On super restricted edge-connectivity of vertex-transitive graphs *

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Abstract Let $X = (V, E)$ be a connected vertex-transitive graph with degree k. Call X super restricted edge-connected, in short, sup-λ', if F is a minimum edge set of X such that $X - F$ is disconnected and every component of $X - F$ has at least two vertices, then F is the set of edges adjacent to a certain edge in X. Wang [Y, Q, Wang, Super restricted edge-connectivity of vertex-transitive graphs, Discrete Mathematics 289 (2004) 199-205] proved that a connected vertex-transitive graph with degree $k > 2$ and girth $g > 4$ is sup-λ'. In this paper, by studying the λ'-superatom of X, we present sufficient and necessary conditions for connected vertex-transitive graphs and Cayley graphs with degree $k > 2$ to be sup-λ'. In particular, sup-λ' connected vertex-transitive graphs with degree $k > 2$ and girth $g > 3$ are completely characterized. These results can be seen as an improvement of the one which is obtained by Wang.
Keywords: Vertex-transitive graph; Restricted edge-connectivity; λ'-optimal; Super restricted edge-connectivity; Cayley graph

1 Introduction

A network can be conveniently modeled as a graph $X = (V, E)$, with vertices representing nodes and edges representing links. A classic measure of network reliability is the edge-connectivity $\lambda(X)$. In general, the larger $\lambda(X)$ is, the more reliable the network is. For $\lambda(X) \leq \delta(X)$, where $\delta(X)$ is the minimum degree of X, a graph X with $\lambda(X) = \delta(X)$ is naturally said to be *maximally edge-connected*, or *λ-optimal* for simplicity. A graph X

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is said to be *vertex-transitive* if for any two vertices u and v in X, there is an automorphism α of X such that $v = \alpha(u)$. Similarly, A graph X is said to be *edge-transitive* if for any two edges e_1 and e_2 in X, there is an automorphism α of X such that $e_2 = \alpha(e_1)$. Mader [7] proved the following beautiful result

Theorem 1.1. *All connected vertex-transitive graphs are maximally edge-connected.*

The problem of exploring edge-connected properties stronger than the maximally edge-connectivity for graphs has been the theme of much research. The first candidate may be the so-called *super edge-connectivity*. A graph X is said to be *super edge-connected*, in short, *sup-λ*, if each of its minimum edge-cut isolates a vertex, that is, every minimum edge-cut is a set of edges incident to a certain vertex in X. By the definitions, a sup-λ graph must be a λ-optimal graph. However, the converse is not true. For example, $K_m \times K_2$ is a λ-optimal graph by Theorem 1.1 but not sup-λ since the set of edges between the two copies of K_m is a minimum edge-cut which does not isolate any vertex.

The concept of sup-λ was originally introduced by Bauer et al. see [1], where combinatorial optimization problems in design of reliable probabilistic graphs were investigated. The following theorem is a nice result of Tindell, see [10], which characterized super edge-connectivity for vertex-transitive graphs.

Theorem 1.2. *A connected vertex-transitive graph X which is neither a cycle nor a complete graph is sup-λ if and only if it contains no clique K_k where k is the degree of X.*

For further study, Esfahanian and Hakimi introduce the concept of restricted edge-connectivity [4]. The concept of restricted edge-connectivity is one kind of conditional edge-connectivity proposed by Harary in [5], and has been successfully applied in the further study of tolerance and reliability of networks, see [3,6,13]. Let F be a set of edges in X. Call F a *restricted edge-cut* if $X - F$ is disconnected and contains no isolated vertices. The minimum cardinality over all restricted edge-cuts is called *restricted edge-connectivity* of X, and denoted by $\lambda'(X)$. It is shown by Wang and Li that the larger $\lambda'(X)$ is, the more reliable the network is [12]. In [4], the authors proved that if a connected graph X of order $n \geq 4$ is not a star $K_{1,n-1}$, then $\lambda'(X)$ is well-defined and $\lambda(X) \leq \lambda'(X) \leq \xi(X)$, where $\xi(X) = min\{d_X(u) + d_X(v) - 2 : uv \in E(X)\}$ is the minimum edge degree of X. Hence, a graph X with $\lambda'(X) = \xi(X)$ is called a *λ'-optimal graph.* Call X *super restricted edge-connected*, in short, *sup-λ'*, if every minimum restricted edge-cut isolates an edge, that is, every minimum restricted edge-cut is a set of edges adjacent to a certain edge with minimum edge degree in X. By the definitions, a sup-λ' graph must be a λ'-optimal graph. However, the converse is not true since there are many λ'-optimal graphs not to be sup-λ'. For example, C_l $(l \geq 6)$, the cycle of length l is a trivial counterexample.

It should be point out that if $\delta(X) \geq 3$, then a λ'-optimal graph must be sup-λ. In fact, a graph X is sup-λ if and only if $\lambda(X) < \lambda'(X)$ [6]. Thus, the concepts of λ-optimal graph, sup-λ graph, λ'-optimal graph and sup-λ' graph describe reliable interconnection structure for graphs at different levels.

In [8], Meng studied behavior of the parameter λ' for connected vertex-transitive graphs. The main result in [8] may be restate as follows:

Theorem 1.3. *Let X be a k-regular-connected vertex-transitive graph which is neither a cycle nor a complete graph. Then X is not λ'-optimal if and only if it contains a $(k-1)$-regular subgraph Y satisfying $k \leq |V(Y)| \leq 2k-3$.*

Recently, Wang [14] concerned the super restricted edge-connectivity of connected vertex-transitive graphs. The main result is

Theorem 1.4. *If X is a connected vertex-transitive graph with degree $k > 2$ and girth $g > 4$, then it is sup-λ'.*

In this paper, by studying the λ'-superatom of X, we present sufficient and necessary conditions for connected vertex-transitive graphs and Cayley graphs with degree $k > 2$ to be sup-λ'. In particular, sup-λ' connected vertex-transitive graphs with degree $k > 2$ and girth $g > 3$ are completely characterized. These results can be seen as an improvement of Theorem 1.4.

We shall closely follow [2] for graph-theoretical terminology and notation not defined here.

2 Preliminary

In this paper, we often refer to the following two graphs, which are called Bowtie and Enhanced ladder: L_p, respectively.

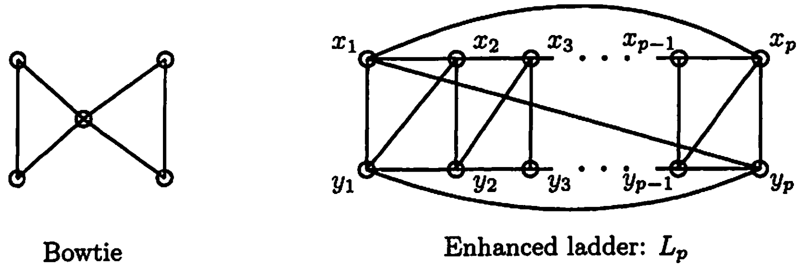

Fig. 1. The defined graphs.

Let $X = (V, E)$ be a graph. For two disjoint non-empty subsets A and B of V, let $[A, B] = \{e = xy \in E : x \in A \textit{ and } y \in B\}$. For the sake of convenience, we write x for the single vertex set $\{x\}$. If $\overline{A} = V \backslash A$, then we write $\omega(A)$ for $[A, \overline{A}]$ and $d(A)$ for $|\omega(A)|$.

A restricted edge cut F of X is called a λ'-cut if $|F| = \lambda'(X)$. It is easy to see that for any λ'-cut F, $X - F$ has exactly two connected non-trivial components. Let A be a proper subset of V. If $\omega(A)$ is a λ'-cut of X, then A is called a *λ'-fragment* of X. It is clear that if A is a λ'-fragment of X, then so is $\overline{A}$. Let $r(X)$=min$\{\ |A|$: *A is a λ'-fragment of X*$\}$. Obviously, $2 \leq r(X) \leq \frac{1}{2}|V|$. A λ'-fragment B is called a *λ'-atom* of X if $|B| = r(X)$. A λ'-fragment C is called a *strict λ'-fragment* if $3 \leq |C| \leq |V(X)| - 3$. If X contains strict λ'-fragments, then the ones with smallest cardinality are called *λ'-superatoms*.

In [16], Xu proved the following two main results.

Theorem 2.1. *Let $X = (V, E)$ be a connected graph with at least four vertices and $X \not\cong K_{1,m}$. Then X is λ'-optimal if and only if $r(X) = 2$.*

Theorem 2.2. *Let $X = (V, E)$ be a connected graph with at least four vertices and $X \not\cong K_{1,m}$. If X is not λ'-optimal, then any two distinct λ'-atoms of X are disjoint.*

By the definition of λ'-superatom, we easily have the following lemma.

Lemma 2.3. *Let $X = (V, E)$ be a connected graph with at least four vertices and $X \not\cong K_{1,m}$. Then X is sup-λ' if and only if it has no λ'-superatoms.*

Cayley graph is an important class of vertex transitive graphs. Let G be a group and S a subset of $G \setminus \{1_G\}$ with $S = S^{-1}$, where 1_G is the identity of G. Define the *Cayley graph* $C(G, S) = (V, E)$, where $V = G$,

$E = \{\{g, gs\} : g \in G, s \in S\}$. It is well known that $C(G, S)$ is connected if and only if $G =< S >$, where $< S >$ is the group generated by S.

The *line graph* of X, denoted by $L(X)$, is a graph with vertex set $E(X)$ and $e_1, e_2 \in E(X)$ are adjacent if and only if they are incident in X.

Recall that an *imprimitive block* for a permutation group Φ on a set T is a proper, non-trivial subset A of T such that if $\varphi \in \Phi$ then either $\varphi(A) = A$ or $\varphi(A) \cap A = \emptyset$. A subset A of $V(X)$ is called an *imprimitive block* for X if it is an imprimitive block for $Aut(X)$ on $V(X)$.

Proposition 2.4. *[11] Let X be a connected graph and Y be the subgraph induced by an imprimitive block A of X.*
(1) If X is vertex-transitive, then so is Y.
(2) If $X = C(G, S)$ is a Cayley graph, and A contains the identity of G, then A is a subgraph of G.

Let X and Y be two graphs. The *lexicographic product* of X by Y, denoted by $X[Y]$, is the graph with vertex set $V(X) \times V(Y)$ and, for two vertices (x_1, y_1) and (x_2, y_2) of $X[Y]$, (x_1, y_1) and (x_2, y_2) are adjacent if and only if either x_1 and x_2 are adjacent in X or $x_1 = x_2$ and y_1 and y_2 are adjacent in Y. We use $X \times Y$ to denote the *cartesian product* of X and Y. M_n denotes the Möbius ladder with n rungs.

3 λ'-superatoms

We first establish some lemmas.

Lemma 3.1. *Let X be a $k(> 2)$-regular graph. If X has a λ'-superatom A, then $|A| \geq k - 1$.*

Proof. Since A is a λ'-superatom, we obtain that $d(A) = \lambda'(X) \leq 2k - 2$. Considering the sum of degrees of all vertices of A, we have

$$\begin{aligned} k|A| &= \sum_{x \in A} d_X(x) \leq |A|(|A| - 1) + d(A) \\ &\leq |A|(|A| - 1) + 2k - 2 \\ &= k|A| - (k - |A| - 1)(|A| - 2). \end{aligned}$$

It follows that $|A| \geq k - 1$ since $|A| \geq 3$. □

Lemma 3.2. *Let X be a k (> 2)-regular graph. If X has a λ'-superatom A, then $\delta(X[A]) \geq 2$.*

Proof. By contradiction, let u be a vertex in A with $d_{X[A]}(u) = 1$. Set $A' = A \backslash \{u\}$. Then both $X[A']$ and $X[\overline{A'}]$ are connected. By the definition of λ'-superatom, we have $|A| \geq 3$, and then $|A'| \geq 2$. Clearly, $|\overline{A'}| = |\overline{A}| + 1 \geq 4$. Thus $[A', \overline{A'}]$ is a restricted edge-cut. Since $k > 2$, we have

$$\lambda'(X) \leq |[A', \overline{A'}]| = |[A, \overline{A}]| + 1 - (d_X(u) - 1) < |[A, \overline{A}]| = \lambda'(X),$$

a contradiction. □

Lemma 3.3. *Let X be a connected vertex-transitive graph with degree $k > 2$. If X has λ'-superatoms, then the intersection of distinct λ'-superatoms is empty except for two cases:*
(1) X is isomorphic to one of the following graphs: $C_m \times K_2$, M_m, $C_m[K_2]$ ($m \geq 4$) or L_p ($p \geq 3$), or
(2) $X \cong L(X_1)$, where X_1 is a 3-regular-connected edge-transitive graph with girth $g \geq 4$.

Proof. If X is not λ'-optimal, then the definitions of λ'-atom and λ'-superatom are the same by Theorem 2.1. By Theorem 2.2, we have that the intersection of distinct λ'-superatoms is empty. Thus, in the following, we assume that X is λ'-optimal.

By contradiction, let X be a λ'-optimal vertex-transitive graph, A_1 and A_2 be two distinct λ'-superatoms with $A_1 \cap A_2 \neq \emptyset$. Then we have the following claims.

Claim 1. $X[A_1 \cup A_2]$ and $X[V \setminus (A_1 \cap A_2)]$ are connected.

In fact, by the definition of λ'-superatom, $X[A_1]$, $X[A_2]$, $X[V \setminus A_1]$ and $X[V \setminus A_2]$ are connected. The results then follow from the facts $A_1 \cap A_2 \neq \emptyset$ and $(V \setminus A_1) \cap (V \setminus A_2) \neq \emptyset$.

Claim 2. $|A_1 \cap A_2| < 3$. If $|A_1 \cap A_2| = 2$, then $X[A_1 \cap A_2] \cong K_2$ and $\omega(A_1 \cup A_2)$ is a λ'-cut.

If not, $|A_1 \cap A_2| \geq 3$. Then by definition, if $X[A_1 \cap A_2]$ is connected, we have that $\omega(A_1 \cap A_2)$ is a restricted edge cut with $3 \leq |A_1 \cap A_2| \leq |V(X)| - 3$. Since A_1 is a λ'-superatom and $A_1 \cap A_2$ is a proper subset of A_1, then we obtain

$$d(A_1 \cap A_2) = |\omega(A_1 \cap A_2)| > d(A_1) = \lambda'(X).$$

Otherwise, if $X[A_1 \cap A_2]$ is not connected, then since X is λ'-optimal and $2k - 2 > k$, we have X is sup-λ, and so $d(A_1 \cap A_2) \geq 2k > \lambda'(X)$. Similarly, if $X[V \setminus (A_1 \cup A_2)]$ is connected, then $d(A_1 \cup A_2) \geq \lambda'(X)$. Otherwise, if $X[V \setminus (A_1 \cup A_2)]$ is not connected, then $d(A_1 \cup A_2) > \lambda'(X)$.

But, from the well-known submodular inequality (see [11]), we have

$$2\lambda'(X) < |\omega(A_1 \cap A_2)| + |\omega(A_1 \cup A_2)| \leq |\omega(A_1)| + |\omega(A_2)| = 2\lambda'(X),$$

it is a contradiction.

If $|A_1 \cap A_2| = 2$, then $|V \setminus (A_1 \cup A_2)| \geq |A_1 \cap A_2| = 2$. Assume that $X[A_1 \cap A_2]$ or $X[V \setminus A_1 \cup A_2]$ are not connected, we have $d(A_1 \cap A_2) > \lambda'(X)$ or $d(A_1 \cup A_2) > \lambda'(X)$. By a similar argument as above, we can obtain a contradiction. Thus, we have that $X[A_1 \cap A_2]$ and $X[V \setminus (A_1 \cup A_2)]$ are connected graphs, $|\omega(A_1 \cap A_2)| \geq \lambda'(X)$ and $|\omega(A_1 \cup A_2)| \geq \lambda'(X)$. By the submodular inequality, we obtain $|\omega(A_1 \cap A_2)| = |\omega(A_1 \cup A_2)| = \lambda'(X)$. Thus, $\omega(A_1 \cup A_2)$ is a λ'-cut.

Claim 3. $|A_1 \setminus A_2| = |A_2 \setminus A_1| < 3$.

If not, consider $A_1 \setminus A_2$. Set $B_1 = V \setminus A_2$. Then $A_1 \setminus A_2 = A_1 \cap B_1$, and

$$|A_1 \cap B_1| = |A_1 \setminus A_2| \geq 3,$$
$$|V \setminus (A_1 \cap B_1)| \geq |A_2| \geq 3,$$
$$|V \setminus (A_1 \cup B_1)| = |A_2 \setminus A_1| \geq 3.$$

By a similar argument to that of Claim 2, we can derive a contradiction.

By Claim 2 and 3, we have $3 \leq |A_1| = |A_2| \leq 4$. Thus, in the remaining proof we only consider $|A_1| = 3$ or $|A_1| = 4$.

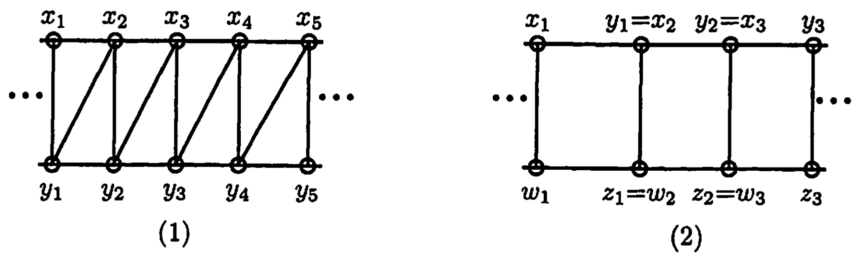

Fig. 2. The proof of Lemma 3.3.

Claim 4. If $|A_1| = 3$, then $X \cong L_p (p \geq 3)$ or $L(X_1)$, where X_1 is a 3-regular-connected edge-transitive graph with girth $g \geq 4$.

Assume $|A_1| = 3$, then $X[A_1] \cong C_3$ by Lemma 3.2. Since A_1 is a λ'-superatom and X is λ'-optimal, we have $3k - 6 = |\omega(A_1)| = \lambda'(X) = 2k - 2$ and $k = 4$. If $|A_1 \cap A_2| = 2$, then $X[A_1 \cap A_2] \cong K_2$ by Claim 2. Assume that $X[A_1]$ is the cycle $x_1x_2y_1$, and $X[A_2]$ is the cycle $y_1y_2x_2$ (see Fig.2 (1)).

By Claim 2, we have $\omega(A_1 \cup A_2)$ is a λ'-cut, and then x_1 is not adjacent to y_2. Since y_1 is contained in two adjacent triangles, by the vertex-transitivity of X, y_2 is contained in two adjacent triangles. For the degree of X is 4, without loss of generality, let x_3 be a vertex such that $y_2x_2x_3$ is a triangle which is adjacent to the triangle $y_1y_2x_2$ (see Fig.2 (1)). Since x_3 is also contained in two adjacent triangles and $k = 4$, there must exist a vertex y_3 such that $y_2y_3x_3$ is a triangle which is adjacent to the triangle $y_2x_2x_3$. Continuing this process, as X is finite, there exists an integer p such that $x_{p+1} = x_1$ and $y_{p+1} = y_1$. Then $X \cong L_p(p \geq 3)$.

If $|A_1 \cap A_2| = 1$, let $X[A_1]$ be the cycle $x_1y_1z_1$ and $X[A_2]$ be the cycle $x_2y_2z_1$. Assume $[\{x_1, y_1\}, \{x_2, y_2\}]$
$\neq \emptyset$, without loss of generality, let $x_1x_2 \in E(X)$. Considering two λ'-superatoms $B_1 = \{x_1, y_1, z_1\}$ and $B_2 = \{x_1z_1x_2\}$, we have $|B_1 \cap B_2| = 2$. Applying a similarly argument as above, we can obtain $X \cong L_p$. Thus $X[A_1 \cup A_2]$ is an induced subgraph which is isomorphic to Bowtie. Let X_1 be a graph with vertices corresponding to the triangles of X, two vertices are adjacent if and only if the two corresponding triangles have exactly one vertex in common in X. Since X is vertex-transitive and Bowtie is an induced subgraph of X, it is not difficult to verify that X_1 is a 3-regular-connected edge-transitive graph with girth $g \geq 4$ and $X \cong L(X_1)$.

Claim 5. If $|A_1| = 4$, then $X \cong C_m \times K_2$, M_m or $C_m[K_2](m \geq 4)$.

In fact, since $4k - 12 \leq |\omega(A_1)| = \lambda'(X) = 2k - 2$, we have $k \leq 5$.

If $k = 3$, then $X[A_1]$ has four edges, and so $X[A_1]$ is isomorphic to a 4-cycle by Lemma 3.2. By Claim 2 and 3, we have $X[A_1 \cap A_2] \cong K_2$. Assume that $X[A_1]$ is the cycle $Q_1 = x_1y_1z_1w_1$, and $X[A_2]$ is the cycle $Q_2 = x_2y_2z_2w_2$, where $y_1 = x_2$ and $z_1 = w_2$ (see Fig.2 (2)).

Since y_1 is in exactly 2 cycles of length 4, by vertex-transitivity of X, y_2 and z_2 must also be in exactly 2 cycles of length 4. As $k = 3$, we see that y_2 and z_2 are in the same cycles of length 4. Let $Q_3 = x_3y_3z_3w_3$ be the cycle of length 4 containing y_2 and z_2 and different from Q_2, where $x_3 = y_2$ and $w_3 = z_2$. Continuing this process, we get a sequence of cycles $Q_i = x_iy_iz_iw_i(i \geq 1)$ with $x_i = y_{i-1}$ and $w_i = z_{i-1}$ such that the intersection of the two consecutive ones is K_2. As X is finite, there exists an integer m such that $y_{m+1} = x_1$ and $w_{m+1} = w_1$ (or $y_{m+1} = w_1$ and $w_{m+1} = x_1$). Then $X \cong C_m \times K_2$ (or M_m). Since A_1 is a λ'-superatom, we have $m \geq 4$.

If $k = 4$, then $X[A_1]$ has five edges, and so $X[A_1]$ is isomorphic to $K_4 \setminus e$ by Lemma 3.2. Since $K_4 \setminus e$ contains 3-cycles, let $u_1u_2u_3$ be a 3-cycle of $X[A_1]$, we easily derive that $\{u_1, u_2, u_3\}$ is a strict λ'-fragment. It contradicts to that A_1 is a λ'-superatom.

If $k = 5$, then $X[A_1]$ has six edges, and so $X[A_1]$ is isomorphic to K_4.

Note that $|A_1 \cap A_2| = 2$, we have $X[A_1 \cap A_2] \cong K_2$. By a similar argument as above, we deduce that $X \cong C_m[K_2] (m \geq 4)$.

In all cases, we obtain contradictions, thus $A_1 \cap A_2 = \emptyset$. □

4 Main results

Now we prove the following main results.

Theorem 4.1. *Let X be a $k(> 2)$-regular-connected vertex-transitive graph with $X \ncong K_{k+1}$ and $|V(X)| \neq 2k$. Then X is not sup-λ' if and only if one of the following conditions holds:*
(1) X contains a $(k-1)$-regular induced subgraph Y satisfying $k \leq |V(Y)| \leq 2k-2$, or
(2) X contains a subgraph $Y \cong K_{k-1}$ $(k > 3)$.

Proof. For condition (1), let $A = V(Y)$. Clearly, $Y = X[A]$ is a connected graph with $|A| \geq 3$ and $k \leq |\omega(A)| \leq 2k-2$. If $X - A$ has at least one component of order at least 3. Write the vertex set of this component as B. Then $\omega(B)$ is a restricted edge-cut with $3 \leq |B| \leq |V(X)| - 3$ and $|\omega(B)| \leq |\omega(A)| \leq 2k-2$. Since $X[B]$ and $X[\overline{B}]$ are connected graphs, we have that X is not sup-λ'. Thus we assume that all components of $X - A$ are isolated edges or isolated vertices. Since $k \leq |\omega(A)| \leq 2k-2$, $X - A$ must be isomorphic to K_1 or K_2. In the case $X - A \cong K_1$, it is easy to see that $X \cong K_{k+1}$, a contradiction. In the case $X - A \cong K_2$, write $V \setminus A = \{x, y\}$. Since Y is $(k-1)$-regular, each vertex in A has exactly one neighbor in $\{x, y\}$. Thus, $|A| = 2k-2$ and $|V(X)| = 2k$, it is a contradiction. For condition (2), by a similar argument as above, we can prove that X is not sup-λ'.

Now we prove the necessity. Assume X is not sup-λ', then X has λ'-superatoms by Lemma 2.3, and let A be a λ'-superatom of X. If X is isomorphic to one of the following graphs: $C_m \times K_2$, M_m, $C_m[K_2]$ $(m \geq 4)$, L_p $(p \geq 3)$ or $L(X_1)$, where X_1 is a 3-regular-connected edge-transitive graph with girth $g \geq 4$, then we can verify that X satisfies condition (1) or (2). Thus, in the following, we can assume that X is not isomorphic to the following graphs: $C_m \times K_2$, M_m, $C_m[K_2]$ $(m \geq 4)$, $G_{2,p}$ $(p \geq 3)$, and $L(X_1)$, where X_1 is a 3-regular-connected edge-transitive graph with girth $g \geq 4$. By Lemma 3.3, we see that A is an imprimitive block of X. It follows from Proposition 2.4(1) that $X[A]$ itself is vertex-transitive, therefore, let t be the degree of $X[A]$. Since $d(A) = |\omega(A)| \leq 2k-2$, we have

$$2k - 2 \geq d(A) = |\omega(A)| = |A|(k-t) \geq (k-1)(k-t),$$

this implies $1 \leq k - t \leq 2$, and then $k - 2 \leq t \leq k - 1$.

If $t = k - 1$, then $X[A]$ is a $(k-1)$-regular induced subgraph of X satisfying $k \leq |V(X[A])| \leq 2k-2$. If $t = k-2$, then $X[A]$ is a $(k-2)$-regular induced subgraph of X satisfying $|V(X[A])| \leq k-1$, that is $X[A] \cong K_{k-1}$. Since $t \geq 2$ by Lemma 3.2, we have $k > 3$. □

For Cayley graphs, we have the following necessary and sufficient condition.

Theorem 4.2. *Let $X = C(G,S)$ be a connected Cayley graph which is neither a cycle nor a complete graph. Then $X = C(G,S)$ is not sup-λ' if and only if one of the following conditions holds:*
(1) $S = S_1 \cup \{t\}$, where t is an element of order 2, and $|S| \leq |< S_1 >| \leq 2|S| - 2$, or
(2) $S = S_1 \cup \{t_1, t_2\}$, where either t_1 and t_2 are elements of order at least 3 with $t_2 = t_1^{-1}$, or t_1 and t_2 are distinct elements of order 2, and $|< S_1 >| = |S| - 1$.

Proof. Write $k = |S|$. For condition (1), let $A =< S_1 >$. It is easy to see that $X[A] = C(A, S_1)$. Thus, $X[A]$ is a connected $(k-1)$-regular subgraph with $|A| \geq 3$ and $k \leq |\omega(A)| \leq 2k-2$. If $X - A$ has at least one component of order at least 3. Write the vertex set of this component as B. Then $\omega(B)$ is a restricted edge-cut with $3 \leq |B| \leq |V(X)| - 3$ and $|\omega(B)| \leq |\omega(A)| \leq 2k-2$. Since $X[B]$ and $X[\overline{B}]$ are connected graphs, we have that X is not sup-λ'. Thus we assume that all components of $X - A$ are isolated edges or isolated vertices. Since $k \leq |\omega(A)| \leq 2k-2$, $X - A$ must be isomorphic to K_1 or K_2. In the case $X - A$ is an isolated vertex x, then $x \in V \setminus A$, and all edges incident with x have the same label t. But then $k = 1$, contradicting $k \geq 3$. Similarly, if $X - A$ is an isolated edge, then $k = 2$, also a contradiction. For condition (2), by a similar argument as above, we can prove that X is not sup-λ'.

Conversely, assume X is not sup-λ', then X has λ'-superatoms by Lemma 2.3, and let A be a λ'-superatom containing the identity element. If X is isomorphic to one of the following graphs: $C_m \times K_2$, M_m, $C_m[K_2]$ $(m \geq 4)$, L_p $(p \geq 3)$ or $L(X_1)$, where X_1 is a 3-regular-connected edge-transitive graph with girth $g \geq 4$, then we can verify that X satisfies condition (1) or (2). Thus, in the following, we can assume that X is not isomorphic to the following graphs: $C_m \times K_2$, M_m, $C_m[K_2]$ $(m \geq 4)$, $G_{2,p}$ $(p \geq 3)$, and $L(X_1)$, where X_1 is a 3-regular-connected edge-transitive graph with girth $g \geq 4$. By Lemma 3.3, we see that A is an imprimitive block of X. It follows from Proposition 2.4(2) that A is subgroup of G. Let $S_1 = A \cap S$. Then $A =< S_1 >$, and $X[A] = C(A, S_1)$. By Theorem 4.1, one of the following conditions occurs:

(1) $X[A]$ is a connected $(k-1)$-regular graph with $k \leq |A| \leq 2k-2$.

In this case, $|S_1| = k-1$. Let $\{t\} = S \setminus S_1$. By the symmetry of S and S_1, t is an element of order 2.

(2) $X[A] \cong K_{k-1}$. In this case, $|S_1| = k-2$. Let $\{t_1, t_2\} = S \setminus S_1$. By the symmetry of S and S_1, we have $\{t_1^{-1}, t_2^{-1}\} = \{t_1, t_2\}$. Thus, either $t_2 = t^{-1}$ or both t_1 and t_2 are elements of order 2. □

The following lemma will be needed which is a simple consequence of Turán's theorem on triangle free graphs.

Lemma 4.3. *A k-regular graph with girth at least four has at least $2k$ vertices, and (up to isomorphism) there exactly one graph with girth four on $2k$ vertices, that is $K_{k,k}$.*

If $g > 3$, there are only three classes of graphs which are not sup-λ'.

Theorem 4.4. *Let X be a connected vertex-transitive graph with degree $k > 2$ and girth $g > 3$. Then X is not sup-λ' if and only if $X \cong C_m \times K_2$, or M_m $(m \geq 4)$, or X contains a subgraph $Y \cong K_{k-1,k-1}$ and $X \not\cong K_{k,k}$.*

Proof. First, we prove the sufficiency. It is easy to verify that X is not sup-λ' if $X \cong C_m \times K_2$, or M_m $(m \geq 4)$. Suppose X contains a subgraph $Y \cong K_{k-1,k-1}$, let $A = V(Y)$. Clearly, $Y = X[A]$ is a connected graph with $|A| = 2(k-1) \geq 4$ and $|\omega(A)| = 2k-2$. If $X - A$ has at least one component of order at least 3. Write the vertex set of this component as B. Then $\omega(B)$ is a restricted edge-cut with $3 \leq |B| \leq |V(X)| - 3$ and $|\omega(B)| \leq |\omega(A)| = 2k-2$. Since $X[B]$ and $X[\overline{B}]$ are connected graphs, we have that X is not sup-λ'. Thus we assume that all components of $X - A$ are isolated edges or isolated vertices. Since $|\omega(A)| = 2k-2$, $X - A$ must be isomorphic to K_2. Since $Y \cong K_{k-1,k-1}$ and $g > 3$, it is not difficult to see that $X \cong K_{k,k}$, a contradiction.

Next, we prove the necessity. Assume that X is not sup-λ', then X has λ'-superatoms by Lemma 2.3, and let A be a λ'-superatom of X. It is easy to see that X is not sup-λ' if $X \cong C_m \times K_2$, or M_m $(m \geq 4)$. Thus, in the following, we assume that $X \not\cong C_m \times K_2$ and M_m $(m \geq 4)$. Since $X \not\cong C_m \times K_2$ and M_m $(m \geq 4)$, and $g > 3$, we see that A is an imprimitive block of X by Lemma 3.3. It follows from Proposition 2.4(1) that $X[A]$ itself is vertex-transitive, therefore, let t be the degree of $X[A]$. Since X is λ'-optimal by the assumption $g > 3$ and Theorem 1.3 (For otherwise, if X is not λ'-optimal, then there exists a $(k-1)$-regular subgraph Y of X satisfying $k \leq |V(Y)| \leq 2k-3$. Clearly, Y contains triangles, a contradiction.), $|A| \geq k-1$ by Lemma 3.1, we have

$$2k - 2 = d(A) = |\omega(A)| = |A|(k-t) \geq (k-1)(k-t),$$

which implies $1 \le k - t \le 2$, and then $k - 2 \le t \le k - 1$.

If $t = k - 1$, then $2t = 2(k - 1) = d(A) = |A|(k - t) = |A|$. Thus,

$$E(X[A]) = |A|t/2 = |A|^2/4.$$

By $g > 3$, it follows from Lemma 4.3 that $X[A]$ is isomorphic to the complete bipartite graph $K_{t,t}$.

If $t = k-2$, then $2(k-1) = d(A) = |A|(k-t) = 2|A|$, that is, $|A| = k-1$. Thus

$$E(X[A]) = |A|t/2 = |A|(|A| - 1)/2.$$

It follows that $X[A]$ is complete, which contradicts the assumption that $g > 3$. □

If $g > 4$, then X is not isomorphic to the following graphs: $C_m \times K_2$ and $M_m (m \ge 4)$, and $K_{k-1,k-1}$ is not an induced subgraph of X. Therefore, Theorem 1.4 follows easily from Theorem 4.4.

Corollary 4.5. *(see Wang* [14]*) If X is a connected vertex-transitive graph with degree $k > 2$ and girth $g > 4$, then it is sup-λ'.*

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