$(2,t)$-choosable graphs

Watcharintorn Ruksasakchai[†] and Kittikorn Nakprasit[‡,1]

[†,‡]Department of Mathematics, Faculty of Science,
Khon Kaen University, Khon Kaen 40002, Thailand

E-mail : [†]watcharintorn1@hotmail.com, [‡]kitnak@hotmail.com

Abstract

A (k,t)-list assignment L of a graph G assigns a list of k colors available at each vertex v in G and $|\bigcup_{v\in V(G)} L(v)| = t$. An L-coloring is a proper coloring c such that $c(v) \in L(v)$ for each $v \in V(G)$. A graph G is (k,t)-choosable if G has an L-coloring for every (k,t)-list assignment L.

Erdős, Rubin, and Taylor proved that a graph is $(2,t)$-choosable for any $t \geq 2$ if and only if a graph does not contain some certain subgraphs. Chareonpanitseri, Punnim, and Uiyyasathian proved that an n-vertex graph is $(2,t)$-choosable for $2n-6 \leq t \leq 2n-4$ if and only if it is triangle-free. Furthermore, they proved that a triangle-free graph with n vertices is $(2,2n-7)$-choosable if and only if it does not contain $K_{3,3}-e$ where e is an edge. Nakprasit and Ruksasakchai proved that an n-vertex graph G that does not contain $C_5 \vee K_{k-2}$ and K_{k+1} for $k \geq 3$ is $(k, kn-k^2-2k)$-choosable. For a non-2-choosable graph G, we find the minimum $t_1 \geq 2$ and the maximum t_2 such that the graph G is not $(2,t_i)$-choosable for $i = 1,2$ in terms of certain subgraphs. The results can be applied to characterize $(2,t)$-choosable graphs for any t.

1 Introduction

A *graph* G is an ordered pair $(V(G), E(G))$, where $V(G)$ is a finite set of vertices and $E(G)$ is a set of unordered pairs of distinct vertices. A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For $X \subseteq V(G)$ or $X \subseteq E(G)$, a graph $G-X$ is obtained by deleting all vertices (or edges) of X from G. For $S \subseteq V(G)$, a subgraph of G induced by S, denoted by $G[S]$, is the graph obtained by deleting all vertices of $V(G)-S$ from G.

[1]Corresponding author

We write $G - u$ and $G - e$ instead of $G - \{u\}$ and $G - \{e\}$ for a vertex u and an edge e. The *core* of a graph G is the subgraph of G obtained by the iterated removal of all vertices of degree 1 from G.

For each vertex v in a graph G, let $L(v)$ denote a list of colors available at v. A (k,t)-list assignment L of a graph G assigns a list of k colors available at each vertex v in G and $|\bigcup_{v\in V(G)} L(v)| = t$. An L-coloring is a proper coloring c such that $c(v) \in L(v)$ for each $v \in V(G)$. A graph G is L-colorable if G has an L-coloring. A graph G is (k,t)-choosable if G is L-colorable for every (k,t)-list assignment L. For $H \subseteq G$, we let $L(H)$ denote $\bigcup_{v\in V(H)} L(v)$ and L_H denote a list restricted to $V(H)$. Given a list assignment L, we call $L(v)$ a *disjoint list* if $L(v) \cap L(u) = \emptyset$ for each $u \in V(G) - \{v\}$. A color b is *k-frequent* if b appears in exactly k lists of vertices. If a graph G is L-colorable for every (k,t)-list assignment L, then G is *(k,t)-choosable.* If a graph G is (k,t)-choosable for every number t, then we say that G is *k-choosable.* The *list chromatic number* of a graph G, denoted by $\chi_l(G)$, is the minimum k such that G is k-choosable. For a non-2-choosable graph G, we let $f(G)$ and $F(G)$ denote the minimum number $t_1 \geq 2$ and the maximum number t_2 such that a graph G is not $(2,t_i)$-choosable for $i = 1$ and 2.

Let $\theta_{p_1,p_2,\ldots,p_r}$ denote a graph obtained by identifying all beginnings and identifying all ends of r disjoint paths having $p_1, p_2, \ldots, p_r$ edges respectively. Two cycles C_m and C_n having exactly one vertex in common is denoted by $C_m \cdot C_n$. Two vertex disjoint cycles C_m and C_n connected by a path P_k is denoted by $C_m \cdot P_k \cdot C_n$.

The concept of list coloring was introduced by Vizing [5] and by Erdős, Rubin, and Taylor [2]. In 1979, Erdős et al. [2] showed that a graph is 2-choosable if and only if its core is isomorphic to K_1, C_{2m+2} or $\theta_{2,2,2m}$.

One can see that this result is equivalent to the following theorem.

Theorem 1.1. *A graph is $(2,t)$-choosable for any $t \geq 2$ if and only if a graph does not contain one of the followings: (a) odd cycle, (b) $C_{2m} \cdot C_{2n}$, (c) $C_{2m} \cdot P_k \cdot C_{2n}$, (d) $\theta_{2,2,2,2m}$, or (e) $\theta_{p,q,r}$ which is not isomorphic to $\theta_{2,2,2m}$ and p, q, r are of the same parity.*

The (k,t)-choosability was first defined by Ganjari et al. [3] in 2002. They used the concept of (k,t)-choosability to generalize a characterization of uniquely 2-list colorable graphs. In 2011, Chareonpanitseri, Punnim, and Uiyyasathian [1] proved that an n-vertex graph is $(2,t)$-choosable for $2n - 6 \leq t \leq 2n - 4$ if and only if it is triangle-free. They also showed that a triangle-free graph with n vertices is $(2, 2n-7)$-choosable if and only if it does not contain $K_{3,3} - e$ where e is an edge. Furthermore, they proved that every n-vertex graph is (k,t)-choosable if $t \geq kn - k^2 + 1$ and every K_{k+1}-free graph with n vertices is (k,t)-choosable for $3 \leq k \leq n-3$ and $t \geq kn - k^2 - 2k + 1$. Nakprasit and Ruksasakchai [4] proved that an n-vertex graph G that does not contain $C_5 \vee K_{k-2}$ and K_{k+1} for $k \geq 3$ is $(k, kn - k^2 - 2k)$-choosable. This result solved a conjecture posed by Chareonpanitseri, Punnim, and Uiyyasathian [1].

Let $W(r_1, r_2, r_3, s_1, s_2, s_3)$ be a subdivision of K_4 as shown in Figure 1. We allow r_i and s_i to be 1. For example, if $r_1 = 1$, then there is no vertex x_i. Let $\mathfrak{F}_1 = \{C_{2m+1}\}$, $\mathfrak{F}_2 = \{C_m \cdot C_n\}$, $\mathfrak{F}_3 = \{C_m \cdot P_k \cdot C_n\}$, $\mathfrak{F}_4 = \{\theta_{2,2,2,2m}\}$, $\mathfrak{F}_5 = \{\theta_{p,q,r}$ which is not isomorphic to $\theta_{2,2,2m}\}$, $\mathfrak{F}_6 = \{W(1,1,r,1,1,s)$: s is even, $s \leq r$, and (r is even) or (r is odd, $s = 2$, $r \neq 3$)$\}$, and $\mathfrak{F} = \bigcup_{i=1}^{6} \mathfrak{F}_i$. In Section 2, we find the values t such that G is $(2,t)$-choosable for G in many classes of graphs including $\mathfrak{F}$. In Section 3, we find $f(G)$ and $F(G)$ for every non-2-choosable graph G in terms of subgraphs in $\mathfrak{F}$. Section 4 gives applications of the results from previous sections including a characterization of $(2,t)$-choosable graph for any t.

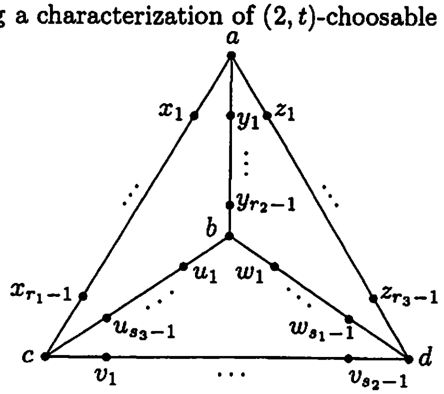

Figure 1: $W(r_1, r_2, r_3, s_1, s_2, s_3)$.

2 $(2,t)$-choosabilities of graphs in some classes

From now on, we let L be a 2-list assignment of a graph G with order $n(G)$.

Lemma 2.1. *If H is a non-2-choosable subgraph of G, then $F(G) \geq F(H) + 2(n(G) - n(H))$.*

Proof. Let L' be a 2-list of H achieving $F(H)$. We can extend L' to L with $|L(G)| = F(H) + 2(n(G) - n(H))$ by assigning disjoint lists to $V(G) - V(H)$. □

Lemma 2.2. *Let H be a subgraph of G and L' be a list assignment of H. Suppose that H is not L'-colorable. Then for each t where $|L'(H)| \leq t \leq |L'(H)| + 2(n(G) - n(H))$ there is a $(2,t)$-list assignment L of G such that G is not L-colorable. In particular we can replace $|L'(H)|$ by $f(H)$ or $F(H)$ in the inequality.*

Proof. We can extend L' to a 2-list assignment L'' of G with $|L''(G)| = |L'(H)| + 2(n(G) - n(H))$ by assigning disjoint lists to vertices outside H. Redefining some colors in $L''(G - H)$ to be redundant with ones in $L'(H)$, we can obtain a $(2,t)$-list assignment of G and reduce $|L(G) - L'(H)|$ as desired. The graph G is not L-colorable because H in not L'-colorable. By definition of $f(H)$ and $F(H)$, there are 2-lists of H which satisfy $f(H)$ and $F(H)$ respectively. □

Theorem 2.3. *(a) An odd cycle is $(2,t)$-choosable if and only if $t \geq 3$.*
(b) If G contains an odd cycle C, then for each t where $2 \leq t \leq 2+2(n(G)-n(C))$ there is a $(2,t)$-list assignment L of G such that G is not L-colorable.
(c) If G contains an odd cycle C, then $F(G) \geq 2 + 2(n(G) - n(C))$.

Proof. The obvious statement (a) implies $F(C) = 2$ for an odd cycle C. The statements (b) and (c) follow from Lemma 2.2. □

Lemma 2.4. *If G is not L-colorable but every proper subgraph H is L_H-colorable, then $|L(u) \cap L(v)| \geq 1$ for each edge uv, every color in $L(G)$ appears in at least 2 lists, and $|L(G)| < n(G)$.*

Proof. By assumption, $G-uv$ is L_{G-uv}-colorable for each edge uv. If $L(u) \cap L(v) = \emptyset$, then G is also L-colorable which is a contradiction. Suppose that there is a color a in $L(G)$ such that a appears in $L(v)$ only. By assumption, $G - v$ is L_{G-v}-colorable. We can extend the coloring to G by using a for the vertex v. Thus every color in $L(G)$ appears in at least 2 lists.

The Hall's theorem implies that there is $S \subseteq V(G)$ such that $|L(S)| < |S|$ to prevent L-coloring of G. We now consider the colors in $L(G - S)$. Since each color in $L(G)$ appears in at least 2 lists, each color in $L(G - S)$ appears in $L(S)$ or in the list of different vertices in $V(G) - S$. Thus $|L(G - S)| - |L(S) \cap L(G - S)| \leq n(G) - |S|$. Hence, $|L(G)| = |L(S)| + |L(G - S)| - |L(S) \cap L(G - S)| < |S| + n(G) - |S| = n(G)$. □

Theorem 2.5. *Let $G = C_m \cdot C_n$. Then*
(a) G is $(2,2)$-choosable if and only if m and n are even,
(b) for $t \geq 3$, G is $(2,t)$-choosable if and only if $t \geq \max\{m+n-1, 2m+1$ if n is odd, $2n+1$ if m is odd$\}$.

Proof. (a) Obvious.

(b) Let $C_m = uv_1v_2\dots v_{m-1}u$ and $C_n = uw_1w_2\dots w_{n-1}u$.

Necessity. Suppose that G is $(2,t)$-choosable for $t \geq 3$. Define a list assignment L of G by $L(u) = \{1,2\}$, $L(v_1) = \{1, a_1\}$, $L(v_i) = \{a_{i-1}, a_i\}$ for $2 \leq i \leq m-2$, $L(v_{m-1}) = \{1, a_{m-2}\}$, $L(w_1) = \{2, a_{m-1}\}$, $L(w_i) = \{a_{m+i-3}, a_{m+i-2}\}$ for $2 \leq i \leq n-2$, $L(w_{n-1}) = \{2, a_{m+n-4}\}$. One can check that G is not L-colorable. If all a_i's are distinct, then $|L(G)| = m+n-2$. We can reduce $|L(G)|$ as needed by defining $a_{m-2} = a_{m+n-4} = 3$ and $a_i = 1$ or 2 for some i. The part of $t \geq \max\{2m+1$ if n is odd, $2n+1$ if m is odd$\}$ follows from Theorem 2.3.

Sufficiency. Let $t \geq \max\{m+n-1, 2m+1$ if n is odd, $2n+1$ if m is odd$\}$. Then $|L(C)| \geq 3$ if C is an odd cycle in G. Theorem 2.3 implies C is L_C-colorable. Thus every proper subgraph H of G is L_H-colorable. Suppose G is not L-colorable. Lemma 2.4 implies $|L(G)| < n(G) = m+n-1$ which contradicts to the assumption. □

Theorem 2.6. *Let $G = C_m \cdot P_k \cdot C_n$. Then*
(a) G is $(2,2)$-choosable if and only if m and n are even,
(b) for $t \geq 3$, G is $(2,t)$-choosable if and only if $t \geq \max\{m+n+k-2, 2m+2k-1$ if n is odd, $2n+2k-1$ if m is odd$\}$.

Proof. (a) Obvious.

(b) Let $C_m = xu_1u_2\dots u_{m-1}x$, $C_n = yv_1v_2\dots v_{n-1}y$ and $P_k = xw_1w_2\dots w_{k-2}y$.

Necessity. Suppose that $3 \leq t \leq m+n+k-3$. Define $L(G)$ by $L(x) = \{1,2\}$, $L(u_1) = \{1, a_1\}$, $L(u_i) = \{a_{i-1}, a_i\}$ for $2 \leq i \leq m-2$, $L(u_{m-1}) = \{1, a_{m-2}\}$, $L(w_1) = \{2, a_{m-1}\}$, $L(w_i) = \{a_{m+i-3}, a_{m+i-2}\}$ for $2 \leq i \leq k-2$, $L(y) = \{a_{m+k-4}, a_{m+k-3}\}$, $L(v_i) = \{a_{m+k+i-4}, a_{m+k+i-3}\}$ for $1 \leq i \leq n-2$, $L(v_{n-1}) = \{a_{m+n+k-5}, a_{m+k-3}\}$. One can check that G is not L-colorable. If all a_i's are distinct, then $|L(G)| = m+n+k-3$. We can reduce $|L(G)|$ as needed by defining $a_{m-2} = a_{m+n+k-5} = 3$ and $a_i = 1$ or 2 for some i. The inequality $t \geq \max\{2m+2k-1$ if n is odd, $2n+2k-1$ if m is odd$\}$ follows from Theorem 2.3.

Sufficiency. The proof is similar to one in Theorem 2.5 (b). □

Lemma 2.7. *Let C_n be L-colorable. Given $u_1, u_k \in V(C_n)$, there exist L-colorings c_1 and c_2 such that $(c_1(u_1), c_1(u_k)) \neq (c_2(u_1), c_2(u_k))$ unless there is a u_1u_k-path P with $|L(P)| = 2$.*

Proof. Let c_1 be an L-coloring of C_n with $c_1(u_1) = a_1$ and $L(u_1) = \{a_1, a_2\}$. First we aim to color vertices in a way that $c_2(u_1) = a_2$. This coloring fails only when $L(u_i) = \{a_i, a_{i+1}\}$ for $1 \leq i \leq n-1$ and $L(u_n) = \{a_n, a_2\}$. In this situation we may assume that $c_1(u_k) = a_{k+1}$ by symmetry. We now aim to use a coloring $c_2(u_i) = a_i$ for each $1 \leq i \leq k$. This plan fails only if $L(u_i) = \{a_k, a_{k+1}\}$ for each $k \leq i \leq n$, $a_{k+1} = a_1$ for odd $n-k$, and $a_k = a_1$ for even $n-k$. This implies $L(u_i) = \{a_1, a_2\}$ for each $k \leq i \leq n$ which completes the proof. □

Theorem 2.8. *Let $G = \theta_{p,q,r}$ and $p \leq q \leq r$. Then*
(a) G is $(2,2)$-choosable if and only if p, q, r are of the same parity,
(b) for $t \geq 3$, G is $(2,t)$-choosable if and only if $t \geq \max\{q+r-1, 2p+1$ if $q+r$ is odd, $2q+1$ if $p+r$ is odd, $2r+1$ if $p+q$ is odd$\}$.

Proof. (a) Obvious.

(b) *Necessity.* The inequality $t \geq \max\{2p+1$ if $q+r$ is odd, $2q+1$ if $p+r$ is odd, $2r+1$ if $p+q$ is odd$\}$ follows from Theorem 2.3. Let $P = uw_1 \ldots w_{p-1}v$, $Q = ux_1x_2 \ldots x_{q-1}v$, and $R = uy_1y_2 \ldots y_{r-1}v$ be paths in $\theta_{p,q,r}$.

Define $L(u) = L(v) = L(w_i) = \{1,2\}$ for $1 \leq i \leq p-1$, $L(x_1) = \{1, a_1\}$, $L(x_i) = \{a_{i-1}, a_i\}$ for $2 \leq i \leq q-2$, $L(y_1) = \{2, a_{q-1}\}$, $L(y_i) = \{a_{q+i-3}, a_{q+i-2}\}$ for $2 \leq i \leq r-2$. Let $L(x_{q-1}) = \{2, a_{q-2}\}$ and $L(y_{r-1}) = \{1, a_{q+r-4}\}$ if p is odd. Let $L(x_{q-1}) = \{1, a_{q-2}\}$ and $L(y_{r-1}) = \{2, a_{q+r-4}\}$ if p is even. One can check that G is not L-colorable. If all a_i's are distinct, then $|L(G)| = q+r-2$. We can reduce $|L(G)|$ as needed by defining $a_{q-2} = a_{q+r-4} = 3$ and $a_i = 1$ or 2 for some i. Thus G is not $(2,t)$-choosable for $3 \leq t \leq q+r-2$.

Sufficiency. Let $t \geq \max\{q+r-1, 2p+1$ if $q+r$ is odd, $2q+1$ if $p+r$ is odd, $2r+1$ if $p+q$ is odd$\}$. Then every odd cycle C (if exists) has $|L(C)| \geq 3$. Consequently, every proper subgraph H of G is L_H-colorable. Suppose G is not L-colorable. If $|L(P)| = 2$ or $|L(Q)| = 2$, then $|L(G)| \leq q+r-2$ by Lemma 2.4. Thus $|L(P)| \geq 3$ and $|L(Q)| \geq 3$. Let $C = G - \{y_1, y_2, \ldots, y_{r-1}\}$. By Lemma 2.7, there there exist L_C-colorings c_1 and c_2 such that $(c_1(u), c_1(v)) \neq (c_2(u), c_2(v))$. In case of $c_1(u) = c_2(u)$, we assign a coloring c to the path R in a way that $c(u) = c_1(u) = c_2(u)$

and $c(v) = c_1(v)$ or $c_2(v)$. Thus G is L-colorable. Now suppose $a_0 = c_1(u) \neq c_2(u) = b_0$ and $a_r = c_1(v) \neq c_2(v) = b_r$. We aim to define a coloring c in a way that $c(u) = a_0$ and $c(v) = a_r$, or $c(u) = b_0$ and $c(v) = b_r$. In a successful case, we can use c_1 or c_2 for C to extend a coloring c to G. If this strategy fails, then $L(y_i) = \{a_i, a_{i+1}\} = \{b_i, b_{i+1}\}$ for $1 \leq i \leq r-1$. Consequently, $L(R) = \{a_0, b_0\}$. Using Lemma 2.4, we have that $|L(G)| \leq p+q-2$ which is a contradiction. □

Theorem 2.9. *Let $G = \theta_{2,2,2,2m}$. Then*
(a) G is $(2,2)$-choosable,
(b) $\theta_{2,2,2,2m}$ is $(2,3)$-choosable if and only if $m = 1$,
(c) for $t \geq 4$, G is $(2,t)$-choosable if and only if $t \geq 2m+3$.

Proof. Let $P_1 = uxv$, $P_2 = uyv$, $P_3 = uzv$, and $P_4 = uw_1w_2\ldots w_{2m-1}v$ be paths in G.

(a) Obvious.

(b) *Necessity.* We define $L(u) = L(x) = L(w_i) = \{1,2\}$ for $1 \leq i \leq 2m-3$, $L(y) = L(v) = L(w_{2m-1}) = \{1,3\}$, $L(z) = L(w_{2m-2}) = \{2,3\}$. One can check that G is not L-colorable.

Sufficiency. Assign a color in $L(u) \cap L(v)$ to both vertices u and v. A coloring of other vertices follows easily.

(c) *Necessity.* Suppose that $4 \leq t \leq 2m+2$. Let L be a list assignment of G such that $L(u) = \{1,2\}$, $L(v) = \{3,4\}$, $L(x) = \{1,3\}$, $L(y) = \{1,4\}$, $L(z) = \{2,3\}$, $L(w_1) = \{2, a_1\}$, $L(w_i) = \{a_{i-1}, a_i\}$ for $2 \leq i \leq 2m-2$, $L(w_{2m-1}) = \{4, a_{2m-2}\}$. One can check that G is not L-colorable. If all a_i's are distinct, then $|L(G)| = 2m+2$. We can reduce $L(G)$ to size t where $t < 2m+2$ by defining $a_i = 1$ for some odd i, and $a_j = 2$ for some even j.

Sufficiency. Assume G is not L-colorable. Note that every proper subgraph H of G is L_H-colorable. Then $|L(G)| \leq n(G)-1$ by Lemma 2.4. Suppose $|L(G)| \leq n(G)-1$. Lemma 2.4 implies every color is 2-frequent except either one color of 4-frequent or two colors of 3-frequent. Let a color a_i be k_i-frequent for $L(u) = \{a_1, a_2\}$ and $L(v) = \{a_3, a_4\}$. Using Lemma 2.4, we have $k_1+k_2 \geq 6$ and $k_3+k_4 \geq 6$ which leads to a contradiction regardless of $L(u) \cap L(v)$. □

Theorem 2.10. *Let $G = \theta_{p,q,r,s}$ where G is not isomorphic to $\theta_{2,2,2,2m}$. Then*

(a) there is $H \subseteq G, H \in \mathfrak{F}_5$ such that $F(H) + 2(n(G) - n(H)) \geq n(G) - 2$,
(b) $F(G) = \max_{H \subseteq G, H \in \mathfrak{F}_1 \cup \mathfrak{F}_5}\{F(H) + 2(n(G) - n(H))\}$.

Proof. (a) Assume $p \leq q \leq r \leq s$. If $\theta_{p,q,r} \neq \theta_{2,2,2m}$, then let $H = \theta_{p,q,r}$, otherwise let $H = \theta_{p,q,s}$. We have $H \subseteq G, H \in \mathfrak{F}_5$ such that $F(H) + 2(n(G) - n(H)) \geq n(G) - 2$.

(b) Assume that $F(G) > \max_{H \subseteq G, H \in \mathfrak{F}_1 \cup \mathfrak{F}_5}\{F(H)+2(n(G)-n(H))\}$. Then every proper subgraph H of G is L_H-colorable. From (a) and Lemma 2.4, $F(G) = n(G) - 1$. We use a similar argument to the proof of Theorem 2.9 (c) to reach a contradiction. □

Theorem 2.11. *Let $G = W(r_1, r_2, r_3, s_1, s_2, s_3)$. Then the followings hold.*
(a) There is $H \subseteq G, H \in \mathfrak{F}_5$ such that $F(H) + 2(n(G) - n(H)) \geq n(G) - 2$. Moreover, if $3 \leq t \leq n - 2$, then G is not $(2, t)$-choosable.
(b) If $G \in \mathfrak{F}_6$, then $F(G) = n(G) - 1 > \max_{H \subseteq G, H \in \mathfrak{F}_1 \cup \mathfrak{F}_5}\{F(H)+2(n(G)-n(H))\}$.
(c) If $G \notin \mathfrak{F}_6$, then $F(G) = \max_{H \subseteq G, H \in \mathfrak{F}_1 \cup \mathfrak{F}_5}\{F(H) + 2(n(G) - n(H))\}$.

Proof. (a) Assume that $s_1 = \min_{1 \leq i \leq 3}\{r_i, s_i\}$. Consider the graph $H = G - \{w_1, w_2, \ldots, w_{s_1-1}\}$. Note that H is a $\theta_{r_1, r_2+s_3, r_3+s_2}$ that is not isomorphic to $\theta_{2,2,2m}$. Then $F(H) \geq r_2 + s_3 + r_3 + s_2 - 2$. Thus $F(H) + 2(n(G) - n(H)) = F(H) + 2(s_1 - 1) \geq r_1 + r_2 + r_3 + s_2 + s_3 + s_1 - 4 = n(G) - 2$. Theorem 2.8 implies G is not $(2, t)$-choosable for $3 \leq t \leq n(G) - 2$.

(b) Let $G \in \mathfrak{F}_6$. Define $L(a) = \{1, 2\}$, $L(b) = \{1, 3\}$, $L(c) = \{1, 4\}$, $L(d) = \{1, 5\}$, $L(u_1) = \{2, 6\}, L(u_{r-1}) = \{r + 3, 5\}$, $L(u_i) = \{i + 4, i + 5\}$ for $2 \leq i \leq r - 2$, $L(z_1) = \{3, r + 4\}, L(z_{s-1}) = \{r + s + 1, 4\}$, and $L(z_j) = \{j + r + 2, j + r + 3\}$ for $2 \leq j \leq s - 2$.

One can check that $|L(G)| = n(G) - 1 > \max_{H \subseteq G, H \in \mathfrak{F}_1 \cup \mathfrak{F}_5}\{F(H) + 2(n(G) - n(H))\}$ and G is not L-colorable. Thus every proper subgraph H of G is L_H-colorable. Lemma 2.4 implies that $|L(G)| < n(G)$. Hence $F(G) = n(G) - 1$. The observation that G is bipartite and $F(G) = n(G) - 1$ completes the proof.

(c) Assume that $F(G) > \max_{H \subseteq G, H \in \mathfrak{F}_1 \cup \mathfrak{F}_5}\{F(H)+2(n(G)-n(H))\}$. Then every proper subgraph H of G is L_H-colorable. From (a) and Lemma 2.4, $F(G) = n(G) - 1$. Assume $r_1 > s_1$. Note that the graph $H = G - \{x_1, x_2, \ldots, x_{r_1-1}\}$ is not $\theta_{2,2,2m}$ unless G is isomorphic to $W(r_1, 1, 1, 2k, 1, 1)$ or $W(r_1, 1, 1, 2, k_1, k_2)$ where $k_1 + k_2$ is even. If H is not $\theta_{2,2,2m}$, then $F(H) + 2(n(G) - n(H)) \geq n(G) - 1$. If G is isomorphic to $W(r_1, 1, 1, 2, k_1, k_2)$ and $k_1 > 1$, then we can find $H' = \theta_{1,3,r_1+k_2} \in \mathfrak{F}_5$ such that $F(H') + 2(n(G) -$

$n(H')) \geq n(G) - 1$. Now it suffices to consider only $W(p,q,r,p,q,r)$ where $q, r \geq 2$ and $W(1,1,r,1,1,s)$.

Suppose $G = W(p,q,r,p,q,r)$ where $q, r \geq 2$. From (a) and Lemma 2.4, $F(G) = n(G) - 1$ and every color is 2-frequent except either one color of 4-frequent or two colors of 3-frequent. Let a color a_i be k_i-frequent for $L(a) = \{a_1, a_2\}, L(c) = \{a_3, a_4\}$, and $L(d) = \{a_5, a_6\}$. Using Lemma 2.4, we have $k_{2i-1} + k_{2i} \geq 5$ for each i = 1, 2, 3. Note that $L(a) \cap L(d) = L(c) \cap L(d) = \emptyset$, otherwise we have a contradiction. By Lemma 2.4, we may assume that a_{2i-1} appears in exactly 2 lists of its neighbors for each i = 1, 2, 3. If $a_1 = a_3$, then $k_1 \geq 4$. If $a_1 \neq a_3$, then a_1, a_3, and a_5 are distinct 3-frequent colors. We have contradictions in both cases.

Consider $G = W(1,1,r,1,1,s) \notin \mathfrak{F}_6$ where $s \leq r$. Let $H = C_{s+2}$ for s is odd, $H = \theta_{2,2,r}$ for r is odd, $s \geq 4$, and $H = C_5$ for $r = 3, s = 2$. In all cases, $H \in \mathfrak{F}_1 \cup \mathfrak{F}_5$ and $F(H) + 2(n(G) - n(H)) \geq n(G) - 1$ which completes the proof. □

3 $(2,t)$-choosabilities of non-2-choosable graphs

From now on, we let G be a non-2-choosable graph.

Lemma 3.1. *(a) If G is not bipartite, then $f(G) = 2$.*
(b) Let G be a non-2-choosable bipartite graph. Then either $f(G) = 3$ or $f(G) = 4$ and $K_{2,m}$ $(m \geq 4)$ is the core of G.

Proof. (a) Obvious.

(b) Since G is non-2-choosable bipartite, G is not (2,2)-choosable and G has a subgraph $H \in \mathfrak{F} = \bigcup_{i=2}^{5} \mathfrak{F}_i$ by Theorem 1.1. Then $f(G) \leq 4$ by Lemma 2.2, Theorems 2.5, 2.6, 2.8 and 2.9. Moreover $f(G) \leq 3$ if $H \neq \theta_{2,2,2,2}$. Suppose $f(G) = 4$. Then $H = \theta_{2,2,2,2}$. If the core of G is not $K_{2,m}$, then G contains a subgraph $H' \in \bigcup_{i=2}^{5} \mathfrak{F}_i$ with $f(H') = 3$. By Lemma 2.2, we have $4 = f(G) \leq f(H') = 3$ which is a contradiction. Hence $K_{2,m}$ $(m \geq 4)$ is the core of G. □

Lemma 3.2. *For a graph G and its 2-list assignment L, denote the inequality $|L(G)| \leq \max_{H \subseteq G, H \in \mathfrak{F}}\{F(H) + 2(n(G) - n(H))\}$ by (A). If each G that is not L-colorable but every proper subgraph K of G is L_K-colorable satisfies the inequality (A), then each G' and its list assignment L', where G' is not L'-colorable, G' also satisfies the inequality (A) for G' and its list assignment L'.*

Proof. Let G' be a graph that is not L'-colorable and $|L'(G')| = F(G')$. Consider a minimal subgraph G of G' that is not L'_G-colorable. Then $F(G') = |L'(G')| \leq |L'(G)| + 2(n(G') - n(G)) \leq \max_{H \subseteq G, H \in \mathfrak{F}}\{F(H) + 2(n(G) - n(H))\} + 2(n(G') - n(G)) \leq \max_{H \subseteq G', H \in \mathfrak{F}}\{F(H) + 2(n(G') - n(H))\}$. □

Theorem 3.3. *$F(G) \leq \max_{H \subseteq G, H \in \mathfrak{F}}\{F(H) + 2(n(G) - n(H))\}$ if G is a non-2-choosable graph.*

Proof. By Lemma 3.2, it suffices to show that each graph G that is not L-colorable but every proper subgraph K is L_K-colorable, has $|L(G)| \leq \max_{H \subseteq G, H \in \mathfrak{F}}\{F(H) + 2(n(G) - n(H))\}$. Since G is not 2-choosable, the graph G contains $H \in \mathfrak{F}$ as a subgraph by Theorem 1.1. If $H \in \mathfrak{F}$ is a core of G, then the inequality immediately follows. We now suppose otherwise.

Case 1: $H \in \mathfrak{F}_2 \cup \mathfrak{F}_3$.

Using Theorems 2.5, 2.6 and Lemma 2.4, we have $F(H) + 2(n(G) - n(H)) \geq (n(H) - 1) + 2(n(G) - n(H)) \geq n(G) - 1 \geq |L(G)|$.

Case 2: $H \in \mathfrak{F}_4$.

Suppose $n(G) > n(H)$. Using Theorem 2.9 and Lemma 2.4, we have $F(H) + 2(n(G) - n(H)) \geq n(H) - 2 + 2n(G) - 2n(H) \geq n(G) - 1 \geq |L(G)|$. Now suppose that $n(G) = n(H)$ and there is $e \in E(G) - E(H)$. If G contains $C_3 \in \mathfrak{F}$, then $F(C_3) + 2(n(G) - n(C_3)) = 2 + 2n(G) - 6 > n(G) - 1 \geq |L(G)|$. If G does not contain C_3, then G contains subgraph H' in Case 1.

Case 3: $H = W(r_1, r_2, r_3, s_1, s_2, s_3)$.

If $n(G) > n(H)$, then $F(H) + 2(n(G) - n(H)) \geq n(H) - 2 + 2n(G) - 2n(H) = 2n(G) - n(H) - 2 \geq n(G) > |L(G)|$. Suppose that $n(G) = n(H)$ and there is $e \in E(G) - E(H)$. Then G contains subgraph H' in Case 1. Now suppose that $G = H$. Thus $F(G) = \max_{H' \subseteq G, H' \in \mathfrak{F}}\{F(H') + 2(n(G) - n(H')\}$ by Theorem 2.11.

Case 4: $H = \theta_{p,q,r,s}$ where H is not isomorphic to $\theta_{2,2,2,2m}$.

If $n(G) > n(H)$, then $F(H) + 2(n(G) - n(H)) \geq n(H) - 2 + 2n(G) - 2n(H) = 2n(G) - n(H) - 2 \geq n(G) > |L(G)|$. Now suppose that $n(G) = n(H)$ and there is $e \in E(G) - E(H)$. Then G contains $\theta_{1,p,q,r,s}$ or a subgraph in previous cases. Suppose G contains $H' = \theta_{1,p,q}$. Then $H' \in \mathfrak{F}$ and $F(H') + 2(n(G) - n(H')) \geq n(G) > |L(G)|$. Now we suppose $G = H$. Thus $F(G) = \max_{H' \subseteq G, H' \in \mathfrak{F}}\{F(H') + 2(n(G) - n(H')\}$ by Theorem 2.10.

Case 5: $H \in \mathfrak{F}_5$.

If H is not the core of G, then G contains a subgraph in previous cases.

Case 6: $H \in \mathfrak{F}_1$.

If H is not the core of G, then G contains a subgraph in previous cases. □

Combining Lemma 2.1 and Theorem 3.3, we have the following corollary.

Corollary 3.4. $F(G) = \max_{H \subseteq G, H \in \mathfrak{F}}\{F(H) + 2(n(G) - n(H))\}$.

Theorem 3.5. *For $f(G) \leq t \leq F(G)$, there is a $(2,t)$-list such that G is not L-colorable.*

Proof. Let H be a subgraph of G such that $F(G) = \max_{H \subseteq G, H \in \mathfrak{F}}\{F(H) + 2(n(G) - n(H))\}$. By Lemma 2.2, Theorems 2.3, 2.5, 2.6, 2.8, 2.9, and 2.11, we have $(2,t)$-list such that G is not L-colorable for each t satisfying $f(H) \leq t \leq F(H) + 2(n(G) - n(H)) = F(G)$. If $f(H) - f(G) \leq 1$, then we have the desired result. Suppose $f(H) - f(G) \geq 2$, then G is not bipartite and $H = \theta_{2,2,2,2}$. Let C be a smallest odd cycle. Suppose $V(H) \subseteq V(C)$. Then C has a chord $e \in E(H)$. Consequently, we have an odd cycle smaller than C which is a contradiction. If $V(C)$ does not contain $V(H)$, then G is not $(2,3)$-choosable by Lemma 2.2. □

4 Application

In [1] Chareonpanitseri, Punnim, and Uiyyasathian proved that an n-vertex graph is $(2,t)$-choosable for $2n-6 \leq t \leq 2n-4$ if and only if it is triangle-free. Furthermore, they proved that a triangle-free graph with n vertices is $(2, 2n-7)$-choosable if and only if it does not contain $K_{3,3} - e$ where e is an edge.

From Corollary 3.4 and Theorem 3.5, we have that G is $(2, 2n-k)$-choosable if and only if $2n-k \geq 2$ and G does not contain (minimal) $H \in \mathfrak{F}$ with $2n(H) - F(H) \leq k$. Let G be an n-vertex graph and $t \geq 2$. From Table 1, we can conclude the followings:

1. If $t \geq 2n-3$, then G is $(2,t)$-choosable.
2. A graph G is $(2,t)$-choosable for $2n-6 \leq t \leq 2n-4$ if and only if G does not contain C_3.
3. A graph G is $(2, t = 2n-7)$-choosable if and only if G does not contain C_3 or $W(1,1,2,1,1,2)$.

4. A graph G is $(2, t = 2n - 8)$-choosable if and only if G does not contain C_3, $W(1,1,2,1,1,2)$, C_5, $C_4 \cdot C_4$, $\theta_{2,2,2,2}$, or $\theta_{1,3,3}$.

5. A graph G is $(2, t = 2n - 9)$-choosable if and only if G does not contain C_3, $W(1,1,2,1,1,2)$, C_5, $C_4 \cdot C_4$, $\theta_{2,2,2,2}$, $\theta_{1,3,3}$, $C_4 \cdot P_2 \cdot C_4$, or $W(1,1,4,1,1,2)$.

One can characterize $(2, 2n-k)$-choosable graphs for any k by this process.

$2n(H) - F(H)$	minimal $H \in \mathfrak{F}$
1	–
2	–
3	–
4	C_3
5	–
6	–
7	$W(1,1,2,1,1,2)$
8	C_5, $C_4 \cdot C_4$, $\theta_{2,2,2,2}$, $\theta_{1,3,3}$
9	$C_4 \cdot P_2 \cdot C_4$, $W(1,1,4,1,1,2)$

Table 1: Minimal graphs $H \in \mathfrak{F}$ with $2n(H) - F(H) = 1, 2, \ldots, 9$

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