1)},$$

$$y_{x'+\xi tz+t^2\xi(i-1),j+f(y'-1)+x'+t\xi(i-1)}x_{\zeta(x'-1)+y'+t'\xi'\zeta'(i-1)+1,j+fz} :$$

$$1 \le j \le f,\ 1 \le x' \le t\xi,\ 1 \le y' \le \zeta,\ 0 \le z \le t-1\}.$$

Finally, let the first subscripts of $x_{i,j}$'s and $y_{i,j}$'s in C_i add expression "$\xi\zeta^2 t'p'$" and "$t'^2\xi'\zeta'p'$", respectively, the second subscripts of $y_{i,j}$'s add expression "$\xi\zeta t'p'$". We get all Type II copies.

For each $1 \le i \le \zeta' q'$, let

$$E_{t'p'+i} = \{x_{\zeta(x'-1)+y'+t'\xi'\zeta'(i-1)+\xi\zeta^2 t'p',j+fz}$$

$$y_{x'+\xi tz+t^2\xi(i-1)+t'^2\xi'\zeta'p',j+f(y'-1)+x'+t\xi(i-1)+\xi\zeta t'p'};$$

$$y_{x'+\xi tz+t^2\xi(i-1)+t'^2\xi'\zeta'p',j+f(y'-1)+x'+t\xi(i-1)+\xi\zeta t'p'}$$

$$x_{\zeta(x'-1)+y'+t'\xi'\zeta'(i-1)+\xi\zeta^2 t'p'+1,j+fz} :$$

$$1 \le j \le f,\ 1 \le x' \le t\xi,\ 1 \le y' \le \zeta,\ 0 \le z \le t-1\}.$$

Notice $\xi\zeta t+1=\xi'\zeta' t'$ $(stuvw+1=s't'u'v'w')$, each of $E_{t'p'+i}$ $(1 \le i \le \zeta' q')$ consists of tf vertex-disjoint Type II copies of $\vec{P}_{2stuvw+1}$. And $\cup_{1\le i\le \zeta' q'} E_{t'p'+i}$ contains $b=t\zeta' q' f$ vertex-disjoint Type II copies of $\vec{P}_{2stuvw+1}$. The first subscripts of $x_{i,j}$'s and $y_{i,j}$'s cover $\{\xi\zeta^2 t'p'+1, \xi\zeta^2 t'p'+2, \cdots, r_1=\xi\zeta^2 t'p'+t'\xi'\zeta'^2 q'\}$ and $\{t'^2\xi'\zeta' p'+1, t'^2\xi'\zeta' p'+2, \cdots, r_2=t'^2\xi'\zeta' p'+t^2\xi\zeta' q'\}$, respectively. And, for each of the first subscripts of $x_{i,j}$'s and $y_{i,j}$'s, the second subscripts of $x_{i,j}$'s and $y_{i,j}$'s cover $\{1,2,\cdots,tf\}$ and $\{1,2,\cdots,\zeta f\}$, respectively. And there are $2\xi\zeta t^2\zeta' q' f$ arcs in $\cup_{1\le i\le \zeta' q'} E_{t'p'+i}$.

Let $F=\cup_{1\le i\le t'p'+\zeta' q'} E_i$. Obviously, F contains $2\xi\zeta t(\zeta t'p'+t\zeta' q')f$ arcs. the first subscripts of $x_{i,j}$'s and $y_{i,j}$'s cover $\{1,2,\cdots,r_1\}$ and $\{1,2,\cdots, r_2\}$, respectively, and for the given first subscripts of $x_{i,j}$'s and $y_{i,j}$'s, the second subscripts of $x_{i,j}$'s and $y_{i,j}$'s cover $\{1,2,\cdots,tf\}$ and $\{1,2,\cdots,\zeta f\}$, respectively. And more, F contains $e=a+b=(\zeta t'p'+t\zeta' q')f$ vertex-disjoint and arc-disjoint $\vec{P}_{2stuvw+1}$ components of $\lambda K^*_{m,n}$. Then the digraph F is a $\vec{P}_{2stuvw+1}$-factor of $\lambda K^*_{m,n}$.

Further, in $\cup_{1\le i\le t'p'} E_i$, the second subscripts of $y_{i,j}$'s with which arcs directed from the set of vertices in X with any given fixed second subscripts "j" of $x_{i,j}$'s to the set of vertices in Y are from "$1+f(x'-1)+j$" to "$\xi\zeta t'p'+f(x'-1)+j$" for $1\le x'\le \zeta$. And in $\cup_{t'p'+1\le i\le t'p'+\zeta' q'} E_i$, the second subscripts of $y_{i,j}$'s with which arcs directed from the set of vertices in X with any given fixed second subscripts "j" of $x_{i,j}$'s to the set of vertices in Y are from "$1+f(y'-1)+\xi\zeta t'p'+j$" to "$f(y'-1)+\xi\zeta t'p'+t\xi\zeta' q'+j=f(y'-1)+\lambda f+j$" for $1\le y'\le\zeta$. Notice the range of the second subscripts of $y_{i,j}$'s is $1\le j\le \zeta f$. By calculation, we can make a conclusion that in F the number of arcs directed from the set of vertices in with any given fixed second subscript to the set of vertices in Y with a given fixed second subscript is precisely λ in every case. By the same reason, the number of arcs directed from the set of vertices in Y with any given fixed second subscript to the set of vertices in X with a given fixed second subscript is also precisely λ in every case.

Define a bijection σ from $X\cup Y$ onto $X\cup Y$ in such a way that $\sigma(x_{i,j})=x_{i+1,j}$ and $\sigma(y_{i,j})=y_{i+1,j}$. For each $\mu\in\{1,2,\cdots,r_1\}$ and each $\nu\in\{1,2,\cdots,r_2\}$, let

$F_{\mu,j}=\{\sigma^{\mu}(x)\sigma^{\nu}(y) : x\in X, y\in Y, xy\in F\}\cup\{\sigma^{\nu}(y)\sigma^{\mu}(x) : x\in X, y\in Y, yx\in F\}$.

As before analysis, we know the graphs $F_{\mu,\nu}$ $(1\le\mu\le r_1,\ 1\le\nu\le r_2)$ are the $\vec{P}_{2stuvw+1}$-factors of $\lambda K^*_{m,n}$ and their union is $\lambda K^*_{m,n}$. Thus, $\{F_{\mu,\nu} : 1\le\mu\le r_1,\ 1\le\nu\le r_2\}$ is a $\vec{P}_{2stuvw+1}$-factorization of $\lambda K^*_{m,n}$.

This proves the lemma.

The proof of Theorem 1.2: By applying Theorem 2.3 and theorem 2.4 with Lemma 2.5 and Lemma 2.6, it can be seen that when the parameters m and n satisfy conditions (1)-(4) in Theorem 1.2, the graph $\lambda K^*_{m,n}$ has a $\vec{P}_{2k+1}$-factorization. This completes the proof of Theorem 1.2.

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