The E_2-Hamiltonian Property of Circuit Graphs of Matroids *

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Abstract

Let G be the circuit graph of any connected matroid. It is proved that the circuit graph of a connected matroid with at least three circuits is E_2-Hamiltonian.

Keywords: Matroid; Circuit graph of matroid; E_2-Hamiltonian

1 Introduction

Let E be a finite set of elements. A collection C of subsets of a finite set E is the set of circuits of a matroid M on E if and only if the following conditions (to be called circuit axioms) are satisfied.
(C1) If C_1 and C_2 are distinct circuits, then $C_1 \not\subseteq C_2$.
(C2) If C_1 and C_2 are circuits and $a \in C_1 \cap C_2$, then there exists a circuit $C_3 \in C$ such that $C_3 \subseteq (C_1 \cup C_2) - a$.

Then $M = (E, C)$ is called a *matroid* on E. We refer to the members of C as *circuits* of matroid M. The family of circuits of M determines a matroid.

A graph is *Hamiltonian* if it contains a Hamilton cycle. We now call a graph G *positively Hamilton*, or *edge-Hamiltonian*, written $G \in H^+$, if every edge of G is in some Hamilton cycle; on the other hand, G is *negatively Hamilton*, written $G \in H^-$, if for each edge of G there is a Hamilton cycle avoiding it. When $G \in H^+$ and $G \in H^-$, we say that G is *uniformly Hamilton*. A graph G is called *E_2-Hamiltonian* if every two edges of G are contained in a Hamilton cycle of G. A graph G with n vertices is called *edge-pancyclic* if for any integer k, $3 \le k \le n$, and any edge of G, there is a cycle of length k containing the edge. Some other definitions and notations about matroid theory can be found in [10].

A subset of E that does not contain any circuit is called an *independent set* of M. A maximal independent set is called a *base* of M, denoted by $B(M)$. Harary et al. considered the properties of tree graphs [3, 2]. Maurer discussed

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the relationship of bases of matroids and graphs and defined the base graph of matroid [8, 9].

The base graph of a matroid $M = (E, \mathcal{B})$ is a graph G such that $V(G) = \mathcal{B}$ and $E(G) = \{BB' | B, B' \in \mathcal{B}, |B \cap B'| = 1\}$ where the same notation is used for the vertices of G and the bases of M.

In 1989, Alspach and Liu studied the properties of paths and circuits in base graphs of matroids [1] and obtained the following results.

Theorem 1. [1] *Let G be the base graph of a simple matroid M. Then G is Hamilton-connected.*

Theorem 2. [1] *The base graph of a simple matroid is edge-pancyclic.*

Later, Liu and Li studied the E_2-Hamiltonian property of matroid base graphs [4].

Theorem 3. [4] *Let $M = (E, \mathcal{B})$ be a matroid and $G = G(M)$ be the base graph of matroid M. If $M = (E, \mathcal{B})$ has at least three bases and there is no minor N of $M = (E, \mathcal{B})$ such that the base graph $G(N)$ of matroid N is isomorphic to W_5, then $G = G(M)$ is E_2-Hamiltonian.*

Now we give the concept of matroid circuit graph. Let G be a graph. The notation $V(G)$ and $E(G)$ will be used for the vertex-set and edge-set of G, respectively. The *circuit graph of a matroid M* is a graph $G = G(M)$ with vertex set $V(G)$ and edge set $E(G)$ such that $V(G) = \mathcal{C}$ and $E(G) = \{CC' | C, C' \in \mathcal{C}, |C \cap C'| \neq 0\}$.

In 2007, Li and Liu discussed the Hamilton property on the circuit graph of matroids, and give the following two theorems.

Theorem 4. [5] *For any connected matroid $M = (E, \mathcal{C})$ which has at least three circuits, the circuit graph $G = G(M)$ is edge-pancyclic.*

Theorem 5. [6] *For any connected matroid $M = (E, \mathcal{C})$ which has at least four circuits, the circuit graph $G = G(M)$ is uniformly Hamilton.*

In this paper, we will prove that if G is a circuit graph of a connected matroid, then G is E_2-Hamiltonian if it contains at least three circuits.

2 Preliminaries

In the following we assume that every matroid has at least one circuit. Let $M = (E, \mathcal{C})$ be a matroid. If $X \subseteq E$, then the matroid on $E - X$ whose circuits are those of M which are contained in $E - X$ is called the *restriction* of M to $E - X$ (or the matroid obtained by deleting X from M) and is denoted by $M \setminus X$ or $M|(E - X)$. There is another derived matroid of importance. If $X \subseteq E$, then the family of minimal non-empty intersections of $E-X$ with circuits of M is the family of circuits of a matroid on $E-X$ called the *contraction* of M to $E-X$ (or the matoird obtained by contracting X from M) and is denoted by M/X. If $X=\{e\}$, we use $M \setminus e$ and M/e to denote the matroid obtained from M by deleting and contracting e, respectively. A matroid obtained from M by limited times of contractions and limited times of deletions is called a *minor* of M.

A sebset S of E is called a *separator* of M if every circuit of M is either contained in S or $E-S$. If $\emptyset$ and E are the only separators of M, then M is said to be *connected.* Let $M=(E,\mathcal{C})$ be a connected matroid. An element e of E is called an *essential* element if $M \setminus e$ is disconnected. Otherwise it is called an *inessential* element. A connected matroid each of whose elements is essential is called a *critically connected* matroid or simply a critical matroid. We now state some lemmas which are used in the proof in next section.

Lemma 1. [10] *If $a \in C_1 \cap C_2$ and $b \in C_1 - C_2$ where C_1, $C_2 \in \mathcal{C}$, then there exists a circuit $C_3 \in \mathcal{C}$ such that $b \in C_3 \subseteq (C_1 \cup C_2) - \{a\}$.*

Lemma 2. [10] *If M is a connected matroid, then for every $e \in E$, either M/e or $M \setminus e$ is also connected.*

Let M be a matroid and let $\mathcal{B}$ denote the family of bases of M. Let $\mathcal{B}^*$ denote the family of complements of members of $\mathcal{B}$ in E. Then $\mathcal{B}^*$ is the family of bases of a matroid, denoted by M^*, called the dual of M. The circuits of M^* are called the *co-circuits* of M.

Lemma 3. [6] *A critical matroid of $rank \geq 2$ contains a co-circuit of cardinality two.*

Lemma 4. [5] *Let M be any matroid on E and $x \in E$. If G and G_1 are circuit graphs of M and $M \setminus x$, respectively, then G_1 is a subgraph of G induced by V_1 where $V_1 = \{C | C \in \mathcal{C}, x \notin C\}$.*

Obviously the subgraph G_2 of G induced by $V_2 = V - V_1$ is a complete graph. G_1 and G_2 are induced subgraphs of G and $V(G_1)$ and $V(G_2)$ partition $V(G)$.

Lemma 5. [5] *For any matroid $M=(E,\mathcal{C})$ which has a 2-cocircuit $\{a,b\}$, the circuit graph of M is isomorphic to that of M/a.*

Lemma 6. [5] *Suppose that $M=(E,\mathcal{C})$ is a connected matroid with an element x such that the matroid $M \setminus x$ is connected and $G=G(M)$ is the circuit graph of matroid M. Let $G_1 = G(M \setminus x)$ be the circuit graph of $M \setminus x$ and G_2 be the subgraph of G induced by V_2 where $V_2 = \{C | C \in \mathcal{C}, x \in C\}$. If the matroid $M \setminus x$ has more than one circuit, then for any edge $C_1C_2 \in E(G)$, there exist a 4-cycle $C_1C_2C_3C_4$ in graph G such that one edge of the 4-cycle belongs to $E(G_1)$ and one belongs to $E(G_2)$ and C_1, C_2 are both adjacent to C_3.*

Lemma 7. *For any connected matroid $M = (E, \mathcal{C})$ which has at least three circuits, and $G = G(M)$ is the circuit graph of M, then for any vertex $C_1 \in G$, there is a subgraph K_3 including it.*

Proof. For any vertex $C_1 \in G$, and any two adjacent vertices C_2 and C_3, $e \in C_1 \cap C_2$, $e' \in C_1 \cap C_3$. If C_2 and C_3 not adjacent, by Lemma 1, there is a circuit C_4 such that $e' \in C_4 \subseteq (C_1 \cup C_2) - \{e\}$. Obviously, C_1, C_3 and C_4 form the complete graph K_3 as a subgraph of G.

□

Lemma 8. [7] *Let $G = G(M)$ be the circuit graph of a connected matroid $M = (E, \mathcal{C})$. If $|V(G)| = n$ and C_1, $C_2 \in V(G)$, then there is a path of length k joining C_1 and C_2 for any k satisfying* $2 \leq k \leq n - 1$.

3 Main Results

Let $E_1 = E(V(G_1))$, $E_2 = E(V(G_2))$ and let $E_3 = E(V(G_1), V(G_2)) = \{CC' | C \in V(G_1), C' \in V(G_2)\}$ be the edge set between $V(G_1)$ and $V(G_2)$. Let $U_{m,n} = (E, \mathcal{C})$ be a matroid such that $E = \{1, 2, \ldots, n\}$ and $\mathcal{C} = \{C | C \subseteq E, |C| = m + 1\}$. Now we give our main result as follows.

Theorem 6. *For any connected matroid $M = (E, \mathcal{C})$ which has at least three circuits, let $G = G(M)$ be the circuit graph of M, then G is E_2-Hamiltonian.*

Proof. We will prove this theorem by induction on $|E|$. When $|E| = 3$, the circuit graph of a connected matroid is trivial or the complete graph K_3, obviously true for the theorem. Suppose the theorem is true for $|E| = n - 1$. We now prove that the result is also true for $|E| = n > 3$. Let $e = C_1C_2$ and $e' = C_1'C_2'$ be any two edges in G. There are two cases to distinguish.

Case 1. There is an element x in M such that $M \backslash x$ is connected. Let G_1 and G_2 be the graphs defined as above. We assume that $|V(G_1)| = n_1$ and $|V(G_2)| = n_2$. If $n_1 = 1$, clearly, the circuit graph of this matroid is a complete graph $K_{n_1+n_2}$, so the theorem is true. If $n_1 \geq 2$, by Lemma 1, there are at least three vertices in G_1, also, there are at least three vertices in G_2. Then there are six subcases to distinguish.

Subcase 1.1. Edges e and e' in E_1.

By the induction hypothesis, there is a Hamilton cycle D in G_1 that contains e and e'. Because $n_1 \geq 3$, we can find an edge $e'' = C_1''C_2''$ in D such that $e'' \neq e$, e'. By Lemma 6, there is a 4-cycle $C_1''C_2''C_3C_4$ in G such that $C_3C_4 \in E_2$. Note that G_2 is a complete graph, then we can find a Hamilton path P connecting C_3 and C_4. Then $D - C_1''C_2'' + C_1''C_4 + P + C_3C_2''$ is a Hamilton cycle in G which contains both e and e'.

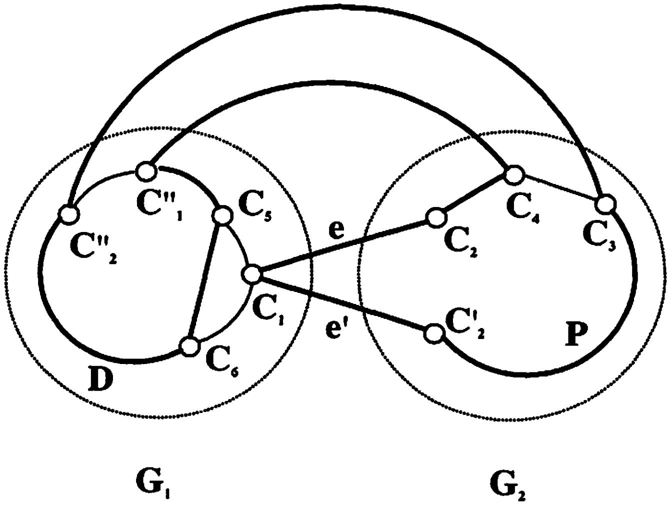

Fig.1 Edge $e, e' \in E_3, C_1 = C_1', C_2 \neq C_2'$.

Subcase 1.2. Edges e and e' in E_2.
In this case the proof is similar to that in Subcase 1.1.

Subcase 1.3. Edges e and e' in E_3.
In this case, without loss of generality, we let $C_1, C_1' \in G_1$ and $C_2, C_2' \in G_2$.
If $C_1 = C_1'$, $C_2 \neq C_2'$. By Lemma 7, we can find two vertices C_5 and C_6 which are adjacent to C_1 and $C_5C_6 \in E_1$. By the induction hypothesis, there is a Hamilton cycle D that contains C_1C_5 and C_1C_6. By Lemma 1, we can choose an edge $C_1''C_2''$ from D such that C_2'' connects a vertex C_3 in G_2 and $C_3 \neq C_2, C_2'$. Thus, we can get a 4-cycle $C_1''C_2''C_3C_4$ such $C_3C_4 \in E_2$. Since G_2 is a complete graph, there is a Hamilton path P connecting C_2 and C_2' and includes C_3C_4. Then $D - C_1C_5 - C_1C_6 + C_5C_6 - C_1''C_2'' + C_2''C_3 + C_1''C_4 + P - C_3C_4 + C_1C_2 + C_1C_2'$ is a Hamilton cycle including e and e'. (See Fig.1)
If $C_1 \neq C_1'$. If $C_2 \neq C_2'$. By Lemma 8, G_1 has a Hamilton path P_1 connecting C_1 and C_1'. Since G_2 is a complete graph, there is a Hamilton path P_2 connecting C_2 and C_2'. Thus $P_1 + C_1C_2 + C_1'C_2' + P_2$ is the Hamilton cycle we want. If $C_2 = C_2'$. By Lemma 8, G_1 has a Hamilton path P_1 connecting C_1 and C_1'. By Lemma 1, for any edge $C_1''C_2'' \in P_1$, we can find a 4-cycle $C_1''C_2''C_3C_4$ such that $C_3, C_4 \neq C_2$. Because G_2 is a complete graph, $G_2 - C_2$ still a complete graph, so we can find a Hamilton path P_2 in $G_2 - C_2$ connecting C_3 and C_4. Then $P_1 - C_1''C_2'' + C_1C_2 + C_1'C_2 + C_1''C_4 + C_2''C_3 + P_2$ is a Hamilton cycle in G that contains e and e'. (See Fig.2)

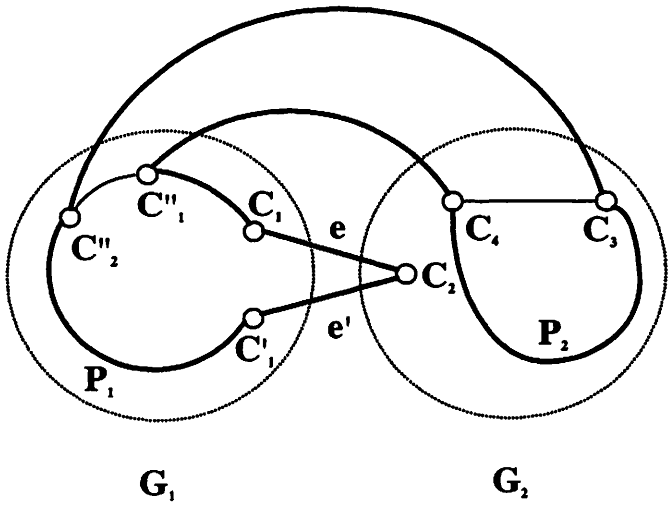

Fig.2 Edge e, $e' \in E_3$, $C_1 \neq C_1'$, $C_2 = C_2'$.

Subcase 1.4. Edge $e \in E_1$ and $e' \in E_2$.

By the induction hypothesis, there is a Hamilton cycle D in G_1 that contains e, and we can find an edge $e'' = C_1''C_2''$ in D such that $e'' \neq e$. From Lemma 6, we can find a 4-cycle $C_1''C_2''C_3C_4$, such that $C_3C_4 \in E_2$ and $C_3C_4 \neq e'$. Because G_2 is a complete graph, we can find a Hamilton path P connecting C_3 and C_4 and including e'. Then the cycle $D - C_1''C_2'' + C_1''C_4 + P + C_3C_2''$ is Hamilton and containing e and e'.

Subcase 1.5. Edge $e \in E_1$ and edge $e' \in E_3$.

Without loss of generality, we let $C_1' \in G_1$, $C_2' \in G_2$. Because $n_1 \geq 3$, we can find an edge $e'' = C_1'C_2''$ in G_1 that adjacent to C_1' and $e'' \neq e$. By induction hypothesis, there is a Hamilton cycle D containing e and e''. By Lemma 1, we can find a 4-cycle $C_1'C_2''C_3C_4$ in G such that $C_3C_4 \in E_2$. Since G_2 is a complete graph, so there is a Hamilton path P connecting C_3 and C_4. Thus the Hamilton cycle in G we want is $D - C_1'C_2'' + C_2''C_3 + P + C_4C_1'$.

Subcase 1.6. Edge $e \in E_2$ and edge $e' \in E_3$.

In this case the proof is similar to that in Subcase 1.5.

Case 2. The matroid M is critically connected. By Lemma 2, for any element e in M, M/e is connected. By Lemma 3, M has a 2-cocircuit $C = \{a, b\}$. By

Lemma 5, the circuit graph of M/a is isomorphic to that of M. By induction hypothesis, the theorem holds.

Thus the theorem follows by induction.

□

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