The hypergraph of Θ-classes and Θ-graphs of partial cubes

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Abstract

Given a partial cube G, the Θ-graph of G has Θ-classes of G as its vertices, and two vertices in it are adjacent if the corresponding Θ-classes meet in a vertex of G. We present a counter-example to the question from [8] whether Θ-graphs of graphs of acyclic cubical complexes are always dually chordal graphs. On a positive side, we show that in the class of ACC p-expansion graphs each Θ-graph is both a dually chordal and a chordal graph. In the proof a fundamental characterization of α-acyclic hypergraphs is combined with techniques from metric graph theory. Along the way, we also introduce a new, weaker version of simplicial elimination scheme which yields yet another characterization of chordal graphs.

Key words: dually chordal graph, doubly chordal graph, acyclic cubical complex, p-expansion graph, isometry.

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1 Introduction

Partial cubes (i.e. isometric subgraphs of hypercubes) were studied quite intensively in recent years (we refer to some very recent papers investigating different aspects of this class of graphs [7, 8, 10, 11, 14, 16, 17, 22, 23]). One of the oldest characterizations of partial cubes, due to Winkler, uses the so-called relation Θ on the edge set of a graph [25]. Winkler proved that partial cubes are precisely those bipartite graphs in which this relation is equivalence. The equivalence classes, called *Θ-classes* or *parallel classes*, contain edges that form a matching, and the distances between endvertices of two edges uv, xy from one class are in the following relation: $d(x,u) = d(y,v) = d(x,v)-1 = d(y,u)-1$. The structure in which different Θ-classes interfere is an issue of many investigations of partial cubes. Though this interference is not completely arbitrary, there are some concepts (such as the *crossing graph* [15]) which show that every graph structure can already be realized within partial cubes (more precisely, every graph is the crossing graph of some partial cube). There are several other natural ways how to analyze the structure of Θ-classes, and one of the seemingly simplest is the intersection graph of Θ-classes, called the *Θ-graph* of a partial cube.

The concept of Θ-graph was used in [8] to obtain good upper bounds for the strong chromatic index of special families of partial cubes, see also [17]. For instance, $s'(G) \leq 2\Delta(G)$ holds for all p-expansion graphs [17] (see [6] for more on p-expansion graphs). The main step in proving the bound was the fact that Θ-graphs of p-expansion graphs are chordal, and thus perfect graphs. Several other connections between some natural classes of partial cubes and their Θ-graphs were then established in [14].

Bandelt and Chepoi introduced the graphs of acyclic cubical complexes as follows [1]. First, a *cubical complex* $\mathcal{K}$ is a set of cubes of arbitrary dimensions which is closed for subcubes and nonempty intersections (our interest here is in graphic cubes, although its geometric realization, called a cubical polyhedron was also considered [24]). One may look at a complex $\mathcal{K}$ as the hypergraph with vertices representing 0-dimensional cubes, and edges representing cubes of $\mathcal{K}$ of larger dimensions. If this hypergraph has an additional property to be α-acyclic (which is a well-known concept in hypergraph theory, that was used for characterizing simplicial complexes, and consequently chordal and dually chordal graphs [4]), then $\mathcal{K}$ is called an *acyclic cubical complex*. In the *underlying graph* of a cubical complex $\mathcal{K}$ two vertices of $\mathcal{K}$ are adjacent whenever they form a 1-dimensional cube, see Bandelt and Chepoi [1]. The underlying graphs of acyclic cubical complexes will be called shortly *ACC graphs*. They are a subclass of the well-known class of median graphs (see Klavžar and Mulder [15] for a survey on median graphs), and were characterized among median graphs in several ways [1].

The cube graph (i.e. intersection graph of maximal hypercubes) of an ACC graph coincides with the line graph of the corresponding cubical complex, which is always a dually chordal graph [1] (in fact, as shown in [6], any dually chordal graph can be realized in this way). Since the cube graph (like the Θ-graph) of a p-expansion graph is always a chordal graph [6], it is natural to look for other similarities between these two intersection concepts. The following question from [8] seems plausible: *Is Θ-graph of every ACC graph a dually chordal graph?*

In this paper we give a negative answer to this question by presenting a counter-example (see Fig. 1). On the positive side, we show that some correlation between these concepts still exists. We prove that if a graph G is an ACC and a p-expansion graph, then $\Theta(G)$ is indeed dually chordal, even more, it is a doubly chordal graph. We failed to obtain a direct proof of this result, but the connection with hypergraphs was helpful – we applied the characterization of dually chordal graphs as the line graphs of α-acyclic hypergraphs, by showing that the naturally defined hypergraph of Θ-classes of an ACC p-expansion graph is indeed α-acyclic (the hypergraph of Θ-classes of a partial cube G is simply the hypergraph on the vertex set of G whose edges correspond to Θ-classes, each hyperedge consisting of the endvertices of all the edges from a Θ-class).

In the next section we present basic definitions and some preliminary results that will be needed. The main result of Section 3 is that the 2-section graph of the hypergraph of Θ-classes of arbitrary ACC graph is chordal. (The proof relies on, as far as we know, a new characterization of chordal graphs, using the so-called almost simplicial elimination scheme). Unfortunately, this hypergraph is not necessarily conformal if G is an ACC graph. However, we prove in Section 4, that by assuming in addition that a graph is p-expansion then the hypergraph of Θ-classes becomes conformal. This result combined with the theorem from Section 3, yields the main positive result of this paper, that the Θ-graph of every ACC p-expansion graph is doubly chordal.

2 Preliminaries

We consider finite, undirected, connected graphs. For $u, v \in V(G)$, let $d_G(u, v)$ (or $d(u, v)$ for short) denote the length of a shortest path (also called *geodesic*) in G from u to v. A subgraph H of a graph G is an *isometric* subgraph if $d_H(u, v) = d_G(u, v)$ for all $u, v \in V(H)$. A subgraph H of a graph G is *convex* if for any two vertices u, v of H all shortest paths between u and v in G lie in H. It is easy to see that the intersection of two convex subgraphs is also convex. A *convex closure* of a subgraph H of G is defined as the smallest convex subgraph of G which contains H. A subgraph H of a graph G is called *gated* in G if for every $x \in V(G)$ there exists a vertex u in H such that $u \in I(x,v)$ for all $v \in V(H)$. If for some x such a u in $V(H)$ exists, it must be unique.

The *hypercube (of dimension k)* or *k-cube* is the graph Q_k with the vertex set $\{0,1\}^k$ where two vertices are adjacent whenever they differ in exactly one position. The 2-cube will be also called the *square*. A graph G is a *partial cube* if G is an isometric subgraph of some Q_k. The most important subclass of partial cubes is the class of *median graphs* [15], and, in turn, ACC graphs form a subclass of median graphs [1]. Among other subclasses of median graphs let us mention trees, hypercubes, and complete grid graphs.

Edges $e = xy$ and $f = uv$ of a graph G are in the Djoković-Winkler relation Θ [9, 25] if $d_G(x,u) + d_G(y,v) \neq d_G(x,v) + d_G(y,u)$. Relation Θ is reflexive and symmetric. If G is bipartite, then Θ can be defined as follows: $e = xy$ and $f = uv$ are in relation Θ if $d(x,u) = d(y,v)$ and $d(x,v) = d(y,u)$. In partial cubes the relation Θ is also transitive and so an equivalence relation, see Winkler [25]. Given an edge ab in a partial cube G we define also the following sets:

$W_{ab} = \{w \in V(G) \mid d(a,w) < d(b,w)\}$,

$U_{ab} = \{w \in W_{ab} \mid w \text{ has a neighbor in } W_{ba}\}$, and

$F_{ab} = \{f \in E(G) \mid ab\Theta f\}$.

Median graphs are defined as the graphs in which for every three vertices there exists a unique vertex, lying on shortest paths between all pairs of the three [21]. They are characterized as the bipartite graphs in which all sets U_{ab} are convex [2]. Note that the sets F_{ab} coincide with Θ-classes of G, and each F_{ab} forms a matching between U_{ab} and U_{ba} which in turn corresponds to an isomorphism between the subgraphs induced by U_{ab} and U_{ba}.

Let G be a (bipartite) graph, and H an isometric subgraph of G. We say that G' is *obtained from G by the peripheral expansion of H* if G' is the graph obtained from the disjoint union of graphs G and H, by addition of $|V(H)|$ edges between H and the subgraph of G isomorphic to H that correspond to an automorphism between the copies of H. We also say that we obtained G' from G by *expanding* H. Mulder characterized median graphs through the peripheral expansion procedure from K_1 in which convex subgraphs are expanded at each step [21].

We will also use some hypergraph notions from [3]. Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph. A hypergraph $\mathcal{H}$ is *conformal* if any subset $S \subseteq V$ of elements is contained in an edge of $\mathcal{H}$ provided any pair of elements of S is. The line graph $L(\mathcal{H})$ of $\mathcal{H}$ is the intersection graph of $\mathcal{H}$ (that is, vertices of $L(\mathcal{H})$ are edges of $\mathcal{H}$, and two vertices are adjacent in $L(\mathcal{H})$ if the corresponding edges intersect). The *2-section* $2SEC(\mathcal{H})$ of a hypergraph $\mathcal{H}$ has the vertices of $\mathcal{H}$ as its vertex set, and two vertices are adjacent in $2SEC(\mathcal{H})$ if they belong to a common edge of $\mathcal{H}$. If $\mathcal{H}$ is conformal, then its edges are exactly the maximal cliques of its 2-section graph. A *cycle* of a hypergraph $\mathcal{H}$ is $x_1, E_1, x_2, E_2, \ldots, x_k, E_k, x_1$ where x_i are distinct vertices and E_i distinct edges of $\mathcal{H}$, such that $x_i, x_{i+1} \in E_i$ for $i = 1, 2, \ldots, k \pmod{k}$, and no member of $\mathcal{H}$ includes three distinct vertices of the cycle. A hypergraph $\mathcal{H}$ is an *α-acyclic* hypergraph if it is conformal and has no cycles. We will use the following characterization of α-acyclic hypergraphs.

Theorem 1 *[4] A hypergraph is α-acyclic if and only if it is conformal and its 2-section graph is chordal.*

We refer to a comprehensive monograph, containing relations between hypergraphs and related classes of graphs, see Brandstädt, Le and Spinrad [5].

Bandelt and Chepoi [1] introduced the graphs of acyclic cubical complexes as presented in the introduction. In addition, ACC graphs are precisely the graphs obtainable by peripheral expansion procedure from K_1 such that a hypercube is expanded at each step. Hence every ACC graph includes a pendant hypercube, that is the k-cube H, having an edge ab, such that $U_{ab} = W_{ab}$ is a subcube of H of dimension $k-1$. Another useful characterization of ACC graphs is that they are median graph that contain no convex bipartite wheels B_r for $r \geq 4$ [1] (where the bipartite wheel B_r is the graph, obtained from the cycle C_{2r} by adding a new vertex and edges between this vertex and every second vertex of the cycle).

Recall that for any vertex $v \in V$, the *open neighborhood* of v is the set $N(v) = \{u \in V \mid uv \in E\}$, and its *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. If $A \subset V$ then $N(A)$ denotes the union of open neighborhoods of vertices of A ($N[A]$ denotes the union of closed neighborhoods, respectively). A vertex $u \in N[v]$ is a *maximum neighbor* of v if for all $w \in N[v]$ the inclusion $N[w] \subseteq N[u]$ holds (note that $u = v$ is not excluded). Let G be a graph and $(v_1, \ldots, v_n)$ the ordering of its vertices. Let G_i be the subgraph of G induced by $\{v_i, \ldots, v_n\}$ and let $N_i[v]$ denote the closed neighborhood of v in G_i. The ordering $(v_1, \ldots, v_n)$ is a *maximum neighborhood ordering* if for all $i \in \{1, \ldots, n\}$ there is a maximum neighbor $u_i \in N_i[v_i]$:

$$N_i[w] \subseteq N_i[u_i], \text{ for all } w \in N_i[v_i].$$

A graph G is a *dually chordal graph* [4] if G has a maximum neighborhood ordering. For a recent paper, dealing with intersection concepts, in particular in relation with dually chordal graphs, we refer to [12].

A vertex v is *simplicial* if $N[v]$ is a clique. A simplicial vertex which has a maximum neighbor is called *doubly simplicial*. Furthermore, if each vertex v_i is doubly simplicial in G_i then such ordering is called *doubly perfect*. A graph G is *doubly chordal* if G has a doubly perfect ordering. Note that doubly chordal graphs can be defined as the graphs which are simultaneously chordal and dually chordal [19]. There are several applications of doubly chordal graphs, cf. [18].

3 2-section graph of the hypergraph of Θ-classes

Let $T(G)$ denote the set of Θ-classes of a partial cube G. We say that Θ-classes $F_1, F_2 \in T(G)$ are *adjacent* if there exist edges $e_1 \in F_1$ and $e_2 \in F_2$ which are adjacent (that is, e_1 and e_2 have an end-vertex in common). By $\Theta(G)$ we denote the intersection graph of Θ-classes of a graph G. That is, the vertex set of $\Theta(G)$ is $T(G)$ and two vertices in $\Theta(G)$ are adjacent whenever the respective Θ-classes are adjacent. Thus the Θ-graph presents an intersection concept in the sense of [20].

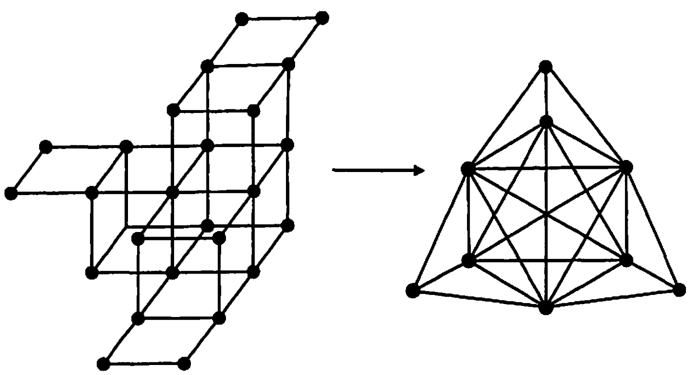

Figure 1: ACC graph and its Θ-graph

We can now present the counter-example to our question from the previous paper [8]. On the left side of Fig. 1 an ACC graph is presented, and on the right side of that figure its Θ-graph is depicted. It is easy to verify that this graph is not dually chordal (in fact, no vertex in this graph has a maximal neighbor). We believe, though we have not proved it, that this is the smallest possible counter-example.

Let G be a partial cube, and $T(G)$ the set of Θ-classes of G. Let $f\colon T(G) \to E(G)$ be any function that maps a Θ-class to one of its edges, and $T'(G)$ be the range of f. A *hypergraph* $\mathcal{H}_G$ *of* Θ*-classes* of a graph G has $V(G)$ as the underlying (vertex) set and $\{U_{ab} \cup U_{ba} : ab \in T'(G)\}$ as its (hyper-)edges. For example, Fig. 2 shows a graph G with five Θ-classes and a representative e_i of every class E_i. For this graph $\mathcal{H}_G =$

$\{\{c, g\}, \{a, b, c, d, e, f\}, \{d, h\}, \{a, b, c, d\}, \{c, d, e, f\}\}$. Next, it is easy to see that the hypergraph $\mathcal{H}_{Q_k}$ of Θ-classes of the k-cube has k copies of $V(G)$ as the set of edges. In particular, one can prove without much effort that $V(G)$ is one of the edges of $\mathcal{H}_G$ if and only if G is a prism (i.e. the Cartesian product of a partial cube with K_2).

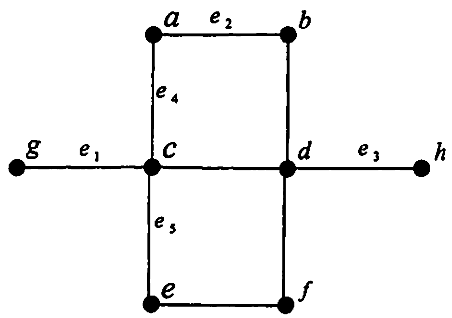

Figure 2: Example

Clearly $\Theta(G)$ coincides with the line graph $L(\mathcal{H}_G)$ of the hypergraph of Θ-classes of G. Hence the following characterization of dually chordal graphs due to Brandstädt et al. will be useful for our purposes.

Theorem 2 *[4] A graph G is dually chordal if and only if it is the line graph of some α-acyclic hypergraph.*

Recall that a graph is *chordal* when every cycle of length more than 3 has a chord (i.e. an edge between non consecutive vertices of the cycle). It is well-known that chordal graphs are characterized by simplicial elimination scheme – an ordering $(v_1, \ldots, v_n)$ of the vertices of V such that for $i = 1, \ldots, n$, v_i is simplicial in G_i. We will show a slightly more general result, by introducing the so-called *almost simplicial vertices* and *scheme.*

We say that x is *almost simplicial* in a graph G if for any pair of non adjacent vertices y, z in $N(x)$ every path from y to z in $G - x$ contains a vertex from $N(x) \setminus \{y, z\}$. (Note that if $N(x)$ is complete, the condition is trivially fulfilled). The corresponding procedure in which we remove one by one a vertex that is almost simplicial in a (sub)graph during the procedure will be called an *almost simplicial elimination scheme.* Here is our first result.

Theorem 3 *A graph G is chordal if and only if it has an almost simplicial elimination scheme.*

Proof. If a graph is chordal then it has a simplicial elimination scheme which is by definition also an almost simplicial elimination scheme. The proof of the converse is by induction on the number of vertices of a graph G. Let x be a vertex that is almost simplicial, and we may assume by induction that $G-x$ is chordal. We need to see that then also G is chordal. Suppose that it is not. Then there are vertices that form a cycle of length at least 4 that has no chords. Since $G-x$ is chordal, this cycle (denote it by C) must contain x, and let y and z be the non adjacent neighbors of x in C. However, since x is almost simplicial, any path from y to z in $G-x$ contains a vertex u from $N(x)\setminus\{y,z\}$. But then ux is a chord of C, a contradiction, which shows that also G is chordal. □

Theorem 3 will now be used in the result about the hypergraph of Θ-classes of an ACC graph. We will also use the following observation from [6].

Lemma 4 *If the intersection I of two hypercubes in an ACC graph G is not a subset of some other intersection of two hypercubes, then I is a cut set in G.*

Theorem 5 *Let G be an ACC graph, and ε a hypergraph of Θ-classes of G. Then $2SEC(\varepsilon)$ is a chordal graph.*

Proof. The proof is by induction on the number of Θ-classes of a graph G. Let E be a Θ-class whose edges belong only to a pendant hypercube R in a graph G. By contracting the Θ-class E we obtain an ACC graph G', and we denote by ε' the hypergraph of Θ-classes of a graph G'. By induction hypothesis $2SEC(\varepsilon')$ is a chordal graph. Let A be the set of vertices from $2SEC(\varepsilon)-2SEC(\varepsilon')$ (in G they induce a cube – contracted part of R) and $N(A)$ be the open neighborhood of this set in a graph $2SEC(\varepsilon)$. Note that by definition, $N(A)$ coincides with $N(x)$ for any $x\in A$. Let $\mathcal{F}=\{F_1,\dots,F_t\}$ be the set of Θ-classes of G such that each $F_i\in\mathcal{F}$ has the following property: for each vertex of the hypercube R there exists an edge from F_i, incident with this vertex. We distinguish two cases.

Case 1: The subgraph of $2SEC(\varepsilon)$ induced by $N[A]=A\cup N(A)$ is a complete subgraph. Therefore all vertices from A are simplicial vertices in the graph $2SEC(\varepsilon)$. Hence $2SEC(\varepsilon)$ is chordal.

Case 2: The subgraph of $2SEC(\varepsilon)$ induced by $A\cup N(A)$ is not a complete subgraph. Then there exist vertices x and y from $N(A)$ in $2SEC(\varepsilon)$ which are not adjacent. To show that vertices of A are almost simplicial in $2SEC(\varepsilon)$, we need to prove, by Theorem 3, that every path between x and y in $2SEC(\varepsilon)$ contains a vertex from $N[A]\setminus\{x,y\}$. Assume to the contrary that there exist vertices x and y that are connected by a path in $2SEC(\varepsilon)$ which lies outside $N[A]\setminus\{x,y\}$. Hence there is clearly such a path between x and y also in G. Since x and y are from $N(A)$, there exist edges $e_i = xa \in F_i$ and $e_j = yb \in F_j$. Note that F_i and F_j are distinct Θ-classes from $\mathcal{F}$, and there is no Θ-class from $\mathcal{F}$ that would be incident with both x and y since they are not adjacent in $2SEC(\varepsilon)$. Let $U_1 = U_{xa} \cup U_{ax}$ and $U_2 = U_{yb} \cup U_{by}$. Note that they are convex subsets and $I = U_1 \cap U_2 \neq \emptyset$ because it contains vertices from R.

Recall that in median graphs every convex subgraph is also gated [13]. Let $z \in I$ be the vertex that is the closest to x in U_2 (since I is gated, z is uniquely determined). Let zz' be an edge in F_i and zz'' be an edge in F_j. Let x' be the neighbor of z on the shortest path from x to z. Denote by A a maximal hypercube that contains the square with vertices z, z', z'', and the common neighbor of z' and z'' (this neighbor is unique, since in partial cubes there can be no induced $K_{2,3}$). Let B be a maximal hypercube that contains the edge $x'z$. Then the intersection of A and B can be maximal or it can be included in some other intersection of two hypercubes, say A' and B'. In either case, by Lemma 4, $A \cap B$ is (or lies in) a cut set C. It is clear that x and y lie in distinct connected components of $G - C$. This implies that any path from x to y necessarily goes through some vertices from two maximal hypercubes A' (or A) and B' (or B), which are all incident with edges from U_1 and U_2, and they are in $N[A]$. We arrive at the contradiction, since we assumed that there is path between x and y that lies outside $N[A]$. This contradiction completes the proof. □

The above result shows that one of the conditions for a hypergraph of Θ-classes to be α-acyclic (cf. Theorem 1) holds in ACC graphs. Hence the question which motivated this paper, was in this sense reasonable. The other condition – conformality – is not always fulfilled. Note that the property in the above theorem is not characteristic for ACC-graphs. For instance, if G is a prism, then $2SEC(\mathcal{H}_G)$ is a complete graph, thus a chordal graph.

4 Θ-graphs of ACC p-expansion graphs

Another subclass of partial cubes are the so-called *p-expansion graphs*, treated in [6]. They are the graphs that can be obtained by successive use of peripheral expansions from K_1 in such a way that at each step the expanded graph H is

- one vertex, or
- a union of maximal hypercubes in G with a nonempty common intersection.

It turns out that also in the second case H is an isometric subgraph, and so the expansion is well-defined [6]. This class was introduced in relation with the cube graph transformation.

We shall prove the conformality property of hypergraphs of Θ-classes of ACC p-expansion graphs. Unfortunately, we cannot extend this property to all p-expansion graphs; see Fig. 3, in which three vertices that violate conformality are marked. As we already know, it is also not true for all ACC graphs (e.g. the hypergraph of Θ-classes of the graph that is depicted on the left side of Fig. 1 is not conformal).

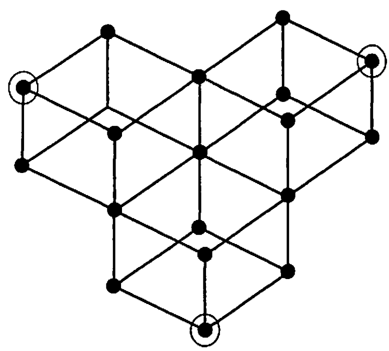

Figure 3: A p-expansion graph

In the proof we shall use the following result, cf. [3].

Theorem 6 *(Gilmore's Theorem) A necessary and sufficient condition for hypergraph $\mathcal{H}$ to be conformal is that for any three edges A_1, A_2, A_3 of $\mathcal{H}$, there exists an edge A_s of $\mathcal{H}$ such that $(A_1 \cap A_2) \cup (A_2 \cap A_3) \cup (A_1 \cap A_3) \subseteq A_s$.*

We will also use the fact that convex subgraphs of median graphs G satisfy the Helly property, cf. [13]. That is, if C is a set of convex subgraphs of a median graph G, where any two elements of C have a nonempty intersection, then $\bigcap_{H \in C} H \neq \emptyset$. The following result can be regarded as the Helly property of the hypergraph of Θ-classes in median graphs.

Proposition 7 *Let G be a median graph and $\{E_1, E_2, \ldots, E_k\}$ a set of Θ-classes which are pairwise adjacent. Then there exists a set of edges $\{e_1, e_2, \ldots, e_k\}$, where $e_i \in E_i$ for every $i \in \{1, 2, \ldots, k\}$ which share a common endvertex.*

Proof. Let G be a median graph. Let $E_1, E_2, \ldots, E_k$ be the set of Θ-classes of G which are pairwise adjacent. Let $x_i y_i$ be an edge of E_i for every

$i \in \{1, 2, \ldots, k\}$. Then $\langle U_{x_i y_i} \cup U_{y_i x_i} \rangle$ are convex subgraphs of median graph G, where $i \in \{1, 2, \ldots, k\}$, hence they have a nonempty intersection. We denote by v a vertex from the intersection. Obviously v is an endvertex of an edge in $F_{x_i y_i}$ for every $i \in \{1, 2, \ldots, k\}$. □

Note that this kind of Helly property does not hold in all partial cubes (for instance, in C_6 all three Θ-classes are pairwise adjacent, but no vertex is incident with all three Θ-classes).

As a by-product we easily deduce the following

Corollary 8 *If G is a median graph, then $\omega(\Theta(G)) = \Delta(G)$.*

This result is again not true in all partial cubes, as for instance $\omega(\Theta(C_6)) = 3$, and $\Delta(C_6) = 2$.

Now we are ready for the main result of this section.

Theorem 9 *Let G be an ACC p-expansion graph. Then the hypergraph $\mathcal{H}_G$ of Θ-classes of G is conformal.*

Proof. We use induction on the number of Θ-classes of a graph G (and conformality trivially holds if the order of a graph G is one). Let G' be obtained from G by expanding a subgraph U. By induction hypothesis the hypergraph $\mathcal{H}_G$ of Θ-classes of a graph G is conformal. We will use the fact that G' is a p-expansion graph, hence one of the two possibilities for U occur. First, if the expanded graph U is one vertex, then $\mathcal{H}_{G'}$ is clearly also conformal, and the result follows. Thus let G' be obtained from a graph G by expanding the union U of maximal hypercubes with a nonempty common intersection. Let E be the new Θ-class obtained in the last expansion step. To prove that $\mathcal{H}_{G'}$ is conformal we will apply Theorem 6. Let E_1, E_2, E_3 be arbitrary Θ-classes of $\mathcal{H}_{G'}$. Let $x_i y_i$ be an arbitrary edge in E_i for every $i \in \{1, 2, 3\}$ and $U_i = \langle U_{x_i y_i} \cup U_{y_i x_i} \rangle$ for every $i \in \{1, 2, 3\}$. (The same notation will be used for the corresponding Θ-classes and hyperedges in G.) We distinguish two cases.

Case 1: Suppose that $E_i = E$ for some $i \in \{1, 2, 3\}$. Without loss of generality let $E = E_1$. Then let $z \in G \setminus U$ be a vertex incident with Θ-classes E_2 and E_3 (if there is no such vertex, the condition from Theorem 6 is already fulfilled by setting $A_s = U_1$). Moreover, let there be a vertex x which is incident with Θ-classes E and E_2 but not with E_3, and a vertex y incident with Θ-classes E and E_3 but not with E_2 (in other cases one of U_2, U_3 can serve as A_s). Observe that vertices x and y are from different maximal hypercubes (which we denote by, say H_x and H_y) from U, and $H_x \cap H_y \neq \emptyset$. Because U_2 and U_3 are convex subsets of G', it follows that $I = U_2 \cap U_3$ is a convex subset, and note that $H_x \cap H_y \subset I$. Since $I \cap (G \setminus U)$ is nonempty (it contains z), we infer there exists a vertex $u \in H_x \cap H_y$ which has a neighbor w in $I \cap (G \setminus U)$. Now consider the edges uu', $u'x'$, $u'y'$ that belong to Θ-classes from E, E_2, E_3, respectively. Since H_x (H_y) is a hypercube, each of its vertices is incident with all Θ-classes that appear in H_x (H_y). Furthermore, vertices u, u', x' induce a path P_3 whose convex closure is a square S_1 which is contained in hypercube H_x. Similarly, vertices u, u', y' form P_3 whose convex closure is a square S_2 in H_y. Let vertices x'', y'' be the remaining vertices of squares S_1, S_2, respectively. By a similar argument we obtain a square S_3, formed by vertices x'', u, w, w', where w' is the other neighbor of w, and a square S_4 formed by vertices y'', u, w, w'', where w'' is the other neighbor of w. We infer that u is the center of a bipartite wheel W isomorphic to B_4 (see Figure 4) which is convex, by construction (namely, because w is not incident with E, and y'' is not incident with E_2). Hence G' is not an ACC graph, a contradiction.

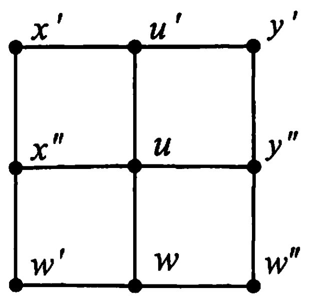

Figure 4: Bipartite wheel B_4 from proof of Theorem 9

Case 2: Suppose that $E_i \neq E$ for every $i \in \{1, 2, 3\}$. Then we may assume that at least two Θ-classes of the three are having an edge that lies in U, for otherwise the hyperedge A_s in G, which includes $\mathcal{A} = (U_1 \cap U_2) \cup (U_2 \cap U_3) \cup (U_1 \cap U_3)$ in G by induction, also includes $\mathcal{A}$ in G' (which is the same as in G). If E_1, E_2, E_3 all have edges in U, then we have two possibilities: either $\mathcal{A}$ is in U in which case the hyperedge of the new Θ-class can serve as A_s. The second possibility is that there is a vertex $x \in U_i \cap U_j$ that lies in $G \setminus U$. Without loss of generality, let $x \in U_2 \cap U_3$. Then we are in the same situation as in Case 1, and the proof follows the same lines. Hence in what follows we may assume that exactly two Θ classes (let these be E_1 and E_2) have some edges in U, while E_3 has no edges in U.

First consider the possibility that both Θ-classes E_1 and E_2 appear in the same hypercube from U, denoted by H_x. Then the hyperedge A_s includes all vertices of H_x in G. We have two possibilities: Θ-class with corresponding hyperedge A_s has edges in H_x. In this case the same (expanded) hyperedge A_s can serve the purpose in G'. If Θ-class with corresponding hyperedge A_s does not appear in H_x, then all vertices of H_x are incident with this Θ-class which is a contradiction with H_x being a maximal hypercube. Hence E_1 and E_2 cannot appear in the same hypercube. Then there exist maximal hypercubes H_x and H_y from U such that E_1 appears in H_x and E_2 appears in H_y. Then the intersection $H_x \cap H_y$ is either maximal or it can be included in some other intersection of two hypercubes from U. Denote by J this maximal intersection. Since E_1 and E_2 appear in different maximal hypercubes and E_3 has no edges in U, there is no edge from E_i, $i = 1, 2, 3$, that would appear in J. Using Lemma 4 we find that J is a cut set. Combining the last two observations we infer that each $U_i, i = 1, 2, 3$ is contained in exactly one connected component of $G' \setminus J$. This implies that $U_1 \cap U_2 \cap U_3$ lies in I which in turn implies that $\mathcal{A}$ lies in one of the U_i, and Gilmore's condition is fulfilled also in this last case. □

Our main result follows from Theorem 1, Theorem 5 and Theorem 9.

Corollary 10 *Let G be an ACC p-expansion graph. Then the Θ-graph of G is a dually chordal graph.*

Recall also the following result.

Theorem 11 *[8] Let G be a p-expansion graph. Then $\Theta(G)$ is a chordal graph.*

By combining Theorem 11 with the fact that doubly chordal graphs are precisely graphs that are chordal and dually chordal (see [4]), we derive the following observation.

Corollary 12 *The Θ-graph of an ACC p-expansion graph is a doubly chordal graph.*

It is easy to observe that $K_{1,3}$ is not obtained as Θ-graph of some median graph. It would be interesting to characterize those doubly chordal graphs that can be obtained as Θ-graphs of ACC p-expansion graphs. The similar question concerning graphs which can be realized as Θ-graphs of all ACC graphs (or median graphs, etc.) is also open.

Some of the results of this paper show certain similarities between the Θ-graph and the cube graph of a partial cube. In the case of ACC graphs G, Theorem 5 shows that the 2-section graph of the hypergraph of Θ-classes of G is chordal. This holds also for the graph G^{Δ} [1] that is defined as the graph obtained from G by adding edges in such a way that cliques are created out of all its maximal hypercubes. The corresponding hypergraph is conformal which can be regarded as the reason that the cube graph of an ACC graph is dually chordal. It would thus make sense to investigate relations between the hypergraph of Θ-classes and the hypergraph of maximal hypercubes (defined on the vertex set of a partial cube with hyperedges consisting of vertices of maximal hypercubes) in other natural classes of partial cubes.

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