On the Dynamic Coloring of Strongly Regular Graphs * †

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Abstract

A proper vertex coloring of a graph G is called a *dynamic coloring* if for every vertex v with degree at least 2, the neighbors of v receive at least two different colors. It was conjectured that if G is a regular graph, then $\chi_2(G) - \chi(G) \leq 2$. In this paper we prove that, apart from the cycles C_4 and C_5 and the complete bipartite graphs $K_{n,n}$, every strongly regular graph G, satisfies $\chi_2(G) - \chi(G) \leq 1$.

1. Introduction

Let G be a graph. We denote the vertex set and the edge set of G by $V(G)$ and $E(G)$, respectively. The number of vertices of G is called the *order* of G. A *proper vertex coloring* of G is a function $c : V(G) \longrightarrow L$, with this property: if $u, v \in V(G)$ are adjacent, then $c(u)$ and $c(v)$ are different. A *vertex k-coloring* is a proper vertex coloring with $|L| = k$. A proper vertex k-coloring of a graph G is called a *dynamic coloring* if for every vertex v with degree at least 2, the neighbors of v receive at least two different colors. The smallest integer k such that G has a dynamic k-coloring is called the *dynamic chromatic number* of G and is denoted by

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$\chi_2(G)$. For every $v \in V(G)$, $N(v)$ denotes the neighbor set of v. Let G be a graph with coloring c. Then $d(v)$ and $c(N(v))$ denote the degree of v and the set of all colors appearing on the neighbors of v, respectively. In this paper we denote the cycle of order n and the complete bipartite graph with part sizes m and n by C_n and $K_{m,n}$, respectively. In a vertex coloring of G, we say that the dynamic property holds for vertex v, if one of the following holds: (i) $d(v) \leq 1$, (ii) $d(v) \geq 2$ and there are at least two vertices with different colors incident with v. A graph G of order n is called ***strongly k-regular*** if there are parameters k, λ and μ such that G is k-regular, every adjacent pair of vertices have λ common neighbors, and every nonadjacent pair of vertices have μ common neighbors. Montgomery [5] conjectured that for every regular graph G, $\chi_2(G) - \chi(G) \leq 2$. In this paper we show that if $G \neq C_4, C_5$ and $K_{k,k}$, then for every strongly regular graph G, $\chi_2(G) - \chi(G) \leq 1$.

Conjecture 1. [5] *For every regular graph G, $\chi_2(G) - \chi(G) \leq 2$.*

Remark 2. If P is the Petersen graph, then clearly $\chi(P) = 3$. We want to show that $\chi_2(P) = 4$. By contradiction suppose that $\chi_2(P) = 3$. Consider the following figure:

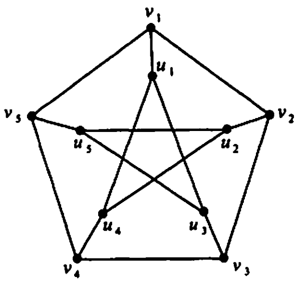

Figure 1

Assume that $c : V(P) \longrightarrow \{1, 2, 3\}$ is a dynamic 3-coloring of P. With no loss of generality, one may assume that $c(v_1) = c(v_3) = 1$, $c(v_2) = c(v_4) = 2$, and $c(v_5) = 3$. Since the dynamic property holds for vertices v_2 and v_3, we conclude that $c(u_2) = c(u_3) = 3$. Thus the dynamic property for u_5 does not hold, a contradiction. By Theorem 1 of [4], $\chi_2(P) \leq 4$. Hence $\chi_2(P) = 4$ and $\chi_2(P) - \chi(P) = 1$.

Theorem 3. *Let G be a strongly k-regular graph with $\mu = 1$ and $G \neq C_5, P$, where P is the Petersen graph. Then $\chi_2(G) = \chi(G)$.*

Proof. If $\lambda > 0$, then every vertex is contained in a triangle and so we have $\chi_2(G) = \chi(G)$. Thus assume that $\lambda = 0$. If $k = 2$, then the assertion is trivial. By [2, p.855], the only strongly 3-regular graphs are $K_{3,3}$ and the Petersen graph. Therefore we can suppose that $k \geq 4$. To the contrary, assume that $\chi_2(G) \neq \chi(G)$. Let c be a vertex $\chi(G)$-coloring of G such that the number of vertices for which the dynamic property holds is maximum. Let $v \in V(G)$ and suppose that the dynamic property does not hold for v. Suppose that $N(v) = \{v_1, \ldots, v_k\}$. Without loss of generality, we can suppose that $c(v) = 1$ and $c(N(v)) = \{2\}$. Note that since $\lambda = 0$, $N(v_i)\backslash\{v\}$ is an independent set for each i, $i = 1, \ldots, k$ and since $\mu = 1$, for every $i, j \in \{1, \ldots, k\}, i \neq j$, $N(v_i) \cap N(v_j) = \{v\}$. Moreover, for every $j, j \neq i$, and $x \in N(v_i)\backslash\{v\}$, $|N(x) \cap N(v_j)| = 1$.

First, we claim that for every $w_i \in N(v_i) \setminus \{v\}$, $c(N(w_i)) = \{2, c_i\}$, where $c_i \in \{1, \ldots, \chi(G)\} \setminus \{2\}$. Clearly, $c(N(w_i)) \neq \{2\}$. Now, by contradiction assume that there are three distinct colors $\{2, x, y\} \subseteq c(N(w_i))$. One of the colors x and y is not 1. With no loss of generality assume that $x \neq 1$. Now, change $c(v_i)$ to color x and next change all colors x in $N(v_i)$ to color 2 and call this coloring c'. Clearly, the dynamic property holds for v. We show that the dynamic property remains for those vertices which had the dynamic property before. Since $\lambda = 0$ and $\mu = 1$, using the equation $k(k - \lambda - 1) = \mu(n - k - 1)$, [7, p.465], we have $n = k^2 + 1$. This implies that $V(G) = N(v) \cup (\cup_{j=1}^{k} N(v_j))$. We note that for every $j, j \neq i$, $c'(v_j) = c(v_j)$ and $c'(N(v_j)) = c(N(v_j))$ and so v_j has the dynamic property in c if and only if v_j has the dynamic property in c'. Now, assume that $j \neq i$ and $z \in N(v_j) \setminus \{v\}$. Since $k \geq 3$, there exists $q \neq j, i$, such that $N(z) \cap N(v_q) = \{a\}$. But $c'(a) \neq 2$ and so the dynamic property holds for z. Obviously, if the dynamic property holds for v_i in coloring c, then it holds for v_i in coloring c'. Now, we would like to show that the dynamic property holds for every $v \in N(v_i)\backslash\{v\}$. We have $\{x, y\} \subseteq c'(N(w_i))$ and so w_i has the dynamic property. Let $z \in N(v_i) \setminus \{v, w_i\}$. Assume that $s \in N(w_i)$ and $c'(s) = x$. Suppose that $s \in N(v_r)$. Since $\mu = 1$, we have $sz \notin E(G)$ and so $N(s) \cap N(z) = \{p\}$. Since $ps \in E(G)$ and $c'(s) = x$, $c'(p) \neq x$ and the dynamic property holds for z. Thus the number of vertices in c' for which the dynamic property holds is more than the number of vertices in c for which the dynamic property holds, a contradiction. Hence, $c(N(w_i)) = \{2, c_i\}$.

Next, we want to prove that $|c(N(v_j) \setminus \{v\})| = k - 1$ for $j = 1, \ldots, k$. To the contrary and with no loss of generality assume that there is a color

$b \in \{1, \ldots, \chi(G)\}$ such that $w_1, u_1 \in N(v_1) \setminus \{v\}$ and $c(w_1) = c(u_1) = b$. Let $N(w_1) \cap N(v_2) = \{w_2\}$. Thus, as we did before, $c(N(w_2)) = \{2, b\}$. Since $\mu = 1$, $w_2 u_1 \notin E(G)$. Thus, w_2 and u_1 should have a common neighbor, say t. But $c(t) = b$, a contradiction. Hence, $\chi(G) \geq k$. Now, Brook's Theorem [7, p.197] implies that $\chi(G) = k$. Since $c(v_j) = 2$ for every j, $1 \leq j \leq k$, we conclude that $3 \in c(N(v_j) \setminus \{v\})$. For every vertex $w \in V(G)$ with $c(w) = 3$, change the color of w to a color from the set $\{1, \ldots, \chi(G)\} \setminus (c(N(w)) \cup \{3\})$ to obtain a vertex $(\chi(G) - 1)$-coloring of G, a contradiction. Thus, every vertex has the dynamic property in c and so $\chi_2(G) = \chi(G)$ and the proof is complete. □

Now, we would like to prove that except for C_4, C_5, and $K_{k,k}$, for every strongly k-regular graph G there is a vertex coloring by $\chi(G)$ colors such that the dynamic property does not hold for at most one vertex of G.

Theorem 4. *Let $G \neq C_4, C_5, K_{k,k}$ be a strongly k-regular graph. Then we can color the vertices of G by $\chi(G)$ colors such that the dynamic property does not hold for at most one vertex.*

Proof. If $\chi(G) = 2$, then G is bipartite. Thus $-k$ is an eigenvalue of G [1, p.53]. If G is a strongly regular graph which is not a complete graph, then it has three distinct eigenvalues, [7, p.466]. Since the eigenvalues of every bipartite graph are symmetric about the origin, we conclude that if $G \neq K_2$ is a strongly k-regular graph, then $\{-k, 0, k\}$ are eigenvalues of G [1, p.53]. This yields that G is a complete multipartite graph [3, p.163]. Hence G is $K_{k,k}$, where $n = 2k$ and $n = |V(G)|$. Thus assume that $\chi(G) \geq 3$. If $\lambda > 0$, then every vertex of G is contained in a triangle. So $\chi_2(G) = \chi(G)$. Thus assume that $\lambda = 0$. If $\mu = k$, then $0, -k = \frac{1}{2}(\lambda - \mu \pm \sqrt{(\lambda - \mu)^2 + 4(k - \mu)})$ [3, p.194] are eigenvalues of G. So by [6, p.399], G is bipartite, a contradiction. Thus we can assume that $\mu \neq k$. Clearly, the assertion holds for $\mu = 0$. Assume that $\mu = 1$. If G is the Petersen graph, then it is not hard to see that there is a vertex 3-coloring such that the dynamic property fails for exactly one vertex. Thus by Theorem 3 we can assume that $\mu \geq 2$. Now, consider a vertex $\chi(G)$-coloring such that the number of vertices of G for which the dynamic property doesn't hold is as small as possible. Let's call this number l. It suffices to show that $l \leq 1$. To the contrary, suppose that $l \geq 2$. Consider that vertex coloring, say c, in which the dynamic property does not hold for exactly l vertices. Assume that v is one of these vertices. So, we can suppose that $c(v) = 1$

and $c(N(v)) = \{2\}$. Let $H = G \setminus (\{v\} \cup (N(v)))$. None of the vertices of H can have color 2, because if $w \in V(H)$ and $c(w) = 2$, then they should have μ common neighbors, a contradiction. Since $\mu \geq 2$, every vertex of H should be adjacent to at least two vertices of $N(v)$. Let $x \in V(H)$. Since $\mu \neq k$, $N(x) \neq N(v)$. Thus the dynamic property holds for vertex x. Now, assume that there exists $y \in N(v)$ such that the dynamic property does not hold for y. So, all neighbors of y should have color 1. Now, by changing $c(y)$ to 3, the dynamic property holds for v in the new coloring. Moreover, since $\mu \geq 2$, every vertex $z \in H$ is adjacent to a vertex with color 2 and also a vertex with a color different from 2 in H. Thus, we obtain a coloring of G such that the number of vertices for which the dynamic property fails is less than l, a contradiction. Hence $l \leq 1$ and the proof is complete. □

We close the paper with the following corollary.

Corollary 5. *If $G \neq C_4, C_5, K_{k,k}$ is a strongly regular graph, then $\chi_2(G) - \chi(G) \leq 1$ and so Conjecture 1 is true for strongly regular graphs.*

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