# The Randić index and girth of triangle-free graphs*

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**Abstract**

The Randić index $R(G)$ of a graph $G$ is defined by $R(G) = \sum_{uv} \frac{1}{\sqrt{d(u)d(v)}}$, where $d(u)$ is the degree of a vertex $u$ in $G$ and the summation extends over all edges $uv$ of $G$. In this work, we give sharp lower bounds of $R(G) + g(G)$ and $R(G) \cdot g(G)$ among $n$-vertex connected triangle-free graphs with the Randić index $R$ and girth $g$.

**Key words:** Randić index; girth; triangle-free; bound

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# 1 Introduction

All the graphs considered in this paper are simple undirected ones. The *girth* of a graph is the minimum length of its cycles. For convenience, girth of a graph which contains no cycle is set to infinity. A *leaf* is a vertex of degree one. For undefined terminology and notations we refer the reader to the book [3] of Bondy and Murty .

The Randić index $R = R(G)$ of a graph $G$ is defined as follows:

$$R = R(G) = \sum_{u,v} \frac{1}{\sqrt{d(u)d(v)}}, \tag{1.1}$$

where $d(u)$ denotes the degree of a vertex $u$ and the summation runs over all edges $uv$ of $G$. It is also known as connectivity index or branching index. Randić [13] in 1975 proposed this index for measuring the extent of branching of the carbon-atom skeleton of saturated hydrocarbons. There is also a good correlation between the Randić index and several physico-chemical properties of alkanes: boiling points, surface areas, energy levels, etc. In 1998 Bollobás and Erdős [2] generalized this index by replacing the square-root by power of any real number, which is called the general Randić index. For a comprehensive survey of its mathematical properties, see the book of Li and Gutman [9], or a survey of Li and Shi [11]. See also the books of Kier and Hall [6, 7] for chemical properties of this index.

There are many results concerning the relations between the Randić index and other graph invariants such as: diameter, radius, average distance, girth, chromatic number, see [8].

Regarding the girth, Aouchiche et al. [1] showed:

**Theorem 1.1** *For any connected graph $G$ on $n \geq 3$ vertices with Randić* *index R and girth g,*

$$R+g \leq \frac{3n}{2},\ R\cdot g \leq \frac{n^2}{2},\ R-g \geq -\frac{n}{2},\ \frac{R}{g} \geq \frac{1}{2},$$

*with equalities if and only if G is $C_n$.*

$$R-g \leq \frac{n}{2}-3,\ \frac{R}{g} \leq \frac{n}{6},$$

*with equalities if and only if G is a regular graph with a triangle.*

They also conjectured that:

**Conjecture 1.2** *[1] For any connected graph on $n \geq 3$ vertices with the Randić index R and girth g,*

$$R+g \geq \frac{n-3+\sqrt{2}}{\sqrt{n-1}}+\frac{7}{2} \quad and \quad R\cdot g \geq \frac{3n-9+3\sqrt{2}}{\sqrt{n-1}}+\frac{3}{2}$$

*with equalities if and only if $G \cong S_n^+$, the graph obtained by adding an edge in an n−vertex star $S_n$.*

Liu, Zhu and Cai [12] showed that the above conjecture is true for unicyclic graphs. Wang, Zhu and Liu [14] showed it is true for bicyclic graphs. This conjecture is proved to be true in general by Li and Liu [10]. Note that the girth of the extremal graph in Conjecture 1.2 is 3. As a continue work, we give the sharp lower bounds of $R+g$ and $R\cdot g$ for triangle-free graphs. The rest of the paper is organized as follows. In Section 2, we give some Lemmas which will be used in the proof of our main result. In Section 3, we give our main result which gives the sharp lower bounds of $R+g$ and $R\cdot g$ for triangle-free graphs.

## 2 Some Lemmas

This section gives some lemmas which will be used in the sequel.

**Lemma 2.1** *Let* $f(d,\ell) := \frac{1}{\sqrt{\ell d}} - \frac{1}{\sqrt{\ell+d-1}} + (d-1)(\frac{1}{\sqrt{d}} - \frac{1}{\sqrt{d+\ell-1}})$. *Then* $f(d,\ell) \geq 0$, *where* $d, \ell$ *are integers and* $d \geq 2, \ell \geq 2$.

**Proof.** Since $\ell d > \ell + d - 1$, we have $\frac{\partial f(d,\ell)}{\partial \ell} = -\frac{d}{2(\ell d)^{3/2}} + \frac{d}{2(d+\ell-1)^{3/2}} > 0$. We want $\frac{\partial f(d,2)}{\partial d} < 0$, that is, $\frac{\partial f(d,2)}{\partial d} = -\frac{1}{2\sqrt{2}d^{3/2}} - \frac{-1+d}{2d^{3/2}} + \frac{1}{\sqrt{d}} + \frac{d}{2(1+d)^{3/2}} - \frac{1}{\sqrt{1+d}} = \frac{1}{2}\left(\frac{d+1-\frac{1}{\sqrt{2}}}{d^{3/2}} - \frac{d+2}{(1+d)^{3/2}}\right) < 0$ or $(d+1-\frac{1}{\sqrt{2}})(d+1)^{3/2} < d^{3/2}(d+2)$, that is to say $(\frac{d+1}{d})^3 < (\frac{d+2}{d+1-1/\sqrt{2}})^2$ or $(1+\frac{1}{d})^3 < (1+\frac{1+1/\sqrt{2}}{d+1-1/\sqrt{2}})^2$. To prove this it suffices to prove $(1+\frac{1}{d})^3 < (1+\frac{1+1/\sqrt{2}}{d})^2$. But, if $d \geq 5$ we have $\frac{1+1/\sqrt{2}}{d+1-1/\sqrt{2}} > \frac{1.6}{d}$. Thus it suffices to prove $1 + \frac{3}{d} + \frac{3}{d^2} + \frac{1}{d^3} < 1 + \frac{3.2}{d} + \frac{2.56}{d^2}$, that is to say $3.2d^2 + 2.56d > 3d^2 + 3d + 1$ or $0.2d^2 - 0.44d - 1 > 0$. Since the positive root of this polynomial is less than 4, we have proven the inequality. We have thus proven that $f(d,\ell) \geq f(d,2) \geq f(+\infty, 2) = 0$ for $d \geq 5$. By direct calculation we know that the result is also valid for $d = 2, 3, 4$. ∎

In 2002, Delorme et al. [4] gave a result of the minimum Randić index among all $n-$vertex connected graphs with the minimum degree at least two.

**Lemma 2.2** *[4]. For any connected graph G of order n with minimum degree* $\delta \geq 2$, *we have*

$$R(G) \geq \frac{2n-4}{\sqrt{2n-2}} + \frac{1}{n-1}$$

*and the bound is tight if and only if* $G \cong K^{\star}_{2,n-2}$ *which arises from complete bipartite graph* $K_{2,n-2}$ *by joining the vertices in the partite set with two vertices by a new edge.*

In 1993, Favaron et al. gave the following result:

**Lemma 2.3** *[5]. For any triangle-free graph G with size m, we have*

$$R(G) \geq \sqrt{m}.$$

# 3 Main results

Now, we come to our main result. Denote by $Y_n$ the graph obtained by linking two edges from an isolated vertex to two leaves of an $(n-1)$-vertex star $S_{n-1}$, see Figure 3.1.

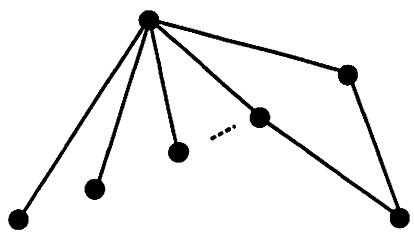

Figure 3.1 The graph $Y_n$

**Theorem 3.1** *For any connected triangle-free graph $G$ on $n \geq 4$ vertices with the Randić index $R$ and girth $g$,*

$$R + g \geq \frac{n - 4 + \sqrt{2}}{\sqrt{n-2}} + 5 \quad and \quad R \cdot g \geq \frac{4n - 16 + 4\sqrt{2}}{\sqrt{n-2}} + 4$$

*with equalities if and only if $G \cong Y_n$.*

**Proof.** By definition of the girth, the results are true for trees. Thus, we only has to consider connected graphs with $g \geq 4$. For $n = 4$ and $n = 5$, it is easy to check that the result is true. Therefore, we only consider $n \geq 6$ in the following. We divide the proof into three cases.

Case A. If $g(G) \geq 5$.

Since the size of $G$ is at least $n$, we have $R + g \geq \sqrt{n} + 5$ and $R \cdot g > 5\sqrt{n}$

by Lemma 2.3. Thus, we have $\sqrt{n}+5 > \sqrt{n-2}+5 > \frac{n-4+\sqrt{2}}{\sqrt{n-2}}+5$ and $5\sqrt{n} > 4\sqrt{n-2}+4 > 4\cdot\frac{n-4+\sqrt{2}}{\sqrt{n-2}}+4$.

Now, it is sufficient to show that $R(G) \geq \frac{n-4+\sqrt{2}}{\sqrt{n-2}}+1$ since $g \geq 4$.

**Case** B. If $\delta(G) \geq 2$.

We have $R(G) \geq \frac{2n-4}{\sqrt{2n-2}}+\frac{1}{n-1}$ by Lemma 2.2. Then

$$\begin{aligned}
&\frac{2n-4}{\sqrt{2n-2}}+\frac{1}{n-1}-(\frac{n-4+\sqrt{2}}{\sqrt{n-2}}+1)\\
&=\sqrt{2}\sqrt{n-1}-\frac{\sqrt{2}}{\sqrt{n-1}}+\frac{1}{n-1}-\sqrt{n-2}+\frac{2-\sqrt{2}}{\sqrt{n-2}}-1\\
&>(\sqrt{2}-1)\sqrt{n-1}+\frac{2-2\sqrt{2}}{\sqrt{n-2}}-1\\
&>0,
\end{aligned}$$

for $n \geq 11$. And it is easy to check that $\frac{2n-4}{\sqrt{2n-2}}+\frac{1}{n-1} > \frac{n-4+\sqrt{2}}{\sqrt{n-2}}+1$ holds for $6 \leq n \leq 10$. Thus, $R(G) \geq \frac{n-4+\sqrt{2}}{\sqrt{n-2}}+1$.

**Case** C. If $g(G)=4$ and $\delta(G)=1$.

We apply induction on $n(\geq 6)$. We assume $R \geq \frac{k-4+\sqrt{2}}{\sqrt{k-2}}+1$ is true for every graph of order $k$, which $k < n$. Let $G$ be the graph of order $n$ with minimum Randić index in this case. Set $V_1=\{u \in V(G)|d(u)=1\}$ and let $u \in V_1$ and $uv \in E(G)$. Then $d(v) \geq 2$. Denote $d(v)=d$ and $N(v)=\{u,u_1,u_2,\cdots,u_{d-1}\}$. Note that $d \leq \Delta(G) \leq n-2$ since $g \geq 4$. Let $N_2=\{u_i|d(u_i) \geq 2, u_i \in N(v)\}$. Then we have $|N_2| \geq 1$ since $G$ is a connected graph with cycle(s). Let $G'=G-\{u\}$. Then $G'$ is connected and $g(G')=4$ and $R(G') \geq \frac{n-5+\sqrt{2}}{\sqrt{n-3}}+1$ by induction. We also have

$$R(G)-R(G')=\frac{1}{\sqrt{d}}+\sum_{i=1}^{d-1}\frac{1}{\sqrt{d(u_i)}}(\frac{1}{\sqrt{d}}-\frac{1}{\sqrt{d-1}}) \tag{3.2}$$

Now, we have the following claim:

**Claim.** $|N_2| \geq 2$.

For otherwise, assume $|N_2| = 1$ and $w \in N_2$. Denote $d(w) = \ell$ and $N(w) = \{w_1, w_2, \cdots, w_{\ell-1}, v\}$. Let $G'' = G - \{ww_1, \cdots, ww_{\ell-1}\} + \{vw_1, \cdots, vw_{\ell-1}\}$ (see Figure 3.2). Then $G''$ is a graph of order $n$ which satisfies the conditions of Case C.

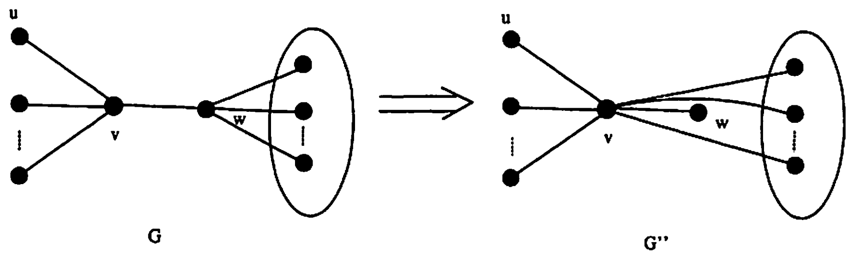


Figure 3.2 Graphs $G$ and $G''$

By Lemma 2.1,

$$\begin{aligned} R(G) - R(G'') &= \frac{1}{\sqrt{\ell d}} - \frac{1}{\sqrt{\ell + d - 1}} + (d-1)\left(\frac{1}{\sqrt{d}} - \frac{1}{\sqrt{d + \ell - 1}}\right) \\ &+ \sum_{i=1}^{\ell-1} \frac{1}{\sqrt{d(w_i)}}\left(\frac{1}{\sqrt{\ell}} - \frac{1}{\sqrt{d + \ell - 1}}\right) \\ &> \frac{1}{\sqrt{\ell d}} - \frac{1}{\sqrt{\ell + d - 1}} + (d-1)\left(\frac{1}{\sqrt{d}} - \frac{1}{\sqrt{d + \ell - 1}}\right) \geq 0, \end{aligned}$$

contradicting to the assumption that $G$ is the graph with minimum Randić index of order $n$ in this case.

Therefore, $|N_2| \geq 2$. Noticing that $\frac{1}{\sqrt{d}} - \frac{1}{\sqrt{d-1}}$ is negative, the latter expression of (3.2) is minimum when $d(u_i), i = 1, \cdots, d-1$ is as small as possible, i.e., $d(u_i) = 1, i = 1, \cdots, d-3$ and, without loss of generality,

$d(u_{d-2}) = d(u_{d-1}) = 2$. We then have

$$
\begin{aligned}
R(G) &\geq R(G') + \frac{1}{\sqrt{d}} + 2 \times \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{d}} - \frac{1}{\sqrt{d-1}}) + (d-3) \times \frac{1}{\sqrt{1}}(\frac{1}{\sqrt{d}} - \frac{1}{\sqrt{d-1}}) \\
&= R(G') + \sqrt{d} - \sqrt{d-1} + (2-\sqrt{2})(\frac{1}{\sqrt{d-1}} - \frac{1}{\sqrt{d}}) \\
&\geq \frac{n-5+\sqrt{2}}{\sqrt{n-3}} + 1 + \sqrt{n-2} - \sqrt{n-3} + (2-\sqrt{2})(\frac{1}{\sqrt{n-3}} - \frac{1}{\sqrt{n-2}}) \\
&= \frac{n-4+\sqrt{2}}{\sqrt{n-2}} + 1,
\end{aligned}
$$

where the equalities hold if and only if $d(v) = d = n-2, d(u) = d(u_1) = d(u_2) = \cdots = d(u_{n-5}) = 1, d(u_{n-4}) = d(u_{n-3}) = 2$, i.e., $G \cong Y_n$, which complete the proof. ∎

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