Hamiltonicity of the Inner Power of a Cycle

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Abstract

Hammack and Livesay introduced a new graph operation $G^{(k)}$ for a graph G, which they called the kth inner power of G. A graph G is Hamiltonian if it contains a spanning cycle. In this paper, we show that $C_n^{(k)}(n \geq 3,\ k \geq 2)$ is Hamiltonian if and only if n is odd and $k = 2$, where C_n is the cycle with n vertices.

Keywords: Inner power; Direct product; Hamiltonian cycle.

1 Introduction

In the study on the problem of cancellation over the direct product, Hammack and Livesay [2] constructed a new graph operation which they called the inner power of a graph. Given a graph G and a positive integer k, the kth inner power of G, denoted by $G^{(k)}$, is the graph defined as follows, where arithmetic on the indices is done modulo k.

$V(G^k) = \{(x_0, x_1, \ldots, x_{k-1}) \colon x_i \in V(G) \text{ for } 0 \leq i < k\}$,

$E(G^k) = \{(x_0, x_1, \ldots, x_{k-1})(y_0, y_1, \ldots, y_{k-1}) \colon x_i y_{i\pm1} \in E(G) \text{ for } 0 \leq i < k\}$.

Since the case $k = 1$ is trivial, we assume $k \geq 2$ in the following. Notice that if $k = 2$, then $(x_0, x_1)(y_0, y_1) \in E(G^{(2)})$ if and only if $x_0 y_1 \in E(G)$ and $x_1 y_0 \in E(G)$. Fig.1 shows two examples, where for convenience the vertices are labeled as xy rather than (x, y). We refer to [2] for various properties of inner powers, such as connectivity, bipartiteness, and their interaction with the direct product.

In this paper, we consider the Hamiltonicity problem of inner powers of cycles. Our main result can be formulated as follows.

Theorem 1. *Let $n \geq 3$ and $k \geq 2$. The inner power $C_n^{(k)}$ is Hamiltonian if and only if n is odd and $k = 2$.*

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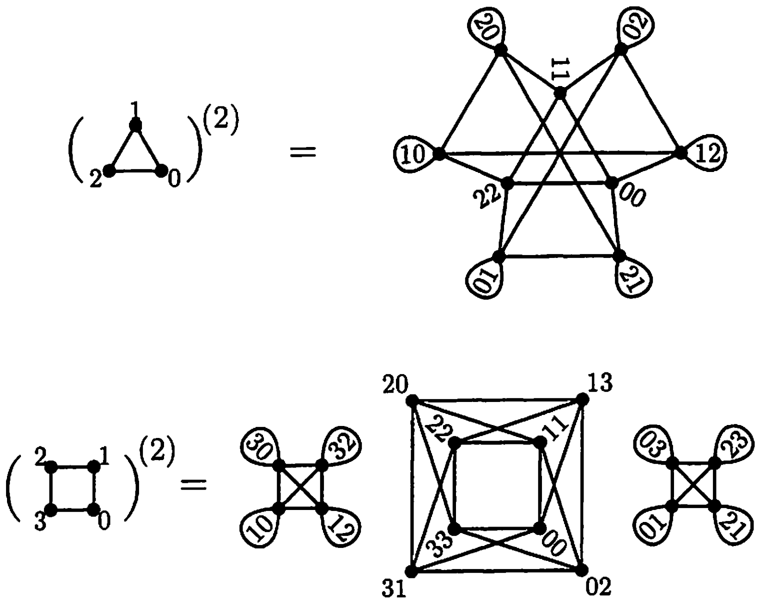


Fig.1 The inner powers $C_3^{(2)}$ and $C_4^{(2)}$.

2 Proof of necessity

We regard the vertices of C_n as the elements of Z_n, where $Z_n = \{[0], [1], \ldots, [n-1]\}$ denotes the set of equivalence class of Z under congruence modulo n. Two vertices $[i]$ and $[j]$ are adjacent in C_n if and only if $[i] - [j] = [\pm 1]$.

Lemma 1. *([2]) $G^{(2)}$ is connected if and only if G is connected and has an odd cycle.*

Lemma 2. *$C_n^{(k)}$ contains an isolated vertex for $n \geq 4$ and $k \geq 3$.*

Proof. Let $x = (x_0, x_1, \ldots, x_{k-1}) = ([0], [1], [1], \ldots, [1])$ and we show that x is an isolated vertex of $C_n^{(k)}$. Suppose to the contrary that there exists a vertex $y = (y_0, y_1, \ldots, y_{k-1})$ such that xy is an edge of $C_n^{(k)}$. It follows by the definition of the inner power that $x_0 y_1 = [0]y_1 \in E(C_n)$ and $x_2 y_1 = [1]y_1 \in E(C_n)$. This is a contradiction since two adjacent vertices $[0]$ and $[1]$ in $C_n (n \geq 4)$ have no common neighbors. □

Lemma 3. *$C_3^{(k)}$ contains a vertex incident with exactly one edge (possibly a loop) for $k \geq 3$.*

Proof. Let

$$x = (x_0, \ldots, x_{k-1}) = \begin{cases} [0],[1],[2],\ldots,[0],[1],[2], & \text{if } 3 \mid k, \\ [0],[1],[2],\ldots,[0],[1],[2],[0], & \text{if } 3 \mid (k-1), \\ [0],[1],[2],\ldots,[0],[1],[2],[2],[0], & \text{if } 3 \mid (k-2). \end{cases}$$

One can easily check that $x_{i-1} \neq x_{i+1}$ for $0 \le i < k$, where arithmetic on the indices is done modulo k. Let $y = (y_0, y_1, \ldots, y_{k-1})$ be an adjacent vertex of x. The definition of the inner power implies that y_i is a common neighbor of x_{i-1} and x_{i+1}, and hence is the unique vertex in C_3 other than x_{i-1} or x_{i+1} for each $i \in \{0, 1, \ldots, k-1\}$. □

Proof of necessity Let $n \ge 3$ and $k \ge 2$. We show that if n is even or $k \ge 3$ then $C_n^{(k)}$ is not Hamiltonian. First, if $k \ge 3$ then by Lemmas 2 and 3, the inner power $C_n^{(k)}$ contains either an isolated vertex (when $n \ge 4$) or a vertex incident with exactly one edge (when $n = 3$). Either case implies that $C_n^{(k)}$ is not Hamiltonian. Now assume $k = 2$ and n is even. By Lemma 1, we see that $C_n^{(k)} = C_n^{(2)}$ is not connected and hence is not Hamiltonian. The necessity part of Theorem 1 follows.

3 Proof of sufficiency

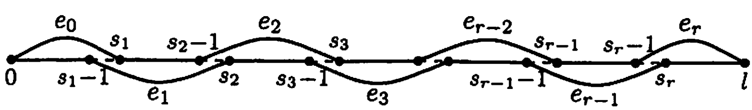
Throughout this section, we assume $n \ge 3$ is odd. We will construct a Hamiltonian path in $C_n^{(2)}$ and then show that it can be adjusted to form a Hamiltonian cycle by edge switching. We need the following lemma, a generalization of the classical Dirac/Ore switching argument.

Lemma 4. *Let G be a graph with a Hamiltonian path $v_0 \ldots v_l (l \ge 2)$. Let $r \ge 0$ and let $s_0 s_1 \ldots s_{r+1}$ be an increasing sequence from $0 = s_0$ to $l = s_{r+1}$. If* (i) *$v_{s_0} v_{s_1} \in E(G)$ and* (ii) *$v_{s_k - 1} v_{s_{k+1}} \in E(G)$ for $k = 1, \ldots, r$ then G is Hamiltonian.*

Proof. Let $e_0 = v_{s_0} v_{s_1}$ and $e_k = v_{s_k-1} v_{s_{k+1}}$ for $k = 1, \ldots, r$. We use the even-indexed edges e_i to build one v_0, v_l-path and the odd-indexed edges to build another v_0, v_l-path. When r is even, the two paths are formed by the following concatenations.

$$e_0, P(s_1, s_2 - 1), e_2, P(s_3, s_4 - 1), e_4, \ldots, P(s_{r-1}, s_r - 1), e_r$$

$$P(0, s_1 - 1), e_1, P(s_2, s_3 - 1), e_3, P(s_4, s_5 - 1), \ldots, e_{r-1}, P(s_r, l)$$



When r is odd, the path starting with e_0 reaches v_{s_r} and ends with $P(s_r, l)$, while the other path reaches v_{s_r-1} and ends with e_r. The union of the above two paths is a Hamiltonian cycle. □

We also need the following two simple lemmas. Let $\lfloor x \rfloor$ denote the largest integer not greater that x.

Lemma 5. *Let $n \geq 3$ be odd. Then $1 \leq \lfloor \frac{m+n}{2n} \rfloor \leq \frac{n-1}{2}$ for $n \leq m \leq n^2 - 1$.*

Proof. Adding n to each term of the inequality $n \leq m \leq n^2 - 1$, we have

$$2n \leq m + n \leq n^2 - 1 + n < n(n+1). \tag{1}$$

Dividing each term of (1) by $2n$, we have

$$1 \leq \frac{m+n}{2n} < \frac{n+1}{2}. \tag{2}$$

Note $(n+1)/2$ is an integer since n is odd. It follows from (2) that

$$1 \leq \lfloor \frac{m+n}{2n} \rfloor \leq \frac{n-1}{2}.$$

□

Lemma 6. *Let a, b, c be nonzero integers with a and b coprime. Then* (i) *$a \nmid c$ implies $a \nmid bc$, and* (ii) *$a \mid c$ and $b \mid c$ implies $ab \mid c$.*

Now we give a construction of a Hamiltonian path in $C_n^{(2)}$. Let

$$x^m = \begin{cases} (i_m, j_m), & \text{for } m \in \{0, 2, \ldots, n^2 - 1\}, \\ (j_m, i_m), & \text{for } m \in \{1, 3, \ldots, n^2 - 2\}, \end{cases} \tag{3}$$

where

$$i_m = \begin{cases} [m], & \text{for } 0 \leq m \leq n-1, \\ [-m-2], & \text{for } n \leq m \leq n^2 - 1, \end{cases} \tag{4}$$

and

$$j_m = \begin{cases} [m], & \text{for } 0 \leq m \leq n-1, \\ [2\lfloor \frac{m-n}{2n} \rfloor - m], & \text{for } n \leq m \leq n^2 - 1. \end{cases} \tag{5}$$

Lemma 7. *No element is repeated in the sequence $x^0 \ldots x^{n^2-1}$.*

Proof. Define a map $\varphi : \{0, 1, \ldots, n^2 - 1\} \mapsto Z_n$ as follows:

$$\varphi(m) = \begin{cases} j_m - i_m, & \text{if } m \text{ is even}, \\ i_m - j_m, & \text{if } m \text{ is odd}. \end{cases} \tag{6}$$

A direct calculation from (4) and (5) shows

$$\varphi(m) = \begin{cases} [0], & 0 \le m \le n-1, \\ [(-1)^m 2\lfloor \frac{m+n}{2n} \rfloor], & n \le m \le n^2-1. \end{cases} \quad (7)$$

Let m and m' be two different integers in $\{0, 1, \ldots, n^2-1\}$. We shall show $x^m \neq x^{m'}$. We may assume $m < m'$.

Case 1: $0 \le m \le n-1$ and $0 \le m' \le n-1$. By (4) and (5), $i_m = j_m = [m]$ and $i_{m'} = j_{m'} = [m']$. Hence, $x^m = ([m], [m])$ and $x^{m'} = ([m'], [m'])$ by (3). Since $0 < m' - m < n$, we see $[m] \neq [m']$ and hence $x^m \neq x^{m'}$, as desired.

Case 2: $0 \le m \le n-1$ and $n \le m' \le n^2-1$. It suffices to show $\varphi(m) \neq \varphi(m')$ since $\varphi(m) \neq \varphi(m')$ implies $x^m \neq x^{m'}$ by (3) and (6). Lemma 5 implies $n \nmid \lfloor \frac{m'+n}{2n} \rfloor$ and hence $n \nmid (-1)^m 2\lfloor \frac{m'+n}{2n} \rfloor$ by Lemma 6 (i). It follows from (7) that $\varphi(m') \neq [0]$ and $\varphi(m) = [0]$, and hence $\varphi(m) \neq \varphi(m')$, as desired.

Case 3: $n \le m \le n^2-1$ and $n \le m' \le n^2-1$.
Subcase 3.1: $m' - m$ is odd. As in Case 2, it suffices to show $\varphi(m) \neq \varphi(m')$. Lemma 5 implies

$$2 \le \lfloor \frac{m+n}{2n} \rfloor + \lfloor \frac{m'+n}{2n} \rfloor \le n-1,$$

and hence

$$n \nmid \lfloor \frac{m+n}{2n} \rfloor + \lfloor \frac{m'+n}{2n} \rfloor.$$

Using Lemma 6 (i), we have

$$n \nmid (-1)^m 2(\lfloor \frac{m+n}{2n} \rfloor + \lfloor \frac{m'+n}{2n} \rfloor). \quad (8)$$

On the other hand, by (7) and the assumption of this case,

$$\begin{aligned} \varphi(m) - \varphi(m') &= [(-1)^m 2\lfloor \frac{m+n}{2n} \rfloor - (-1)^{m'} 2\lfloor \frac{m'+n}{2n} \rfloor] \\ &= [(-1)^m 2(\lfloor \frac{m+n}{2n} \rfloor + \lfloor \frac{m'+n}{2n} \rfloor)]. \end{aligned}$$

Therefore, by (8), $\varphi(m) - \varphi(m') \neq [0]$, as desired.

Subcase 3.2: $m' - m$ is even. We assume that both m and m' are even (the case that both m and m' are odd can be settled in the same way). From this assumption and (3), we know $x^m = (i_m, j_m)$ and $x^{m'} = (i_{m'}, j_{m'})$. If

$i_m \neq i_{m'}$ then $x^m \neq x^{m'}$, completing the proof. Now assume $i_m = i_{m'}$, i.e., $[-m-2] = [-m'-2]$ by (4) and thus $n|(m'-m)$, yielding $2n|(m'-m)$ by Lemma 6 (ii). Let $m' = m + 2np$. It follows from the inequality $n \leq m < m' \leq n^2 - 1$ that $0 < 2p < n$ and hence $[2p] \neq [0]$. From (5), we have

$$\begin{aligned} j_{m'} &= [2\lfloor \frac{m'-n}{2n} \rfloor - m'] \\ &= [2\lfloor \frac{m-n}{2n} \rfloor + 2p - m] \\ &= j_m + [2p] \\ &\neq j_m, \end{aligned}$$

completing the proof. □

Lemma 8. *Any two consecutive elements in the sequence $x^0 \ldots x^{n^2-1}$ are adjacent.*

Proof. By definitions of the inner power and the sequence $x^0 \ldots x^{n^2-1}$, we need to show that both $i_0 \ldots i_{n^2-1}$ and $j_0 \ldots j_{n^2-1}$ are walks in C_n, that is any two consecutive elements in either sequence are adjacent.

We first show that $i_0 \ldots i_{n^2-1}$ is a walk. From (4), we see $i_n - i_{n-1} = [-n-2] - [n-1] = [-1]$, implying i_{n-1} is adjacent to i_n. It is clear that consecutive elements among $i_0 \ldots i_{n-1}$ or among $i_n \ldots i_{n^2-1}$ are adjacent in C_n.

Now consider the sequence $j_0 \ldots j_{n^2-1}$. It is clear that consecutive elements among $j_0 \ldots j_{n-1}$ are adjacent in C_n. Since $j_n - j_{n-1} = [-n] - [n-1] = [1]$ by (5), we see that j_{n-1} is adjacent to j_n. Let m be arbitrary with $n \leq m < n^2 - 1$. If $\lfloor \frac{m+1-n}{2n} \rfloor = 1 + \lfloor \frac{m-n}{2n} \rfloor$, then $j_{m+1} - j_m = [1]$, otherwise $\lfloor \frac{m+1-n}{2n} \rfloor = \lfloor \frac{m-n}{2n} \rfloor$ and hence $j_{m+1} - j_m = [-1]$. Either case implies that j_m is adjacent to j_{m+1}. It follows from the arbitrariness of m that consecutive elements among $j_n \ldots j_{n^2-1}$ are adjacent in C_n. Lemma 8 follows. □

Lemma 9. *Let $r = (n-3)/2$, $s_0 = 0$, and $s_k = (2k+1)n - 1$ for $k = 1, \ldots, r+1$. Then* (i) *$x^{s_0}x^{s_1} \in E(C_n^{(2)})$ and* (ii) *$x^{s_k - 1}x^{s_{k+1}} \in E(C_n^{(2)})$ for $k = 1, \ldots, r$.*

Proof. From (3)-(5), we know $x^{s_0} = x^0 = (i_0, j_0) = ([0],[0])$. Since $s_1 = 3n - 1$ and $n \geq 3$ is odd, we see that s_1 is even and $s_1 \geq n$. Hence,

$$\begin{aligned} x^{s_1} &= x^{3n-1} \\ &= (i_{3n-1}, j_{3n-1}) \\ &= ([-(3n-1)-2], [2\lfloor \frac{3n-1-n}{2n} \rfloor - (3n-1)]) \\ &= ([-1],[1]). \end{aligned}$$

Hence, $x^{s_0}x^{s_1} \in E(C_n^{(2)})$.

If $r = 0$ we are done. Now assume $r \geq 1$. Let $k \in \{1, \ldots, r\}$. Since $s_k - 1 = (2k+1)n - 2 \geq n$, from (4) and (5) we have

$$i_{s_k-1} = i_{(2k+1)n-2} = [-((2k+1)n-2)-2] = [0],$$

and

$$\begin{aligned} j_{s_k-1} &= j_{(2k+1)n-2} \\ &= [2\lfloor\frac{(2k+1)n-2-n}{2n}\rfloor - ((2k+1)n-2)] \\ &= [2\lfloor\frac{kn-1}{n}\rfloor + 2] \\ &= [2k]. \end{aligned}$$

Noting $s_k - 1 = (2k+1)n - 2$ is odd, we see $x^{s_k-1} = (j_{s_k-1}, i_{s_k-1}) = ([2k],[0])$. By a similar argument we can show $x^{s_{k+1}} = ([-1],[2k+1])$. Therefore, $x^{s_k-1}x^{s_{k+1}} \in E(C_n^{(2)})$. Lemma 9 follows. □

Proof of sufficiency By Lemmas 7 and 8, we see that $x^0 \ldots x^{n^2-1}$ is a Hamiltonian path in $C_n^{(2)}$. Let $r = (n-3)/2$, $s_0 = 0$, and $s_k = (2k+1)n-1$ for $k = 1, \ldots, r+1$ as in Lemma 9. Let $l = n^2 - 1$. It is clear that $s_0 s_1 \ldots s_{r+1}$ is an increasing sequence with $s_0 = 0$ and $s_{r+1} = n^2 - 1 = l$. From Lemma 9, we see that the graph $C_n^{(2)}$ satisfies the condition of Lemma 4, implying $C_n^{(2)}$ is Hamiltonian.

4 A conjecture

By Γ, Γ_0 we denote, respectively, the set of finite simple graphs and the set of finite graphs in which loops are admitted. In [1], Gravier obtained a necessary and sufficient condition for the Hamiltonicity of the direct product of two Hamiltonian graphs in Γ. Let $\mathcal{H}$ be the set of graphs G such that G is Hamiltonian and if $|V(G)|$ is even then there exists a Hamiltonian cycle C_n of G with two chords $[r][s]$ and $[u][v]$ where r and s are even, and u and v are odd.

Theorem 2. *([1]) Let $G_1, G_2 \in \Gamma$ be two Hamiltonian graphs. The graph $G_1 \times G_2$ is Hamiltonian if and only if either G_1 or G_2 belongs to $\mathcal{H}$.*

We define $\mathcal{H}_0$ by admitting both $[r][s]$ and $[u][v]$ with identical endpoints in the definition of $\mathcal{H}$. We end this paper by putting forward the following conjecture.

Conjecture 1. *Let $G \in \Gamma_0$ be a Hamiltonian graph. The inner power $G^{(2)}$ is Hamiltonian if and only if G belongs to $\mathcal{H}_0$.*

Remark 1. *Let G_i $(1 \leq i \leq 4)$ be Hamiltonian graphs with four vertices, as illustrated in Fig. 2. With the aid of GraphTheory package in Maple, we find that $G_i^{(2)}$ is Hamiltonian for $i = 1, 2, 3$ while $G_4^{(2)}$ is not Hamiltonian.*

Note $G_1, G_2, G_3 \in \mathcal{H}_0$ and $G_4 \notin \mathcal{H}_0$. Moreover, for any Hamiltonian graph G with four vertices, $G \in \mathcal{H}_0$ implies G contains a span subgraph isomorphic to G_i for some $i \in \{1, 2, 3\}$, while $G \notin \mathcal{H}_0$ implies G is isomorphic to some span subgraph of G_4. This proves Conjecture 1 is true when $|V(G)| = 4$.

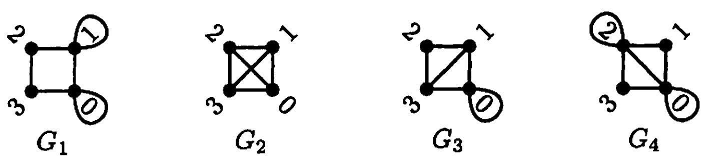


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Fig.2 Four graphs and Hamiltonicity of their second inner powers.

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