A note on Z_5-connectivity in 3-edge-connected graphs

Mingfang Huang[1,2], Xiangwen Li[1] *
[1]Department of Mathematics
Huazhong Normal University
Wuhan 430079, China
[2]School of Science
Wuhan University of Technology
Wuhan 430070, China

Abstract

Jaeger *et al.*[J. Combin. Theory, Ser B, 56 (1992) 165–182] conjectured that ever 3-edge-connected graph is Z_5-connected. Let G be a 3-edge-connected simple graph on n vertices and A an abelian group with $|A| \geq 3$. If a graph G^* is obtained by repeatedly contracting nontrivial A-connected subgraphs of G until no such a subgraph left, we say G can be A-reduced to G^*. It is proved in this paper that G is A-connected with $|A| \geq 5$ if one of the following holds: (i) $n \leq 15$; (ii) $n = 16$ and $\Delta \geq 4$; or (iii) $n = 17$ and $\Delta \geq 5$. As applications, we also show the following results.

(1) For $|A| \geq 5$ and $n \geq 17$, if $|E(G)| \geq \binom{n-15}{2} + 31$, then G is A-connected.

(2) For $|A| \geq 4$ and $n \geq 13$, if $|E(G)| \geq \binom{n-11}{2} + 23$, then either G is A-connected or G can be A-reduced to the Petersen graph.

1 Introduction

The graphs considered in this paper are finite, loopless, and may have multiple edges. We follow the notations and terminology in [1] except otherwise stated.

Denote by $\delta(G)$ (or short δ) and $\Delta(G)$ (or short Δ) the minimum degree and maximum degree of a graph G, respectively. A *k-path* is a path of

*Supported by the Natural Science Foundation of China (11171129); corresponding author: Email address: xwli68@yahoo.cn.

length k. A *k-cycle* is a cycle of length k. We also say C_n for a n-cycle. The girth of a graph G, denoted by $g(G)$, is the length of its shortest cycle. An edge cut $X \subseteq E(G)$ of G is a *trivial* edge cut if one of the components of $G - X$ is a K_1. For two subsets $V_1, V_2 \subseteq V(G)$ such that $V_1 \cap V_2 = \emptyset$, let $e_G(V_1, V_2)$ ($e(V_1, V_2)$ for short) denote the number of edges of G with one endpoint in V_1 and the other endpoint in V_2. If H_1 and H_2 are two vertex disjoint subgraphs of G, we write $e(H_1, H_2)$ instead of $e(V(H_1), V(H_2))$.

Let G be a graph, and D an orientation of G. If an edge $e \in E(G)$ is directed from a vertex u to a vertex v, then let $tail(e) = u$ and $head(e) = v$. For a vertex $v \in V(G)$, denote by $E^+(v)$ the set of edges with tail at v and $E^-(v)$ the set of edges with head at v. Let A be a nontrivial additive abelian group and A^* the set of all the nonzero elements in A. Define $F(G, A) = \{f \mid f : E(G) \to A\}$ and $F^*(G, A) = \{f \mid f : E(G) \to A^*\}$.

For a given function $f \in F(G, A)$, let $\partial f : V(G) \to A$ be given by, for all $v \in V(G)$,

$$\partial f(v) = \sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e),$$

where "$\sum$" refers to the addition in A.

A function $b : V(G) \to A$ is called an *A-valued zero-sum function* on G if $\sum_{v \in V(G)} b(v) = 0$. The set of all A-valued zero-sum functions on G is denoted by $Z(G, A)$. For a given $b \in Z(G, A)$, a function $f \in F^*(G, A)$ is called a *nowhere-zero (A, b)-flow* if G has an orientation $D(G)$ such that $\partial f = b$. A graph G is *A-connected* if for any $b \in Z(G, A)$, G has a nowhere-zero (A, b)-flow. A *nowhere-zero A-flow* is an $(A, 0)$-nowhere-zero flow. More specifically, a *nowhere-zero k-flow* is a nowhere-zero Z_k-flow, where Z_k is the cyclic group of order k. Tutte in [15] proved that G admits a nowhere-zero A-flow with $|A| = k$ if and only if G admits a nowhere-zero k-flow.

The concept was first introduced by Tutte [14], and the theory of nowhere-zero flows provides an interesting way to investigate the coloring of planar graphs. The long standing open problems in this area are Tutte's 3-, 4- and 5-flow conjectures. Jaeger *et al.* [5] successfully generalized nowhere-zero flow problems to group connectivity. For group connectivity, Jaeger *et al.* posed two famous conjectures: Z_3- and Z_5-connectivity conjectures. On the other hand, group connectivity plays an important role to study nowhere-zero flow problems. For example, Thomassen [13] used the all generalized Tutte-orientation, which is equivalent to Z_3-connectivity, to confirm the weak 3-flow conjecture. The following conjecture of Jaeger *et al.* [5] is still open.

Conjecture 1.1 *Every 3-edge-connected graph is Z_5-connected.*

Lai and Zhang in [9] proved that every 3-edge-connected planar graph is Z_5-connected. Yang and Li [16] proved that Conjecture 1.1 holds for graphs on at most 11 vertices. On the other hand, Lai in [8] proved that the family consisting of all A-connected graphs is a complete family. This provides an approach as the induction proof in such a way that if a graph can be contracted to a small graph by contracting Z_5-connected subgraph into a vertex and such a small graph is also Z_5-connected, then the origin graph is Z_5-connected. Motivated by these observations, we investigate small 3-edge-connected graphs and present the following theorem.

Theorem 1.2 *Let G be a 3-edge-connected graph on n vertices and A an abelian group with $|A| \geq 5$. If one of the following holds:*
(i) $n \leq 15$; or
(ii) $n = 16$ and $\Delta \geq 4$; or
(iii) $n = 17$ and $\Delta \geq 5$,
then G is A-connected.

Theorem 1.2 tells us that Conjecture 1.1 holds for every 3-edge-connected graphs at most 15 vertices. The proof techniques used in this note are splitting of edges and deletion of one vertex developed by Lai in [8] and Luo [12]. For large 3-edge-connected graphs, such two techniques will give us a lot of computations and the new technique will be needed. We will prove Theorem 1.2 in Section 3 and applications of Theorem 1.2 will be discussed in Section 4.

2 Preliminaries

For a subset $X \subseteq E(G)$, the contraction G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting all loops generated in this process. Note that even if G is simple, G/X may have multiple edges. For simplicity, we write G/e for $G/\{e\}$, where $e \in E(G)$. If H is a subgraph of G, then G/H denotes $G/E(H)$.

We state some known results in [5, 8] on group connectivity which are useful as follows.

Lemma 2.1 *Let G be a graph and A be an abelian group with $|A| \geq 3$. Then each of the following holds*
(1) K_1 is A-connected.
(2) If G is A-connected and $e \in E(G)$, then G/e is A-connected.
(3) Let H be a subgraph of G. If H is A-connected and G/H is A-connected, then G is A-connected.
(4) Let C_n be a cycle of n vertices. Then C_n is A-connected if and only if $|A| \geq n+1$.

Let G be a graph. Denote by $\tau(G)$ the maximum number of edge-disjoint spanning trees of G, and let $\mathscr{T}$ denote a family of graphs such that a graph $G \in \mathscr{T}$ if and only if $\tau(G) \geq 2$ or G is a cycle of length 3. A graph G^* is called the T-reduction of G if it is obtained from G by repeatedly contracting nontrivial subgraphs of G in $\mathscr{T}$ until no subgraph in $\mathscr{T}$ left.

Lemma 2.2 *Let G be a graph and A be an abelian group. Then each of the following holds.*

(1) ([5]) *If $\tau(G) \geq 2$, then G is A-connected with $|A| \geq 4$.*

(2) ([16]) *Let G^* be the T-reduction of G. If G^* is nontrivial, then* $2|V(G^*)| - |E(G^*)| \geq 3$.

Let v be a vertex of a graph G and u, w be two neighbors of v in G. If $d(v) \geq 4$, let $G_{[vu,vw]}$ be the graph by deleting vu and vw and adding a new edge uw. The following lemma was first proved by Lai in [8] and reformulated by Chen *et al.* in [2].

Lemma 2.3 *Let A be an abelian group with $|A| \geq 3$ and $d(v) \geq 4$. If $G_{[vu,vw]}$ is A-connected, then so is G.*

Let v be a vertex of a graph G with $d(v) = 3$ and u, w be two neighbors of v in G. Define $G(v, uw)$ be the graph by removing vertex v and adding a new edge uw. The following technique is due to Luo *et al.* in [12].

Lemma 2.4 *Let A be an abelian group with $|A| \geq 3$ and $d(v) = 3$. Let $b \in Z(G, A)$ and $b(v) \neq 0$. If $G(v, uw)$ is A-connected, then there exists an orientation D of G and a function $f \in F^*(G, A)$ such that $\partial f = b$ under the orientation D.*

Let G be a 2-edge-connected graph on n vertices. Lai in [7] proved that if $n \leq 17$, then either G admits a nowhere-zero 4-flow or G can be contracted to the Petersen graph. Yang and Li in [17] gave group connectivity version for 3-edge-connected graphs as follows.

Theorem 2.5 *Let G be a 3-edge-connected simple graph on n vertices. If $n \leq 11$ and A an abelian group with $|A| \geq 4$, then either G is the Petersen graph or G is A-connected.*

Lai *et al.* in [10] proved that if A is an abelian group with $|A| \geq 5$, then the Petersen graph is A-connected. Thus we obtain next lemma immediately from Theorem 2.5.

Lemma 2.6 *Let G be a 3-edge-connected simple graph on $n \leq 11$ vertices and A an abelian group with $|A| \geq 5$. Then G is A-connected.*

On the other hand, Kochol gave smallest counterexamples to the 5-flow conjecture, a recent one is stated as follows.

Theorem 2.7 ([6]) *A smallest counterexample to the 5-flow conjecture has girth at least 11.*

3 Proof of Theorem 1.2

Let G be a graph and u, v be two vertices of G. Denote by $d_G(u, v)$ the distance between u and v in G. For $i \geq 1$, let $D_i(G) = \{v \in V(G) : d(v) = i\}$. We establish the following lemmas.

Lemma 3.1 *Suppose that G is a simple graph on n vertices with $\delta \geq 2$. If $g(G) \geq 5$, then each of the following holds:*

(1) If $|D_2| = s$, then $n \geq 2g(G) - s$.

(2) $n \geq 1 + \delta\Delta$. In particular, if $|D_2| = s$, then $n \geq 3(\Delta - s) + 2s + 1$. Furthermore, if $|D_2| = 1$ and the vertex of degree 2 is not adjacent to v with $d(v) = d$, then $n \geq 1 + 3d$.

(3) If $g(G) \geq 6$, then $n \geq 2 + (\Delta + \delta - 2)\delta$. In particular, if $|D_2| = 1$ and the vertex of degree 2 is not adjacent to v with $d(v) = d$, then $n \geq 4 + 3d$.

(4) If $g(G) \geq 7$, then $n \geq 1 + \Delta + \Delta(\delta - 1)\delta$.

Proof. (1) Let $k = g(G)$ and $C : v_1v_2 \ldots v_kv_1$ be a cycle of G. Note that each $v \in V(C) \setminus D_2$ has a neighbor in $V(G) \setminus V(C)$ for $i \in \{1, 2, \ldots, k\}$. Thus, G contains at least $2k - s$ vertices.

(2) Let $\Delta = l$. Pick a vertex $u \in V(G)$, and let $N(u) = \{u_1, u_2, \cdots u_l\}$. Since $g(G) \geq 5$, u_i is not adjacent to u_j and $N(u_i) \cap N(u_j) = \{u\}$ for $1 \leq i < j \leq l$. Thus, we obtain that $n \geq 1 + l + (\delta - 1)l = 1 + \delta l$. The proof for the other two cases are similar.

(3) Pick a vertex $u \in V(G)$ with $d(u) = \Delta$. Let v be a neighbor of u. Define $N_k^*(u) = \{w \mid d_{G-uv}(u, w) = k\}$. Since $g(G) \geq 6$, we can obtain that $|N_1^*(u)| = \Delta - 1$ and $|N_1^*(v)| \geq \delta - 1$ and $|N_2^*(u)| \geq (\Delta - 1)(\delta - 1)$ and $|N_2^*(v)| \geq (\delta - 1)^2$ and $N_i^*(u) \cap N_j^*(v) = \emptyset$ for $1 \leq i, j \leq 2$. Therefore, $n \geq 2 + |N_1^*(u)| + |N_1^*(v)| + |N_2^*(u)| + |N_2^*(v)| \geq 2 + (\Delta - 1) + (\delta - 1) + (\Delta - 1)(\delta - 1) + (\delta - 1)^2 = 2 + (\Delta + \delta - 2)\delta$. The proof for the other case is similar.

(4) Pick a vertex $u \in V(G)$ with $d(u) = \Delta$. Define $N_k(u) = \{w \mid d_G(u, w) = k\}$. Since $g(G) \geq 7$, we can obtain that $|N_1(u)| = \Delta$ and $|N_2(u)| \geq \Delta(\delta - 1)$ and $|N_3(u)| \geq \Delta(\delta - 1)^2$ and $N_i(u) \cap N_j(u) = \emptyset$ for $1 \leq i < j \leq 3$. Therefore, $n \geq 1 + |N_1(u)| + |N_2(u)| + |N_3(u)| \geq 1 + \Delta + \Delta(\delta - 1) + \Delta(\delta - 1)^2 = 1 + \Delta + \Delta(\delta - 1)\delta$. ■

Lemma 3.2 *Suppose that G is 3-edge-connected with $g(G) \geq 5$, $u \in V(G)$, and $\{u_1, u_2, u_3, u_4\} \subseteq N(u)$. Let X be a nontrivial edge cut X of $G_{[uu_1,uu_2]}$. If $|V(G)| \leq 16$, then $|X| \geq 3$. Moreover, $|X| = 3$ if and only if $G - X$ has a component $H \cong K_2$ with one vertex of degree 3 and the other of degree 2.*

Proof. Since G is 3-edge-connected, $G_{[uu_1,uu_2]}$ is connected. Suppose otherwise that $G_{[uu_1,uu_2]}$ contains a nontrivial edge cut X with $1 \leq |X| \leq 3$ and let $H_1, H_2, \ldots, H_t$ be the components of $G - X$ such that $u_1 \in V(H_1)$ and $u_3 \in V(H_2)$.

We first show that each component of $G - X$ contains a cycle. Suppose otherwise that $G - X$ has a component, say H_1, which is a tree. Then $\sum_{v \in V(H_1)} d_{H_1}(v) = 2(|V(H_1)| - 1)$. On the other hand, since each vertex of H_1 has degree at least three except one, $\sum_{v \in V(H_1)} d_{H_1}(v) \geq 3|V(H_1)| - 4$. Thus, $2(|V(H_1)| - 1) \geq 3|V(H_1)| - 4$, which implies that $|V(H_1)| = 2$, $|X| = 3$. Moreover $H \cong K_2$ with one vertex of degree 3 and the other vertex of degree 2.

Suppose that $|V(H_1)| = \min_{1 \leq i \leq t} |V(H_i)|$. Since $g(G) \geq 5$, $|V(H_1)| \geq 5$ since H_1 contains a cycle. In the case when $|V(H_1)| = 5$. Since $g(H_1) \geq 5$, H_1 is a 5-cycle and $e_G(H_1, G_2 \setminus V(H_1)) \geq 4$, contrary to that $|X| \leq 3$. Let $|V(H_1)| = 6$. Since H_1 contains a cycle of length at least 5, $e_G(H_1, G \setminus V(H_1)) \geq 4$, contrary to that $|X| \leq 3$. If $|V(H_1)| = 7$, then $t = 2$. In this case, it is easy to verify that $e_G(H_1, G \setminus V(H_1)) \geq 4$, contrary to that $|X| \leq 3$. Suppose that $|V(H_1)| = 8$. Since $|V(G)| \leq 16$, $|V(H_1)| = |V(H_2)| = 8$. If H_1 contains a 5-cycle C, the subgraph of H_1 induced by $V(H_1) \setminus V(C)$ is either a 2-path or a 1-path and a vertex or three independent vertices. In each case, $e_G(H_1, G \setminus V(H_1)) \geq 4$, contrary to that $|X| \leq 3$. If H_1 contains a k-cycle, where $k = 6, 7, 8$, the proofs are similar. ■

Utilizing the technique in the proof of Lemma 3.2, we can prove next two lemmas. We left the detail to the readers.

Lemma 3.3 *Suppose that G is 3-edge-connected with $g(G) \geq 5$, $v \in V(G)$ and $N(v) = \{v_1, v_2, v_3\}$. Let X be a nontrivial edge cut X of $G(v, v_1v_2)$. If $|V(G)| \leq 14$, then $|X| \geq 3$. Moreover, $|X| = 3$ if and only if $G - X$ has a component $H \cong K_2$ with one vertex of degree 3 and the other of degree 2.*

Lemma 3.4 *Suppose that G is connected and $\delta(G) \geq 3$. If $g(G) \geq 5$ and $|V(G)| \leq 14$, then G is a 3-edge-connected graph.*

Lemma 3.5 *Let A an abelian group with $|A| \geq 5$. Suppose that G is a 2-edge-connected simple graph with $|D_2(G)| \leq 1$. If G is not A-connected with $|V(G)|$ minimized, then $g(G) \geq 5$.*

Proof. Suppose otherwise that G contains a k-cycle C, where $3 \leq k \leq 4$. Contracting C into one vertex and contracting all cycles of length less than 5 generated in the process, denote by G' the resulting graph. Then $|V(G')| \leq |V(G)| - 2$. If $|V(G')| = 1$, then $G' \cong K_1$, which is A-connected by Lemma 2.1. Thus, by Lemma 2.1 again, G is A-connected, a contradiction. Thus, assume that $|V(G')| \geq 2$. In this case, it is easy to see that $|D_2(G')| \leq 1$ and $|D_1(G')| = 0$. Since the edge connectivity will not decrease under contraction, G' satisfies the hypothesis of our lemma. By assumption, G' is A-connected. By Lemma 2.1, G is A-connected. This is a contradiction. ■

Denote by P^+ the graph obtained from the Petersen graph by replacing one edge of the Petersen graph with a 2-path.

Lemma 3.6 *Let G be a 2-edge-connected simple graph on $n \leq 11$ vertices and A an abelian group with $|A| \geq 5$. If $|D_2(G)| = 1$, then either G is A-connected or G is the graph P^+.*

Proof. Suppose that G is not A-connected with $|V(G)| = n$ minimized. Let v be the vertex of degree 2 of G. Since $|D_2(G)| = 1$, $n \geq 3$. If $3 \leq n \leq 4$, then G contains at least one 2-cycle. Contracting this 2-cycle and repeatedly contracting 2-cycle generated in the process, we finally obtain a K_1 which is A-connected by Lemma 2.1. By Lemma 2.1 again, G is A-connected. Therefore, we assume that $n \geq 5$.

Consider the case when $5 \leq n \leq 8$. By Lemma 3.5, we assume that $g(G) \geq 5$. By Lemma 3.1(1), $n \geq 2g(G) - 1 = 9$, a contradiction.

In the case when $n = 9$. We claim that $g(G) \leq 5$. Suppose otherwise that $g(G) = k \geq 6$. Let C be a k-cycle of G. For each vertex $u \in V(G)\backslash V(C)$, there is no more than one edge from u to $V(C)$ since $g(G) \geq 6$. Hence there are no more than $9 - k$ edges from $V(G)\backslash V(C)$ to $V(C)$. On the other hand, since G contains a vertex v of degree 2, there are at least $k - 1$ edges from $V(C)$ to $V(G)\backslash V(C)$. Thus, $k - 1 \leq 9 - k$, and hence $k \leq 5$, a contradiction. Therefore, $g(G) \leq 5$.

By Lemma 3.5, $g(G) = 5$. Let $C : v_1v_2v_3v_4v_5v_1$ be a cycle of length 5 of G. If $v \notin V(C)$, then for $i = 1, \cdots, 5$ each v_i has a neighbor vertex $u_i \in V(G)\backslash V(C)$. Since G has no cycle of length 3 or 4, $u_i \neq u_j$ for $i \neq j$. Thus, $n \geq 10$, a contradiction. Thus, $v \in V(C)$. We assume, without loss of generality, that $v_5 = v$. Observe the vertex u_2. If u_2 were adjacent to a vertex of $V(C) \setminus \{v_2\}$, then G would contains a cycle of length less than 5, contrary to our assumption. Since $n = 9$, $N(u_2) \setminus \{v_2\} \subset \{u_1, u_4, u_3\}$. Since $d(u_2) \geq 3$, there are at least two vertices in $\{u_1, u_3, u_4\}$ adjacent to u_2. It follows that $u_1u_2 \in E(G)$ or $u_2u_3 \in E(G)$. In the former case, we get a 4-cycle $v_1u_1u_2v_2v_1$; in the latter case, we get a 4-cycle $v_2u_2u_3v_3v_2$. In each case, we conclude that $g(G) \leq 4$, contrary to Lemma 3.5. Thus, $n \geq 10$.

In the case when $n = 10$. Note that every G contains even number of vertices with odd degree. Since G contains one vertex of degree 2, it contains a vertex u with even degree at least 4. It follows that $\Delta(G) \geq 4$. By Lemma 3.5, $g(G) \geq 5$. By Lemma 3.1(2), $n \geq 3(\Delta - 1) + 2 + 1 = 12$, contrary to our assumption that $n \leq 10$.

We are left with the case when $n = 11$. By Lemma 3.5, $g(G) \geq 5$. It follows that $\Delta(G) = 3$ for otherwise, by Lemma 3.1(2), $n \geq 12$, contrary to that $n = 11$. Let $e = uv$ be an edge incident with the vertex v of degree 2. Define $G' = G/e$ and v_e be the vertex obtained by identifying u and v. Then G' is a 2-edge-connected cubic graph with $|V(G')| = 10$. Since $g(G) \geq 5$, $g(G') \geq 4$. We now show that $g(G') = 5$. If $g(G') \geq 6$, by Lemma 3.1(1), $|V(G')| \geq 12$ and hence $n \geq 13$, contrary to that $n = 11$.

If $g(G') = 4$, then v_e lies in a 4-cycle of G' since $g(G) \geq 5$. Thus, v lies in a 5-cycle: $vv_1v_2v_3v_4v$ of G. Since $d(v_i) = 3$ for $i = 1, 2, 3, 4$, each v_i has a neighbor vertex $u_i \in V(G) \setminus V(C)$. Let $\{w_1, w_2\} = V(G) \setminus (V(C) \cup \{u_1, u_2, u_3, u_4\})$. Let H be the subgraph of G induced by $\{u_1, u_2, u_3, u_4\}$. We consider two possibilities.

If $w_1w_2 \notin E(G)$, then H contains only one edge. It follows that there are two distinct vertices u_i, u_j such that $d_H(u_i) = d_H(u_j) = 0$. We get a 4-cycle of $G : u_iw_1u_jw_2u_i$, contrary to that $g(G) \geq 5$.

If $w_1w_2 \in E(G)$, then there are two edges in $E(H)$. Suppose that there exists a vertex u_i such that $d_H(u_i) = 0$. Then G contains a 3-cycle $u_iw_1w_2u_i$, a contradiction. Therefore, $d_H(u_i) \geq 1$ for $i = 1, 2, 3, 4$. Since $g(G) \geq 5$, $u_1u_3, u_2u_4 \in E(H)$. Without loss of generality, assume $u_1w_1 \in E(G)$. Since u_3 is adjacent to either w_1 or w_2, in each case G contains a 3-cycle or 4-cycle, a contradiction.

Therefore, $g(G') = 5$. In this case, let $C : v_1v_2v_3v_4v_5v_1$ be a cycle of length 5 of G'. Since G' is a cubic graph, each v_i has a neighbor vertex u_i. Since $g(G') = 5$, $u_i \neq u_j$ for $i \neq j$ and $\{u_1, u_2, \dots u_5\} \subseteq V(G')\setminus V(C)$. Note that $|V(G')| = 10$, $V(G') = \{v_1, \dots, v_5, u_1, \dots, u_5\}$. Since G' is cubic and $g(G') = 5$, $d_{G'}(u_i) = 3$ for each i and u_i has two neighbors in $\{u_1, u_2, \dots, u_5\} \setminus \{u_i\}$. It follows that $u_1u_3, u_1u_4, u_2u_4, u_2u_5, u_3u_5 \in E(G')$. This means that G' is the Petersen graph, and so G is the graph P^+. ■

Lemma 3.7 *Let G be a 3-edge-connected cubic simple graph on $n = 12$ or 14 vertices and A be an abelian group with $|A| \geq 5$. If $g(G) = 5$, then G is A-connected.*

Proof. Let $b \in Z(G, A)$. Assume first that $b \equiv 0$. Since $g(G) = 5$, by Theorem 2.7, G admits a nowhere-zero 5-flow. Therefore, we assume that there is a vertex v with $b(v) \neq 0$. Let $N(v) = \{v_1, v_2, v_3\}$ and $N(v_i)\setminus\{v\} = \{v_{i_1}, v_{i_2}\}$ for $i = 1, 2, 3$. let $\{u_1, \cdots u_l\} = V(G) \setminus (N(v) \cup N(v_1) \cup N(v_2) \cup N(v_3))$ where $l = 2$ or 4.

Claim. There exist two vertices v_{ik} and v_{jt} such that $v_{ik}v_{jt} \in E(G)$ where $i \neq j$, $i, j \in \{1, 2, 3\}$ and k, $t = 1, 2$.

Proof of Claim. By way of contradiction, we assume that $v_{ik}v_{jt} \notin E(G)$ for all i, j, k, t.

In the case when $n = 12$. Then $l = 2$. It follows that there are 12 edges from $V(G)\backslash\{u_1, u_2\}$ to $\{u_1, u_2\}$. On the other hand, there are at most 6 edges from $\{u_1, u_2\}$ to $V(G)\backslash\{u_1, u_2\}$. This is a contradiction. Therefore, there are two vertices v_{ik} and v_{jt} with $i \neq j$ such that $v_{ik}v_{jt} \in E(G)$.

In the case when $n = 14$. Then $l = 4$. It follows that there are precise 12 edges between $V(G)\backslash\{u_1, \cdots u_4\}$ and $\{u_1, \cdots u_4\}$. It implies that the subgraph induced by u_1, u_2, u_3 and u_4 contains no edge. We assume, without loss of generality, that $u_1v_{11}, u_1v_{21}, u_1v_{31} \in E(G)$. Since $d(v_{11}) = 3$, we may assume $u_2v_{11} \in E(G)$. Since $d(u_2) = 3$, u_2 must be adjacent to two vertices of $v_{12}, v_{21}, v_{22}, v_{31}$ and v_{32}. On the other hand, since $g(G) = 5$, u_2 cannot be adjacent to one of v_{12}, v_{21}, v_{31}. It implies that $u_2v_{22}, u_2v_{32} \in E(G)$. Since $d(v_{12}) = 3$, v_{12} cannot adjacent to one of u_1 and u_2. So, $v_{12}u_3, v_{12}u_4 \in E(G)$. Since $d(v_{21}) = 3$ and $v_{21}u_1 \in E(G)$, v_{21} is adjacent to either u_3 or u_4. By symmetry, assume that $v_{21}u_3 \in E(G)$. It follows that $v_{32}u_3, u_4v_{31}, u_4v_{22} \in E(G)$. This means that $g(G) = 6$. This contradiction proves our claim. □

By the claim, G contains a 5-cycle: $vv_iv_{ik}v_{jt}v_jv$. Let $G_1 = G(v, v_iv_j)$. Then G_1 contains a 4-cycle $v_iv_{ik}v_{jt}v_jv_i$. Contracting this 4-cycle and repeatedly contracting all cycles of length at most 4 generated in process, we obtain the resulting graph, denoted by G_2. Then $|V(G_2)| \leq n - 1 - 3 \leq 10$. By Lemma 3.3, G_2 is 2-edge-connected. By Lemma 3.6, G_2 is A-connected. By Lemma 2.1, G_1 is A-connected. By Lemma 2.4, there exist an orientation $D(G)$ and a function $f \in F^*(G, A)$ such that $\partial f = b$. Therefore, G is A-connected. ■

Lemma 3.8 *Let G be a 3-edge-connected simple graph on $13 \leq n \leq 16$ vertices and A be an abelian group with $|A| \geq 5$. If $g(G) = 5$ and $\Delta(G) = 4$, then G is A-connected.*

Proof. Let $d(v) = 4$ and $N(v) = \{v_1, v_2, v_3, v_4\}$. Let $N(v_i)\backslash\{v\} = \{v_{i1}, \cdots, v_{il_i}\}$ for $i = 1, 2, 3, 4$ and $l_i \geq 2$. Let $\{u_1, \cdots u_l\} = V(G) \setminus (N(v) \cup N(v_1) \cup N(v_2) \cup N(v_3) \cup N(v_4))$. Then $l = n - (5 + l_1 + l_2 + l_3 + l_4) \leq 16 - 13 = 3$. It follows that there are at most 12 edges from $\{u_1, \cdots, u_l\}$ to $V(G)\backslash\{u_1, \cdots, u_l\}$. We claim that there exist two vertices v_{ik} and v_{jt} with $i \neq j$ such that $v_{ik}v_{jt} \in E(G)$. Suppose otherwise. There are at least 16 edges from $V(G)\backslash\{u_1, \cdots, u_l\}$ to $\{u_1, \cdots, u_l\}$, a contradiction. So v lies in a 5-cycle of G. We assume, without loss of generality, that G contains 5-cycle $C = vv_1v_{1k}v_{2t}v_2v$. Let $G_1 = G_{[vv_1, vv_2]}$. Then G_1 contains a 4-cycle

$L: v_1v_{1k}v_{2t}v_2v_1$. Contracting L into one vertex and repeatedly contract all cycles of length at most 4 generated in process, we finally get a resulting graph denoted by G_2 and let $v*$ be the vertex into which H is contracted, where $G_2 = G/H$. Then $|V(G_2)| \leq n - 3 = 13$ and $g(G_2) \geq 5$.

Let $|V(G_2)| \leq 11$. If $v \notin V(G_2)$, then by Lemma 3.2, G_2 is 3-edge-connected. By Lemma 2.6, G_2 is A-connected. Thus, assume that $v \in V(G_2)$. By Lemma 3.3, G_2 is 2-edge-connected and $d_{G_2}(v) = 2$. Since $d_{G_2}(v^*) \geq 4$, G_2 is not the graph P^+. By Lemma 3.6, G_2 is A-connected.

Thus, assume that $12 \leq |V(G_2)| \leq 13$. In this case, since $n \leq 16$, it follows that G_2 is the graph obtained just by contracting L, v^* is the vertex into which L is contracted and $15 \leq n \leq 16$. Note that v is not adjacent to v^* in G_2. Assume that $g(G_2) \geq 6$. By Lemma 3.1(3), $|V(G_2)| \geq 4 + 3d_{G_2}(v^*) \geq 16$, a contradiction. Thus, $g(G_2) = 5$. By Lemma 3.1(2), $|V(G_2)| \geq 1 + 3d_{G_2}(v^*) \geq 13$. Thus, $|V(G_2)| = 13$ and $d_{G_2}(v^*) = 4$ and $n = 16$. Note that v is not adjacent to v_L. Let $N(v^*) = \{v_1', v_2', v_3', v_4'\}$. Since $|V(G_2)| = 13$, $V(G_1) = N(v^*) \cup N(v_1') \cup N(v_2') \cup N(v_3') \cup N(v_4')$ and $v \in V(G_2) \setminus \{v^*, v_1', v_2', v_3', v_4'\}$. It follows that v^* and v lie in a 5-cycle $L_1 : v^*v_i'vuv_j'v^*$ of G_2. Thus, either v_3 or v_4 is adjacent to v^*. It implies that G contains a cycle of length at most 4, contrary to that $g(G) = 5$. ■

Lemma 3.9 *Let G be a graph and A an abelian group with $|A| \geq 3$. Suppose that $N_G(v) = \{v_1, v_2, v_3\}$ and $u \neq v_3$. Let $b \in Z(G, A)$ such that $b(v) \neq 0$ and $b(u) \neq 0$. Assume that $G_v = G(v, v_1v_2)$ and $N_{G_v}(u) = \{u_1, u_2, u_3\}$. Let $G_u = G_v(u, u_1u_2)$. If G_u is A-connected, then there exists an orientation D of G and a function $f \in F^*(G, A)$ such that $\partial f = b$ under the orientation D.*

Proof. Define b_v: $V(G_v) \mapsto A$ by

$$b_v(z) = \begin{cases} b(v) + b(v_3) & \text{if } z = v_3 \\ b(z) & \text{otherwise} \end{cases}$$

Then $b_v \in Z(G_v, A)$ and $b_v(u) \neq 0$. Since G_u is A-connected, by Lemma 2.4, there exists an orientation $D(G_v)$ of G_v and a function $f_v \in F^*(G_v, A)$ such that $\partial f_v = b_v$. Without loss of generality, assume that the new edge v_1v_2 is oriented from v_1 to v_2. Define an orientation of G from $D(G_v)$ and extend f_v to $f \in F^*(G, A)$ as follows. Orient the edge v_1v from v_1 to v and the edge vv_2 from v to v_2 and the edge vv_3 from v to v_3. For any other edges in G orient the same way as in $D(G_v)$. Define

$$f(w) = \begin{cases} f_v(v_1v_2) & \text{if } w = v_1v, vv_2 \\ b(v) & \text{if } w = vv_3 \\ f_v(w) & \text{otherwise} \end{cases}$$

Then $f \in F^*(G, A)$ such that $\partial f = b$. ■

Lemma 3.10 *Let G be a 3-edge-connected simple graph on $n \leq 15$ vertices and A an abelian group with $|A| \geq 5$. Then G is A-connected.*

Proof. Suppose otherwise that G is not A-connected with $|V(G)| = n$ minimized. Suppose that $n \leq 11$. By Lemma 2.6, G is A-connected. Therefore, we assume that $n \geq 12$. By Lemma 3.5 , we assume that $g(G) \geq 5$. By Lemma 3.1(2), $n \geq 1+\delta\Delta \geq 1+3\Delta$. Since $n \leq 15$, it follows that $\Delta \leq 4$.

Assume that $\Delta = 4$. Since $n \geq 1+3\Delta$, it follows that $n \geq 13$. Suppose that $g(G) \geq 6$. By Lemma 3.1(3), $n \geq 17$, a contradiction. Therefore, $g(G) = 5$. By Lemma 3.8, G is A-connected.

Assume that $\Delta = 3$. Since there are even number of the vertices of odd degree, it follows that $n = 12$ or 14. If $g(G) = 5$, then G is A-connected by Lemma 3.7. So we suppose that $g(G) \geq 6$. By Lemma 3.1(3), $n \geq 14$. If $g(G) \geq 7$, then by Lemma 3.1(4), $n \geq 22$, a contradiction. Thus, $g(G) = 6$. Now we prove that G is A-connected when G is a 3-edge-connected cubic graph with $n = 14$ and $g(G) = 6$ (Fig. 1).

Let $b \in Z(G, A)$. If $b \equiv 0$, then by Theorem 2.7, G admits a nowhere-zero 5-flow. Therefore, we assume that there exists a vertex v such that $b(v) \neq 0$. Let $V(G) = \{v, v_1, v_2, \ldots, v_{13}\}$ as depicted in Fig. 1. Since $b(v) \neq 0$, there must exist a vertex v_i such that $b(v_i) \neq 0$ for some $i \in \{1, \ldots, 13\}$. As depicted in Fig. 1, the vertices v_i and v must lie in a 6-cycle of G. By symmetry, we discuss only $v_i = v_1$ in level 2; $v_i = v_4$ in level 3; $v_i = v_{10}$ in level 4.

In the case when $v_i = v_1$, consider this 6-cycle: $vv_1v_4v_{10}v_6v_2v$. Define $G_{v_1} = G - \{v, v_1\} + \{v_2v_4\}$. Then G_{v_1} contains a 4-cycle L_1: $v_4v_{10}v_6v_2v_4$. We contract this 4-cycle into a vertex v_{L_1} and denote by G_2 the resulting graph. It is easy to see that G_2 contains a 3-cycle: $v_{L_1}v_{11}v_7v_{L_1}$. We contract this 3-cycle and repeatedly contract all cycles of length less than 5, finally we obtain a K_1, which is A-connected by Lemma 2.1. Repeatedly utilizing Lemma 2.1, G_{v_1} is A-connected.

In the case when $v_i = v_4$, define $G_{v_4} = G - \{v, v_4\} + \{v_1v_2, v_1v_{10}\}$. Then G_{v_4} contains a 4-cycle L_1: $v_1v_2v_6v_{10}v_1$. As the argument above, we contract this 4-cycle into a vertex and repeatedly contract all cycles of length less than 5, finally we obtain a K_1, which is A-connected by Lemma 2.1. Repeatedly utilizing Lemma 2.1, G_{v_4} is A-connected.

In the case when $v_i = v_{10}$, define $G_{v_{10}} = G - \{v, v_{10}\} + \{v_1v_2, v_4v_6\}$. Then $G_{v_{10}}$ contains a 4-cycle L_1: $v_1v_2v_6v_4v_1$. As the argument above, we contract this 4-cycle into a vertex and repeatedly contract all cycles of length less than 5, finally we obtain a K_1, which is A-connected by Lemma 2.1. Repeatedly utilizing Lemma 2.1, G_{v_4} is A-connected.

By the arguments above, we may assume that G_{v_i} is A-connected for $1 \leq i \leq 13$. By Lemma 3.9, there exists an orientation D of G and a function $f \in F^*(G, A)$ such that $\partial f = b$ under the orientation D. Therefore, G is A-connected. ■

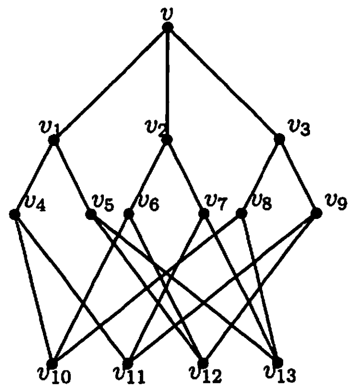

Fig. 1: Cubic graph on 14 vertices with girth 6

Proof of Theorem 1.2 Let G be a 3-edge-connected simple graph on $n \leq 17$ vertices and A an abelian group with $|A| \geq 5$. By Lemma 3.10, we assume that $n = 16$ with $\Delta(G) \geq 4$ or $n = 17$ with $\Delta(G) \geq 5$. We wish to show that G is A-connected.

Suppose otherwise that G is not A-connected with $|V(G)|$ minimized. By Lemmas 3.10 and 3.5, $g(G) \geq 5$. By Lemma 3.1(2), $17 \geq n \geq 1 + 3\Delta$. It follows that $\Delta \leq 5$.

Assume first that $\Delta = 4$. By our assumption, $n = 16$. If $g(G) \geq 6$, then by Lemma 3.1(3), $n \geq 17$, a contradiction. Therefore, $g(G) = 5$. By Lemma 3.8, G is A-connected.

Assume then that $\Delta = 5$. If $g(G) \geq 6$, then by Lemma 3.1(3), $n \geq 20$, a contradiction. Therefore, $g(G) = 5$. Let $d(v) = 5$ and $N(v) = \{v_1, v_2, v_3, v_4, v_5\}$. Let $N(v_i)\backslash\{v\} = \{v_{i1}, \cdots, v_{il_i}\}$ and $l_i \geq 2$ for $i = 1, \cdots, 5$. Let H be the subgraph induced by $N(v) \cup N(v_1) \cup N(v_2) \cup N(v_3) \cup N(v_4) \cup N(v_5)$. It follows that $|V(G)\backslash V(H)| = n - (6 + l_1 + l_2 + l_3 + l_4 + l_5) \leq 1$. Therefore, there are at most 5 edges from $V(G)\backslash V(H)$ to $V(H)$. If $v_{ik}v_{jt} \notin E(G)$ for all $i \neq j$, then there would be at least 20 edges from $V(H)$ to $V(G)\backslash V(H)$. This is a contradiction. This implies that there exist two vertices, say v_{11} and v_{21}, such that $v_{11}v_{21} \in E(G)$. It follows that v lies in a 5-cycle of G: $vv_1v_{11}v_{21}v_2v$. Let $G_1 = G_{[vv_1, vv_2]}$. Then G_1 contains a 4-cycle: $v_1v_{11}v_{21}v_2v_1$. Contracting this 4-cycle and repeatedly contracting all cycles of length at most 4 generated in processing, finally we obtain the resulting graph, denoted by G_2. It follows that $|V(G_2)| = |V(G_1)| - 3 \leq 14$ and $g(G_2) \geq 5$. By Lemma 3.4 and Theorem 3.10, G_2 is A-connected. By

Lemma 2.1, G_1 is A-connected. By Lemma 2.3, G is A-connected. ■

4 Applications

Recently, the problems of nowhere-zero flow and group connectivity have been studied extensively. In particular, degree conditions are used to guarantee the existence of group connectivity. For the literature, some results can be seen in [3, 4, 10, 11, 18, 16]. Lai [7] first used edge condition to guarantee the existence of nowhere-zero 4-flows. As applications of Theorem 1.2, we consider edge condition and group connectivity and prove the following result. First we establish the following lemma.

Lemma 4.1 *Let G be a 3-edge-connected simple graph on n vertices and $p \geq 2$ be an integer. If $|E(G)| \geq \binom{n-p+1}{2} + 2p - 1$, then G can be T-reduced to G^* with $|V(G^*)| \leq p-1$.*

Proof. Suppose otherwise that $|V(G^*)| \geq p$ and let $V(G^*) = \{v_1, v_2, \cdots, v_c\}$. We choice a graph G such that $|V(G^*)| = c \geq p$ with $|E(G)|$ maximized. Let H_i denote the preimage of v_i in G. Since $|E(G)|$ is maximized, all the H_i's are complete subgraphs and at most one H_i is a nontrival subgraph of G with $|V(H_i)| = n - c + 1$. Thus

$$|E(G)| = |E(H_i)| + |E(G^*)| = \binom{n-c+1}{2} + |E(G^*)|. \qquad (1)$$

Since $c \geq p \geq 2$ and G is 3-edge-connected, G^* is nontrival. By Lemma 2.2, $2|V(G^*)| - |E(G^*)| \geq 3$, together with Equation (1), we obtain

$$\binom{n-p+1}{2} + 2p - 1 \leq \binom{n-c+1}{2} + 2c - 3. \qquad (2)$$

Evaluating this inequality, we get $2n(c-p) \leq (c-p)(p+c+3) - 4$, which implies that $c > p$. Since $n \geq c$, we get $n \leq p + 3 - \frac{4}{c-p}$ which implies that $n \leq p+2$.

If $n = p+2$, then $c - p \leq 2$. This implies that $n \leq p+3-\frac{4}{c-p} \leq p+1$, a contradiction. If $n = p+1$, then $c = p+1$. It follows that $n \leq p+3-\frac{4}{c-p} = p-1$, a contradiction. Thus, $n \leq p$. On the other hand, $n \geq c > p$. This is a contradiction. ■

Theorem 4.2 *Let G be a 3-edge-connected simple graph on n vertices and A an abelian group with $|A| \geq 5$. If $|E(G)| \geq \binom{n-15}{2} + 31$ where $n \geq 17$, then G is A-connected.*

Proof. Let $|A| \geq 5$. By Lemma 4.1, when $p = 16$, $|E(G)| \geq \binom{n-15}{2} + 31$ and the T-reduction G^* of G has order at most 15. By Theorem 1.2, G^* is A-connected. Thus, G is A-connected. ■

If a graph G^* is obtained by repeatedly contracting nontrivial A-connected subgraphs of G until no such a subgraph left, we say G can be A-reduced to G^*. Note that if T-reduction G^* of G is A-connected with $|A| \geq 4$, then G is A-connected.

Theorem 4.3 *Let G be a 3-edge-connected simple graph on n vertices and A an abelian group with $|A| \geq 4$. If $|E(G)| \geq \binom{n-11}{2} + 23$ where $n \geq 13$, then either G is A-connected or G can be A-reduced to the Petersen graph.*

Proof. Let $|A| \geq 4$. When $p = 12$, by Lemma 4.1, $|E(G)| \geq \binom{n-11}{2} + 23$ and the T-reduction G^* of G has order at most 11. By Theorem 2.5, either G^* is A-connected or G^* is the Petersen graph. Thus, either G is A-connected or G can be A-reduced to the Petersen graph. ■

References

[1] J. A. Bondy and U. S. R. Murty, Graph Theory with Application, North-Holland, New York, 1976.

[2] J. Chen, E. Eschen and H. -J. Lai, Group connectivity of certain graphs, *Ars Combin.*, **89** (2008) 141–158.

[3] G. Fan and C. Zhou, Ore condition and nowhere zero 3-flows, *SIAM J. Discrete Math.* **22** (2008), 288–294.

[4] G. Fan and C. Zhou, Degree sum and nowhere zero 3-flows, *Discrete Math.* **308** (2008), 6233–6240.

[5] F. Jaeger, N. Linial, C. Payan and M. Tarsi, Group connectivity of graphs-a nonhomogeneous analogue of nowhere-zero flow properties, *J. Combin. Theory*, Ser B, **56** (1992) 165–182.

[6] M. Kochol, Smallest counterexample to the 5-flow conjecture has girth at least eleven, *J. Combin. Theory*, Ser B, **100** (2010) 381–389.

[7] H. -J. Lai, The size of graphs without nowhere-zero 4-flows, *J. Graph Theory*, **19** (1995), 385–395.

[8] H. -J. Lai, Group connectivity of 3-edge-connected chordal graphs, *Graphs and Combin.*, **16** (2000), 165–176.

[9] H. -J. Lai and X. Zhang, Group chromatic number of graphs without K_5-minor, *Graphs and Combin.*, **18** (2002) 147–154.

[10] H. -J. Lai and X. Yao, Group connectivity of graphs with diameter at most 2, *European J.Combin.*, **27** (2006) 436–443.

[11] X. Li, H. -J. Lai and Y. Shao, Degree condition and Z_3-connectivity, *Discrete Math.*, **312** (2012) 1658–1669.

[12] R. Luo, R. Xu, J. Yin and G. Yu, Ore-condition and Z_3-connectivity, *European J. Combin.*, **29** (2008) 1587–1595.

[13] C. Thomassen, The weak 3-flow conjecture and the weak circular flow conjecture, *J. Combin. Theory*, Ser. B, **102** (2012) 521–529.

[14] W. T. Tutte, A contribution on the theory of chromatic polynomial, *Canad. J. Math.*, **6** (1954), 80–91.

[15] W. T. Tutte, On the algebraic theory of graph colorings, *J. Combin. Theory*, **1** (1966), 15–50.

[16] X. Yao, X. Li and H. -J. Lai, Degree conditions for group connectivity, *Discrete Math.*, **310** (2010) 1050–1058.

[17] F. Yang and X. Li, Group connectivity in 3-edge-connected graphs, *Graphs and Combin.*, 28 (2012) 743–750.

[18] X. Zhang, M. Zhan, R. Xu, Y. Shao, X. Li and H. -J. Lai, Degree sum condition for Z_3-connectivity in graphs, *Discrete Math.*, **310** (2010) 3390–3397.