ON THE FIRST UNKNOWN VALUE OF TWO FUNCTIONS FOR CONVEX LATTICE v-GONS

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ABSTRACT. Let $a(v)$ and $g(v)$ denote the least possible area and the least possible number of lattice points in the interior of a convex lattice v-gon, respectively. Many lower and upper bounds for $a(v)$ and $g(v)$ are known for every v. However, the exact values of these two functions are only known for $v \leq 10$ and $v \in \{12, 13, 14, 16, 18, 20, 22\}$. The purpose of this paper is to answer the following Open Question 1 from [13]: What is the exact value of $a(11)$? We answer this question by proving that $a(11) = 21.5$. On our way to achieve the goal we also prove that $g(11) = 17$.

1. INTRODUCTION

A *lattice point* in the plane is a point with integer coordinates. A *lattice segment* is a line segment whose endpoints are lattice points. The *lattice length* of such a segment is one less than the number of all lattice points of that segment. Any line passing through two lattice points is called a *lattice line*. A *lattice polygon* is a simple (hence non-degenerated) polygon whose vertices are lattice points. A convex lattice polygon with v vertices is called a *v-gon*. The set of all v-gons is denoted by $\mathcal{K}_v$.

By $v = v(K)$, $b = b(K)$, $g = g(K)$ and $a = a(K)$ we denote the number of vertices, boundary lattice points, interior lattice points and the area of a lattice polygon K, respectively. We will also deal with the number $G = G(K) = b(K) + g(K)$ and use the following standard notation. For a set S we denote by $\operatorname{conv} S$, ∂S, $\operatorname{cl} S$ the convex hull, the boundary and the closure of the set S, respectively.

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Problems of finding relationships between the numbers v, b, g, G and a are of great interest and have been investigated by many authors, see among others [2, 8, 9, 10, 11, 12, 13, 14, 15].

For positive integers $v \geq 3$ define the function

$$a(v) = \min\{a(K) : K \in \mathcal{K}_v\}.$$

Since Andrews [1] the natural problem of finding the values of the function $a(v)$ has been investigated by many authors. As a result some lower and upper estimates for $a(v)$ have been obtained, see among others [3, 4, 5, 6, 7, 13, 15]. For example, in [4] one can find the following estimates

$$\frac{1}{8\pi^2} < \frac{a(v)}{v^3} \leq \frac{1}{54}(1 + o(1)).$$

The exact values of $a(v)$ are only known for a few, relatively small, v; namely for $v \leq 10$ and $v \in \{12, 13, 14, 16, 18, 20, 22\}$, see [5, 15]. For the first yet unknown value, $a(11)$, Simpson [15] has obtained that

$$19.5 \leq a(11) \leq 21.5 \tag{1.1}$$

and Rabinowitz [13, Open Question 1] has asked: What is the exact value of $a(11)$? The main purpose of this paper is to answer this question by proving the following theorem.

Theorem 1.1. $a(11) = 21.5$.

In our proof of Theorem 1.1 we use the function

$$g(v) = \min\{g(K) : K \in \mathcal{K}_v\}$$

which is closely related to $a(v)$ in the following way.

Fact 1.2. [15] *For* $v \geq 3$,

$$a(v) = g(v) + v/2 - 1. \tag{1.2}$$

From Fact 1.2 and inequalities (1.1) we immediately get that

$$15 \leq g(11) \leq 17. \tag{1.3}$$

Now it is clear that in order to establish Theorem 1.1 it suffices to show that $g(11) = 17$, and this will be done in Section 3. In this way we also find the first unknown value of the function $g(v)$. Since we will have to refer to some values of the function $g(v)$ let us collect the first few of them which are as follows: $g(3) = 0$, $g(4) = 0$, $g(5) = 1$, $g(6) = 1$, $g(7) = 4$, $g(8) = 4$, $g(9) = 7$, $g(10) = 10$, see [12].

2. Preliminaries

We will work with classes of equivalent lattice polygons. To understand when two lattice polygons are equivalent we need to review some definitions. An *affine transformation* is a linear transformation followed by a translation. A *unimodular transformation* is one that preserves area. If the entries of the matrix corresponding to a unimodular transformation are integers then the transformation is known as an *integral unimodular transformation.* Such a transformation has the property that it preserves convexity and the number of lattice points in a set.

Two lattice polygons are said to be *lattice equivalent*, or just *equivalent*, if one can be transformed into the other via an integral unimodular affine transformation. In particular, two lattice polygons are lattice equivalent if one can be transformed into the other via a *shear about a line l*, that is, an integral unimodular transformation that leaves all the points on the line l fixed. For example, a shear about the x-axis in the plane is given by the equations

$$\begin{aligned} x' &= x + ky\,, \quad k \in \mathbb{Z}, \\ y' &= y. \end{aligned}$$

The interior hull and the outer hull of a convex lattice polygon are two key concepts used in this paper. Let us recall their definitions. Let K be a convex lattice polygon in the plane. Denote by $H = H(K)$ the convex hull of all lattice points in the interior of K. Following [12] we call $H(K)$ the *interior hull* of K. Note that $H(K)$ might degenerate into a segment, a point or even the empty set. We will also use $H^n = H^n(K)$ defined by

$$H^1(K) = H(K) \quad \text{and} \quad H^n(K) = H(H^{n-1}(K)) \quad \text{for} \quad n \geq 2.$$

Suppose that the interior hull of a convex lattice polygon K is a lattice polygon. Let u be an edge of $H(K)$. By $l(u)$ and $h(u)$ we respectively denote the lattice line containing u and the open half-plane that is exterior to $H(K)$ and bounded by $l(u)$. The following two facts will be very useful.

Fact 2.1. [12] *Let K be a convex lattice polygon with interior hull H. If $v(K) \geq 7$, then $v(H) \geq \left\lceil \frac{v(K)}{2} \right\rceil$. If $v(K) \geq 9$, then $b(H) \geq \left\lceil \frac{2}{3} v(K) \right\rceil$.*

Fact 2.2. [12] *Let H be the interior hull of a convex lattice polygon K. If u is an edge of H, then $h(u)$ contains at most two vertices of K.*

Recall from [9] that the *outer hull* of a convex lattice polygon H, denoted by $\mathcal{O}(H)$, is the closed convex region bounded by the lattice lines exterior to H, parallel to and closest to the edges of H. In the proofs of the next two theorems we will also use the following notation. Let $g(u) \subset h(u)$ denote the open halfplane bounded by the lattice line $l_g(u)$ which is parallel to and closest to u.

Theorem 2.3. *Let the interior hull of a convex lattice polygon K be a lattice polygon H. Then the vertices of K must lie on the boundary of $\mathcal{O}(H)$.*

Proof. From the definition of $\mathcal{O}(H)$ it follows that the vertices of K cannot lie in the interior of $\mathcal{O}(H)$. We will show that they cannot lie in the exterior of $\mathcal{O}(H)$ either. Suppose that there is a vertex X of K that lies in the exterior of $\mathcal{O}(H)$. Then $X \in g(u_0)$ for some $u_0 \subset \partial H$. Let A and B be the endpoints of the edge u_0 and let m be its lattice length. First notice that the triangle ABX cannot contain any interior lattice points. Indeed, if there were a lattice point Y in the interior of ABX then $\mathrm{conv}(H \cup Y)$ would be a convex lattice polygon contained in K and larger than H, contradicting the maximality of H. A similar argument reveals that the segments AX and BX contain no lattice points in their relative interiors. Thus $g(ABX) = 0$ and $b(ABX) = m + 2$.

Let C be any lattice point on $l_g(u_0)$. Applying a similar argument to that in the case of the triangle ABX one can see that the triangle ABC contains $m + 2$ boundary lattice points and no interior lattice points. By Pick's Theorem both triangles should have the same area, but obviously their areas are different since they have the same base and different altitudes, a contradiction. The proof is complete. $\square$

It was pointed out in [9] that the outer hull is not always a lattice polygon. The following theorem explains why we are interested only in outer hulls which are lattice polygons.

Theorem 2.4. *Let K be a convex lattice polygon. There exists a convex lattice polygon P such that $H(P) = K$ if and only if $\mathcal{O}(K)$ is a lattice polygon.*

Proof. The sufficient condition is obvious. We prove the necessary condition. Let K be the interior hull of a convex lattice polygon P. Assume that $\mathcal{O}(K)$ is not a lattice polygon. Then at least one vertex of $\mathcal{O}(K)$, say B, is not a lattice point. In order to obtain a lattice polygon we need to cut away the vertex B. Thus, we have to find two lattice points, say B_1 and B_2, lying on two edges adjacent to B such that the intersection of the triangle BB_1B_2 and the polygon K is the empty set. Considering two cases we will show that this is not possible.

First consider the case when $B = l_g(u_1) \cap l_g(u_2)$ for two consecutive edges u_1 and u_2 of K. Let $A = u_1 \cap u_2$, $X_1 = l_g(u_1) \cap l(u_2)$ and $X_2 = l_g(u_2) \cap l(u_1)$, see Fig. 1.

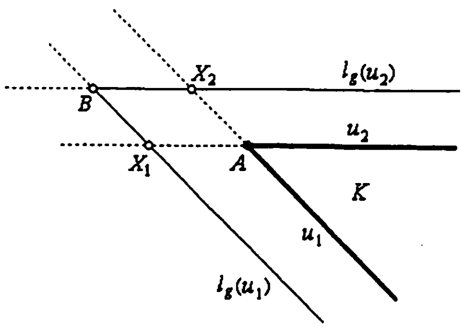

Fig. 1.

Notice that B_1 cannot lie in the relative interior of the segment X_1B since $l_g(u_2)$ would not be the lattice line closest to $l(u_2)$. Similarly B_2 cannot lie in the relative interior of the segment X_2B. The point B_1 cannot coincide with X_1, otherwise X_1A would be a lattice segment and therefore between B and X_2 there would have to be a lattice point. This, however, is impossible since $l_g(u_1)$ would not be the lattice line closest to $l(u_1)$. For the same reason B_2 cannot coincide with X_2. Hence B_1 and B_2 would have to lie out of BX_1 and BX_2, respectively. Moreover B_1B_2 cannot intersect K. Therefore the lattice points B_1, B_2 and A are not collinear. One can easily see that $B_1 + (B_2 - A)$ would be a lattice point in the interior of the quadrilateral BB_1AB_2, which of course cannot happen.

Suppose now that $B = l_g(u_1) \cap l_g(u_k)$, where $u_1, u_2, \ldots, u_k$, $k \geq 3$, are consecutive edges of K. Notice that this happens when the lattice lines $l_g(u_i)$ parallel to and closest to $l(u_i)$, $i = 2, \ldots, k-1$, do not intersect the interior of $\mathcal{O}(K)$. Let $X_1 = l_g(u_1) \cap l(u_2)$ and $X_2 =$

$l_g(u_k) \cap l(u_2)$. The lattice points B_1 and B_2 cannot lie in the relative interiors of the segments X_1B and X_2B, respectively. Otherwise $l_g(u_2)$ would intersect the interior of $\mathcal{O}(K)$. It is clear now that a construction of triangle BB_1B_2 having empty intersection with K is impossible. The proof is complete. $\square$

In view of Theorem 2.3 we need a procedure of trimming the outer hull to a desired polygon. By *cutting away* or *trimming a vertex* A of a convex lattice polygon K with nonempty interior hull, we mean an operation that replaces K by the polygon $\mathrm{cl}(K \setminus \Delta_A)$, where Δ_A is the triangle with vertices at A and its closest lattice points lying on the edges adjacent to A. We say that a convex lattice polygon K *admits a trimming at its vertex* A if $\Delta_A \cap H(K) = \emptyset$, and *admits a proper trimming at its vertex* A if, in addition, we have

$$v(\mathrm{cl}(K \setminus \Delta_A)) = v(K) + 1.$$

Alternatively, we say that *the vertex* A *can be cut away* or that it *can be cut away properly*, respectively.

We notice some useful observations about trimming. First, if a convex lattice polygon K admits a trimming at its vertex A, then Δ_A cannot contain any interior lattice points. Second, K admits a proper trimming at A if and only if the vertices of Δ_A, different from A, lie in the relative interiors of the edges adjacent to A. Third, if AB is an edge of K of lattice length 2 and K admits a proper trimming at A and at B, then after cutting away the vertex A, the polygon $\mathrm{cl}\,(K \setminus \Delta_A)$ does not admit a proper trimming at B.

We want to draw the reader's attention to the difference between cutting away a single vertex of $\mathcal{O}(H)$ and the process of trimming $\mathcal{O}(H)$ to a required polygon. This is illustrated in the case of the lattice pentagon H with vertices $(0,0)$, $(0,2)$, $(1,2)$, $(2,1)$ and $(2,0)$. It is easy to check that the outer hull of H is a convex lattice pentagon which admits a proper trimming at every vertex. Therefore, one could get an impression that $\mathcal{O}(H)$ could be trimmed to a decagon. However, it is easy to observe that we can cut away properly at most three vertices in a row, obtaining an octagon.

The following two facts will be very helpful.

Fact 2.5. [9] *Suppose that H is a convex lattice polygon with interior lattice points. Let u be an edge of H with no lattice points in its relative interior and let v be an edge of $\mathcal{O}(H)$ parallel to u and lying* *in $h(u)$. Then v cannot contain two lattice points in its relative interior.*

Fact 2.6. [10] *For any convex lattice polygon K with interior lattice points we have*

$$b(K) \leq 2g(K) - v(K) + 10. \tag{2.1}$$

3. Proof of the main result

We have already mentioned that in order to establish Theorem 1.1, which is the main result of this paper, it suffices to prove the following theorem.

Theorem 3.1. *A convex lattice* 11*-gon must contain at least* 17 *interior lattice points.*

Proof. In connection with inequalities (1.3) the theorem is proved by considering two claims in which we show that there is no convex lattice 11-gon K with $g(K) = 15$ or $g(K) = 16$.

Claim 1. *A convex lattice* 11*-gon cannot contain* 15 *interior lattice points.*

Proof of Claim 1. Suppose to the contrary that there exists a convex lattice 11-gon K with 15 interior lattice points. We first find necessary conditions for the existence of such a polygon and next show that no polygon satisfying these conditions exists.

The interior lattice points of K are all lattice points of $H = H(K)$. Thus

$$b(H) + g(H) = g(K) = 15 \tag{3.1}$$

and obviously $v(H) \leq b(H) \leq 15$. By Fact 2.1 we have $v(H) \geq 6$ and $b(H) \geq 8$. We must also have $v(H) \leq 8$, otherwise we would get $b(H) \geq v(H) \geq 9$ and $g(H) \geq g(9) = 7$ and equality (3.1) would not be valid. Clearly, when $v(H) = 6$ we have $g(H) \geq g(6) = 1$, for $v(H) \geq 7$ we have $g(H) \geq g(7) = 4$. From the above conditions we get 15 possible realizations of the numbers $v(H)$, $b(H)$ and $g(H)$, which we collect in Table 1.

Case	$v(H)$	$b(H)$	$g(H)$	$G(H)$=$g(K)$	Case	$v(H)$	$b(H)$	$g(H)$	$G(H)$=$g(K)$
1.1	6	8	7	15	1.9	7	9	6	15
1.2	6	9	6	15	1.10	7	10	5	15
1.3	6	10	5	15	1.11	7	11	4	15
1.4	6	11	4	15	1.12	8	8	7	15
1.5	6	12	3	15	1.13	8	9	6	15
1.6	6	13	2	15	1.14	8	10	5	15
1.7	6	14	1	15	1.15	8	11	4	15
1.8	7	8	7	15					

Table 1

By Fact 2.6 we can immediately eliminate Cases 1.5, 1.6, 1.7 and 1.15. In the remaining cases we will show that it is not possible to construct a convex lattice 11-gon with the interior hull H having the required parameters collected in the ordered triple

$$t(H) = (v(H), b(H), g(H)).$$

Case 1.1: $t(H) = (6, 8, 7)$ *and 1.2:* $t(H) = (6, 9, 6)$. Let H be a convex lattice hexagon with 8 or 9 boundary lattice points. We try to trim $\mathcal{O}(H)$ to an 11-gon. Obviously this would not be possible if $v(\mathcal{O}(H)) \leq 5$. So assume that $v(\mathcal{O}(H)) = 6$. Now we need to cut away properly five vertices of $\mathcal{O}(H)$ in a row.

Since the polygon H has at least three edges of lattice length 1, Fact 2.5 implies that $\mathcal{O}(H)$ has at least three edges with at most one lattice point in their relative interiors. Even if we could cut away properly each vertex of $\mathcal{O}(H)$ separately, in view of the observations following the definition of trimming it is impossible to cut away properly five of them in a row. Thus, we can obtain at most a decagon. This completes the proof in this case.

We want to refer to the above observations later on and therefore we formulate the following fact.

Fact 3.2. *Let H be a convex lattice hexagon with* 8 *or* 9 *boundary lattice points. Then $\mathcal{O}(H)$ cannot be trimmed to any convex lattice 11-gon.*

Case 1.3: $t(H) = (6, 10, 5)$. A reasoning similar to that in Cases 1.1 and 1.2 allows us to assume that H has only two edges of lattice length 1 and four edges of lattice length 2. The lattice lengths of the edges of H can occur in the following three sequences: 122122, 121222 and 112222. The three possible situations are symbolically illustrated in Fig. 2. We want to warn the reader that here we pay more attention to the lattice lengths than to the placement of the vertices.

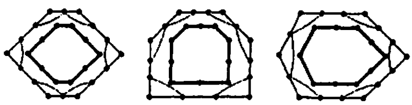

Fig. 2.

In the case of the first outer hull we can cut away properly only four vertices in a row. The second outer hull cannot be trimmed to an 11-gon either. Only in the case of the sequence 112222 there is a chance to obtain an 11-gon. Using a similar reasoning one can easily observe that the following fact is true.

Fact 3.3. *Let H be a convex lattice hexagon. The necessary condition for a possibility of trimming $\mathcal{O}(H)$ to a convex lattice 11-gon is that H has at least four consecutive edges of lattice length at least 2.*

We continue the analysis of Case 1.3. Recall that $g(H) = 5$. This and $g(5) = 1$ imply that $v(H^2) \leq 4$. Thus H^2 is either a lattice segment or a lattice polygon for which

$$t(H^2) \in \{(4,5,0),(4,4,1),(3,5,0),(3,4,1),(3,3,2)\}.$$

The interior lattice points of H are collinear. Obviously they form a lattice segment of lattice length 4. Using a similar argument to that in [11, The x-axis Lemma] we can transform H by means of an integral unimodular affine transformation in such a way that its interior lattice points are mapped into points with coordinates $(1,0),\ldots,(5,0)$. It is clear that such a transformation carries H into a lattice hexagon lying in the strip S bounded by the lines $y = -1$ and $y = 1$ with two vertices on each line and two vertices with coordinates $(0,0)$ and $(6,0)$. From now on, whenever we place a polygon in the strip S it will lie in the half-plane $x \geq 0$ with vertex $(0,1)$. In order to get such a placement previous transformations should be (if necessary) followed by a shear about the x-axis. Clearly, the four edges adjacent to the vertices $(0,0)$ and $(6,0)$ have lattice length 1. Thus, by Fact 3.3 this situation cannot happen.

The interior lattice points of H form a triangle. Obviously, $\mathcal{O}(H^2)$ has 3 edges. After cutting away all three vertices of $\mathcal{O}(H^2)$ we get three edges of lattice length 1. Therefore by Fact 3.3 we can eliminate this case.

The interior lattice points of H form a quadrilateral. When H^2 is a quadrilateral we have to cut away at least two vertices of $\mathcal{O}(H^2)$ to obtain a hexagon. The resulting hexagon will never have four consecutive edges of lattice lengths 2 and again by Fact 3.3 such a situation cannot happen. This completes the consideration of Case 1.3.

From Fact 3.3 and the above observations we have the following fact.

Fact 3.4. *Let H^2 denote the interior hull of a convex lattice hexagon H. If $v(H^2) \le 4$, then $\mathcal{O}(H)$ cannot be trimmed to a convex lattice 11-gon.*

Case 1.4: $t(H) = (6, 11, 4)$. In view of Fact 3.4 this case is eliminated.

Case 1.8: $t(H) = (7, 8, 7)$. Here by Fact 2.1 we have $v(H^2) \ge 4$. Thus

$$t(H^2) \in \{(4,4,3),(4,5,2),(4,6,1),(4,7,0),(5,5,2),(5,6,1),(6,6,1)\}.$$

When $t(H^2) = (4, 4, 3)$ it follows from Fact 2.5 that every edge of $\mathcal{O}(H^2)$ contains at most one lattice point in its relative interior. Thus it is impossible to trim $\mathcal{O}(H^2)$ to a heptagon. Using a similar argument one can see that $\mathcal{O}(H^2)$ cannot be trimmed to a heptagon when $t(H^2) = (4, 5, 2)$.

When $t(H^2) = (5, 5, 2)$ we place H^2 in the strip S. If $(0, 0)$ and $(3, 0)$ are vertices of H^2 we can assume that $(1, 1)$ is also a vertex and then the last vertex is either $(1, -1)$ or $(2, -1)$. If only one of the points $(0, 0)$ and $(3, 0)$, say $(3, 0)$, is a vertex of H^2, then on both lines $y = -1$ and $y = 1$ there are two vertices of H^2. Clearly, in this way we get three non-equivalent polygons. By [11, Theorems 1 and 5] we get five more polygons related to the remaining cases $t(H^2) \in \{(4, 6, 1), (4, 7, 0), (5, 6, 1), (6, 6, 1)\}$. All candidates for H^2 are presented in Fig. 3.

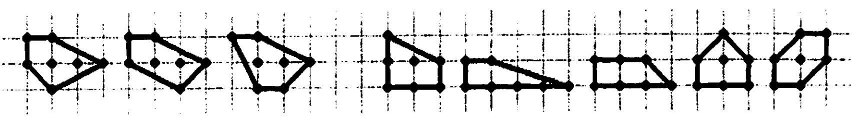

Fig. 3.

It is easy to check that only the outer hulls of the last three polygons, called Q_0, P_0 and H_0, can be trimmed to heptagons.

For the quadrilateral Q_0 we obtain 15 heptagons shown in Fig. 4.

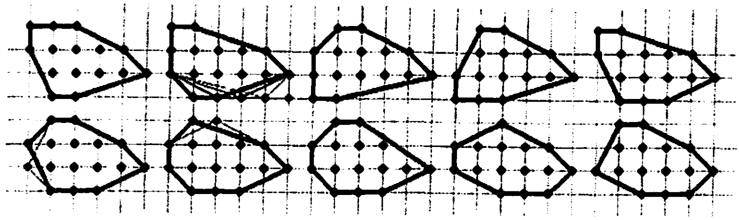

Fig. 4.

Only for two of them the outer hulls are lattice polygons but they cannot be trimmed to an 11-gon.

For P_0 and H_0 we obtain the following 10 polygons but none of their outer hulls can be trimmed to an 11-gon.

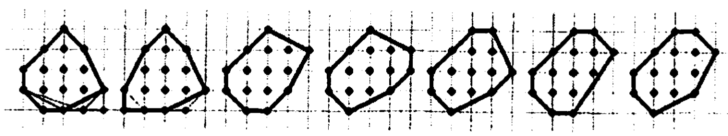

Fig. 5.

Case 1.9: $t(H) = (7, 9, 6)$. We have the following four possibilities:

$$t(H^2) \in \{(4,4,2), (4,5,1), (4,6,0), (5,5,1)\}.$$

In Case 1.8 we noticed that the first two triples do not lead to any heptagon.

By [11, Theorem 5] only pentagon P_1 with vertices $(0,1)$, $(1,2)$, $(2,2)$, $(2,1)$ and $(1,0)$ has triple $(5,5,1)$. Obviously, $\mathcal{O}(P_1)$ cannot be trimmed to a heptagon with 9 boundary lattice points.

By [11, Theorem 1] there are two realizations of the triple $(4,6,0)$. Namely, the trapezoid T_1 with vertices $(1,1)$, $(1,2)$, $(2,2)$, $(4,1)$ and the rectangle R_1 with vertices $(1,1)$, $(1,2)$, $(3,2)$, $(3,1)$. The reader can easily check that $\mathcal{O}(T_1)$ cannot be trimmed to a heptagon. In the case of $\mathcal{O}(R_1)$ a symmetry consideration restricts the resulting polygons to the six heptagons in Fig. 6, the first one with one edge of lattice length 3 and the remaining with two edges of lattice length 2. None of their outer hulls can be trimmed to an 11-gon.

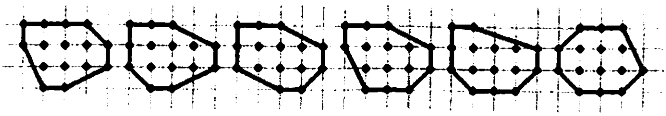

Fig. 6.

Case 1.10: $t(H) = (7, 10, 5)$. Fact 2.1 and $g(5) = 1$ imply that $v(H^2) = 4$. Thus $t(H^2) \in \{(4,5,0),(4,4,1)\}$. By [11, Theorems 1 and 5] H^2 is equivalent to one of the three possible quadrilaterals shown with their outer hulls in Fig. 7.

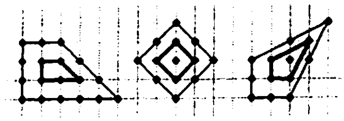

Fig. 7.

It is obvious that only the first outer hull can be trimmed to a heptagon. No matter how we trim $\mathcal{O}(H^2)$ to a heptagon we obtain heptagons equivalent to the one shown in Fig. 8. Its outer hull cannot be trimmed to an 11-gon.

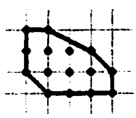

Fig. 8.

Case 1.11: $t(H) = (7, 11, 4)$. In this case we have equality in (2.1). On the other hand equality in (2.1) cannot happen for heptagons, see [10]. This eliminates Case 1.11.

Case 1.12: $t(H) = (8, 8, 7)$. All possible lattice polygons with $v \geq 4$ and $G = 7$ are shown in the picture in Case 1.8. Only the outer hulls of Q_0 and H_0 can be trimmed to an octagon. One can see that the outer hulls of the resulting octagons, shown in Fig. 9, are not lattice polygons.

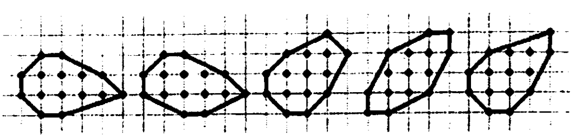

Fig. 9.

Case 1.13: $t(H) = (8, 9, 6)$. We can repeat here the argument from Case 1.9 and assume that $t(H^2) \in \{(4,6,0),(5,5,1)\}$. The outer hull of the pentagon P_1 cannot be trimmed to an octagon. When H^2 is equivalent to the rectangle R_1 the outer hull of the only obtained octagon, shown in Fig. 10, cannot be trimmed to an 11-gon.

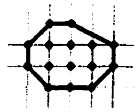

Fig. 10.

Case 1.14: $t(H) = (8, 10, 5)$. This case can be ruled out since a convex lattice octagon cannot contain 5 interior lattice points, see [12, The Octagon Anomaly]. This completes the proof of Claim 1. □

Claim 2. *A convex lattice* 11*-gon cannot contain* 16 *interior lattice points.*

Proof of Claim 2. Suppose that there exists a convex lattice 11-gon K with 16 interior lattice points. In this case we have $b(H) + g(H) = g(K) = 16$, $v(H) \leq b(H) \leq 16$ and again by Fact 2.1 $v(H) \geq 6$ and $b(H) \geq 8$. Moreover $v(H) \leq 9$ and when $v(H) = 9$ we have $g(H) \geq g(9) = 7$. From the constraints and observations made in the analysis of Claim 1 and also by Facts 3.2 and 2.6 we get the following remaining possible realizations of the numbers $v(H)$, $b(H)$ and $g(H)$.

Case	$v(H)$	$b(H)$	$g(H)$	$G(H)=g(K)$	Case	$v(H)$	$b(H)$	$g(H)$	$G(H)=g(K)$
2.1	6	10	6	16	2.7	7	11	5	16
2.2	6	11	5	16	2.8	8	8	8	16
2.3	6	12	4	16	2.9	8	9	7	16
2.4	7	8	8	16	2.10	8	10	6	16
2.5	7	9	7	16	2.11	9	9	7	16
2.6	7	10	6	16					

Table 2

Case 2.1: $t(H) = (6, 10, 6)$. In view of Fact 3.4 we may assume that $v(H^2) \geq 5$. Thus $t(H^2) = (5, 5, 1)$ and H^2 is equivalent to the pentagon P_1. Since $b(\mathcal{O}(P_1)) = 10$ it is impossible to trim $\mathcal{O}(P_1)$ to a hexagon with $b(H) = 10$.

Case 2.2: $t(H) = (6, 11, 5)$ *and 2.3:* $t(H) = (6, 12, 4)$. Here $v(H^2) \leq 4$ and both cases are eliminated by Fact 3.4.

Case 2.4: $t(H) = (7, 8, 8)$. In this case

$$t(H^2) \in \{(4,4,4), (4,5,3), (4,6,2), (4,7,1), (4,8,0), \\ (5,5,3), (5,6,2), (5,7,1), (6,6,2)\}.$$

A similar argument to that in Case 1.8 reveals that the triples $(4,4,4)$ and $(4,5,3)$ do not lead to any heptagon.

We place the polygons with triples $(4,6,2)$, $(5,6,2)$ and $(6,6,2)$ in the strip S. Vertices of the polygons have to be on both lines $y=-1$ and $y=1$ and can be at $(0,0)$ and/or $(3,0)$. It is easy to check that there are only two non-equivalent hexagons having the required parameters. We can assume that in the case of $(5,6,2)$ the existing edge of lattice length 2 can lie either on the y-axis or on the line $y=1$. In this way we obtain four non-equivalent pentagons. Any quadrilateral with the triple $(4,6,2)$ has one edge of lattice length 3 or two edges of lattice length 2. All polygons with two interior lattice points obtained in this case are presented in Fig. 11.

Fig. 11.

The interior lattice points of H^2 with $t(H^2)=(5,5,3)$ can be collinear or not. If they are collinear we place H^2 in the strip S. If H^2 has two vertices on the x-axis, then we get three nonequivalent pentagons. If H^2 has only one vertex on the x-axis we get one polygon. If the interior lattice points of H^2 are not collinear, then we can assume that H^3 is the triangle with vertices $(1,0)$, $(2,0)$ and $(1,1)$. To obtain a required polygon we have to cut away all three vertices of $\mathcal{O}(H^3)$. All pentagons with three interior lattice points obtained here are presented in Fig. 12.

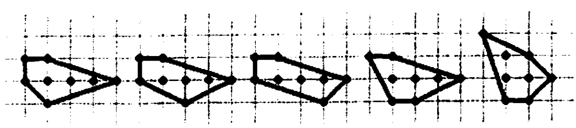

Fig. 12.

In our consideration of the remaining triples $(4,7,1)$, $(4,8,0)$ and $(5,7,1)$ we use the known characterization of convex lattice polygons with at most one interior lattice point, see [11, Theorems 1 and 5].

Fig. 13 collects only those polygons, obtained in this case analysis, whose outer hulls are lattice polygons.

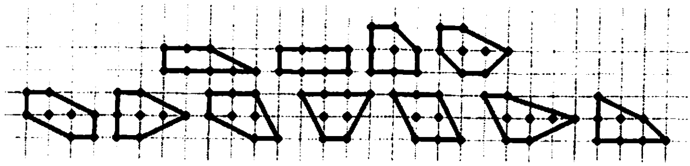

Fig. 13.

Only the outer hulls of the top four polygons can be trimmed to a heptagon but only in the case of the two heptagons shown in Fig. 14 their outer hulls are lattice polygons.

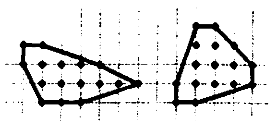

Fig. 14.

However, no vertex of the resulting two polygons can be cut away properly.

Case 2.5: $t(H) = (7, 9, 7)$. We can repeat here the reasoning from Case 1.8. The outer hulls of Q_0, P_0 and H_0 can be trimmed to 24 non-equivalent heptagons. Only nine of them (shown in Fig. 15) have outer hulls which are lattice polygons. The resulting outer hulls can be trimmed at most to an octagon.

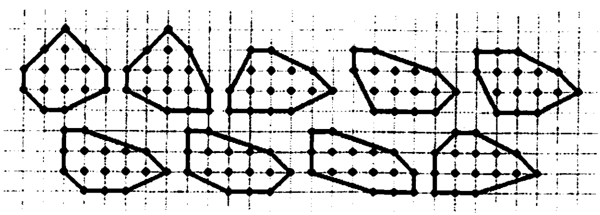

Fig. 15.

Case 2.6: $t(H) = (7, 10, 6)$. As in Case 1.9 we obtain in Fig. 16 four heptagons with 10 boundary and 6 interior lattice points. None of their outer hulls can be trimmed to a convex lattice 11-gon.

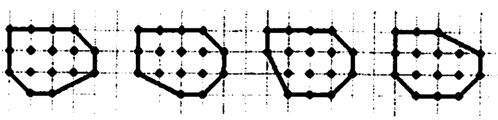

Fig. 16.

Case 2.7: $t(H) = (7, 11, 5)$. As in Case 1.10, $\mathcal{O}(H^2)$ is the quadrilateral with vertices $(0,0)$, $(0,3)$, $(2,3)$, $(5,0)$. Since $b(\mathcal{O}(H^2)) = 13$ it is impossible to trim $\mathcal{O}(H^2)$ to a heptagon H with $b(H) = 11$.

Case 2.8: $t(H) = (8, 8, 8)$. We can repeat here the reasoning from Case 2.4. In this case only two outer hulls can be trimmed to an octagon. We obtain 8 different octagons but the outer hull of one of them, shown in Fig.17, is a lattice polygon. This outer hull does not admit a proper trimming at any vertex.

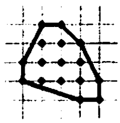

Fig. 17.

Case 2.9: $t(H) = (8, 9, 7)$. Following Case 1.12 we obtain 3 different octagons, see Fig. 18. None of their outer hulls admits a proper trimming at any vertex.

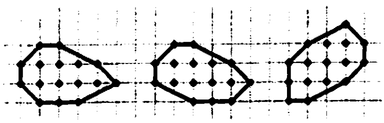

Fig. 18.

Case 2.10: $t(H) = (8, 10, 6)$. Just like in Case 1.13 we trim the outer hull of the rectangle R_1 to the only octagon given in Fig. 19. Its outer hull can be trimmed at most to a decagon.

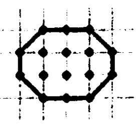

Fig. 19.

Case 2.11: $t(H) = (9, 9, 7)$. In Fig. 20 we show the unique nonagon with the required parameters, see [12]. Its outer hull is also a nonagon that does not admit a proper trimming at any vertex.

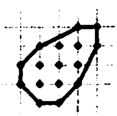

Fig. 20.

This completes the proof of Theorem 3.1, namely, that a convex lattice 11-gon must contain at least 17 interior lattice points. Since there exists an example of a convex lattice 11-gon with 17 interior lattice points, see [12], we proved that $g(11) = 17$ and in view of Fact 1.2 also that $a(11) = 21.5$. □

Remark 3.5. It would be nice to eliminate the readable asymmetry between the numbers of known values of the functions $a(v)$ and $g(v)$ for odd and even values of v.

References

[1] G.E. Andrews, A lower bound for the volume of strictly convex bodies with many boundary points, *Trans. Amer. Math. Soc.* **106** (1963) 270-279.

[2] J.R. Arkinstall, Minimal requirements for Minkowski's theorem in the plane I, *Bull. Austral. Math. Soc.* **22** (1980) 259-274.

[3] I. Bárány, J. Pach, On the number of convex lattice polygons, *Comb. Prob. Comp.* **1** (1992) 295-302.

[4] I. Bárány, N. Tokushige, The minimum area of convex lattice n-gons, *Combinatorica* **24** (2004) 171-185.

[5] Cai Tian-Xin, On the minimum area of convex lattice polygons, *Taiwanese J. Math.* **1** (1997) 351-354.

[6] H. Chaix, Sur les points frontières des compacts strictement convexes de $\mathbb{R}^2$ appartenant à un réseau, *C. R. Acad. Sci. Paris Sér. A* **285** (1977) 887-889.

[7] C.J. Colbourn, R.J. Simpson, A note on bounds on the minimum area of convex lattice polygons, *Bull. Austral. Math. Soc.* **45** (1992) 237-240.

[8] K. Kołodziejczyk, Hex-triangles with one interior H-point, *Ars Combin.* **70** (2004) 33-45.

[9] K. Kołodziejczyk, D. Olszewska, On some conjectures by Rabinowitz, *Ars Combin.* **79** (2006) 171-188.

[10] K. Kołodziejczyk, D. Olszewska, A proof of Coleman's conjecture, *Discrete Math.* **307** (2007) 1865-1872.

[11] S. Rabinowitz, A census of convex lattice polygons with at most one interior lattice point, *Ars Combin.* **28** (1989) 83-96.

[12] S. Rabinowitz, On the number of lattice points inside a convex lattice n-gon, *Congr. Numer.* **73** (1990) 99-124.

[13] S. Rabinowitz, $O(n^3)$ bounds for the area of a convex lattice n-gon, *Geombinatorics*, **2** (1993) 85-88.

[14] P.R. Scott, On convex lattice polygons, *Bull. Austral. Math. Soc.* **15** (1976) 395-399.

[15] R.J. Simpson, Convex lattice polygons of minimum area, *Bull. Austral. Math. Soc.* **42** (1990) 353-367.