A Note on Upper Generalized Exponents of Tournaments

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Abstract. For any $n \geq 7$, we prove that there exists a tournament of order n, such that for each pair of distinct vertices there exists a path of length 2.

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1. Introduction

Let T be a tournament (a digraph in which every pair of distinct vertices is joined by exactly one arc). We denote the in-neighborhood and out-neighborhood of a vertex u in T by $N_T^-(u)$ and $N_T^+(u)$ respectively.

Let D be a digraph with vertex set $V(D)$ and arc set $E(D)$. D is called primitive if there exists positive integer p such that for each ordered pair of vertices u and v there is a walk of length p from u to v (notation $u \xrightarrow{p} v$). The minimum such p is called the exponent of D.

Definition 1.1 [1]. Let D be a digraph and X be a subset of $V(D)$. The "set exponent" $\exp_D(X)$ is defined to be the smallest positive integer p such that for each vertex y of D there exists a walk of length p from at least one vertex in X to y.

Definition 1.2 [1]. Let D be an arbitrary digraph of order n and $1 \leq k \leq n$ then we define

$$F(D,k) = \max\left\{\exp_D(X) \mid X \subseteq V(D), |X| = k\right\}.$$

$F(D,k)$ is called the "k^{th} upper generalized exponent" of D.

Let PT_n be the set of all primitive tournament of order n, and let $E(n,k) = \left\{F(T,k) \middle| T \in PT_n\right\}$ be the upper generalized exponent set for primitive tournament of order n.

In [4], Zhou and Shen gave the upper generalized exponent sets for tournaments on $n(n \geq 7)$ vertices (as the following theorem).

Theorem 1.1[4]. *For* $n \geq 7$ *and* $4 \leq k \leq n-1$ *or* $n \geq 24$ *and* $k = 2,3$,

$$E(n,k) = \begin{cases} \{2,3,4,\cdots,n+1\} & k = 2, \\ \{2,3,4,\cdots,n-k+2\} & 3 \leq k \leq [n/2]+1, \\ \{1,2,3,4,\cdots,n-k+2\} & [n/2]+2 \leq k \leq n-1. \end{cases}$$

We notice that in this theorem, the limit $n \geq 7$ is necessary on the condition that $4 \leq k \leq n-1$, but it must be strengthened to $n \geq 24$ for $k = 2,3$. It was proved but not constructed that $2 \in E(n,2)$ in [4]. In order to show $2 \in E(n,2)$, one needs to find a tournament T on n vertices with $F(T,2) = 2$. And it was also pointed out in [2] for some special n ($n \geq 7$ and $n \equiv 3(\bmod 4)$), $F(T,2) = 2$. In this paper, we will prove that $2 \in E(n,2)$ for any $n \geq 7$.

2. Main Results

Let $n \geq 7$, the notation T_n is used to denote the specified digraph of order n with $V(T_n) = \{1,2,3,\cdots,n\}$ and

$$E(T_n) = \begin{cases} \{(i, i+k(\bmod n)) \mid 1 \leq i \leq n, k = 1,2,4,6,\cdots,n-5,n-3\} & \textit{if } n \textit{ is odd}, \\ E(T_{n-1}) \cup \{(i,n) \mid 3 \leq i \leq n-1, i \neq n-2\} \cup \{(n,1),(n,2),(n,n-2)\} & \textit{if } n \textit{ is even}. \end{cases}$$

Note: We assume that $n \equiv n(\bmod n)$.

It is obvious that T_n contains the subgraph T_{n-1} if $n(n \geq 7)$ is even.

Lemma 2.1. *For any* $n \geq 7$, T_n *is a tournament.*

Proof. There are two cases to consider.

Case 1. $n(n \geq 7)$ is odd. Let $X = \{1,2,4,6,\cdots,n-5,n-3\}$ and $Y = \{n-k \mid k \in X\}$ hence $X \cap Y = \phi$ and $X \cup Y \cup \{n\} = \{1,2,3,\cdots,n\}$. For any $i, j \in V(T_n)$ and $i \neq j$, if $j \notin N^+_{T_n}(i) = \{i+k(\bmod n) \mid k \in X\}$ which implies $j \not\equiv i+k(\bmod n)$ for any $k \in X$, there exists $t \in Y$ such that $j \equiv i+t(\bmod n)$. Since $t \in Y$ there exists $k' \in X$ such that $t = n-k'$, thus $j \equiv i+n-k'$ $(\bmod n)$, so $i \equiv j+k'(\bmod n)$ and hence $i \in N^+_{T_n}(j)$. It is easy to see that $j \in N^+_{T_n}(i)$ and $i \in N^+_{T_n}(j)$ are impossible to be established at the same time. Consequently T_n is a tournament.

Case 2. $n(n \geq 7)$ is even. T_n contains the subgraph T_{n-1} which is a tournament. According to the constructing of T_n, it is easy to see that T_n is a tournament.

The proof is completed. ■

Theorem 2.1. *Let* $n \geq 7$*, there exists a path of length 2 for each pair of distinct vertices* i, j *in* T_n.

Proof. There are two cases to consider.

Case 1. $n(n \geq 7)$ is odd. For $\{i, j\} \subseteq V(T_n)$, we are going to prove that there exists $i \xrightarrow{2} j$. Since $j \in V(T_n) \setminus \{i\}$ we only need to prove that $V(T_n) \setminus \{i\} \subseteq \bigcup_{t \in N^+_{T_n}(i)} N^+_{T_n}(t)$.

For $i+1(\bmod n)$, $i+2(\bmod n)$, $i+4(\bmod n) \in N^+_{T_n}(i)$,

$$N^+_{T_n}(i+1(\bmod n)) = \{i+1+k(\bmod n) \mid k = 1,2,4,6,\cdots,n-3\}$$
$$= \{i+k(\bmod n) \mid k = 2,3,5,7,\cdots,n-2\},$$
$$N^+_{T_n}(i+2(\bmod n)) = \{i+2+k(\bmod n) \mid k = 1,2,4,6,\cdots,n-3\}$$
$$= \{i+k(\bmod n) \mid k = 3,4,6,8,\cdots,n-1\},$$
$$N^+_{T_n}(i+4(\bmod n)) = \{i+4+k(\bmod n) \mid k = 1,2,4,6,\cdots,n-3\}$$
$$= \{i+k(\bmod n) \mid k = 5,6,8,10,\cdots,n-1,1\}.$$

Hence $V(T_n) \setminus \{i\} = N^+_{T_n}(i+1(\bmod n)) \cup N^+_{T_n}(i+2(\bmod n)) \cup N^+_{T_n}(i+4(\bmod n))$

$$\subseteq \bigcup_{k \in N^+_{T_n}(i)} N^+_{T_n}(k).$$

Case 2. $n(n \geq 7)$ is even. We know that T_n contains the subgraph T_{n-1}. According to Case 1 T_{n-1} there is a path of length 2 from each vertex i to each vertex $j \neq i$. Now we only need to prove the existence of $n \xrightarrow{2} j$ and $j \xrightarrow{2} n$ for any $j \in V(T_{n-1})$. In other words, we only need to prove $V(T_{n-1}) \subseteq \bigcup_{t \in N^+_{T_n}(n)} N^+_{T_n}(t)$ and $V(T_{n-1}) \subseteq \bigcup_{t \in N^-_{T_n}(n)} N^-_{T_n}(t)$.

① If $t \in N^+_{T_n}(n)$,

$$N^+_{T_n}(t) = N^+_{T_{n-1}}(t) = \{t+k(\bmod n-1) \mid k = 1,2,4,6,\cdots,n-6,n-4\}.$$

For $1,2,n-2 \in N^+_{T_n}(n)$, $N^+_{T_n}(1) = \{2,3,5,7,\cdots,n-5,n-3\}$, $N^+_{T_n}(2) = \{3,4,6,8,\cdots,n-4,n-2\}$, $N^+_{T_n}(n-2) = \{n-1,1,3,5,\cdots,n-5\}$.

Hence $V(T_{n-1}) = N^+_{T_n}(1) \cup N^+_{T_n}(2) \cup N^+_{T_n}(n-2) \subseteq \bigcup_{t \in N^+_{T_n}(n)} N^+_{T_n}(t)$.

② If $t \in N^-_{T_n}(n)$,

$N^+_{T_n}(t) = N^+_{T_{n-1}}(t) \cup \{n\} = \{t + k(\mathrm{mod}\, n-1) \mid k = 1,2,4,6,\cdots,n-6,n-4\} \cup \{n\}$,

$N^-_{T_n}(t) = V(T_n) \setminus \left(N^+_{T_n}(t) \cup \{t\}\right) = \{t + k(\mathrm{mod}\, n-1) \mid k = 3,5,7,\cdots,n-5,n-3,n-2\}$.

For $3,4,5,7 \in N^-_{T_n}(n)$, $N^-_{T_n}(3) = \{6,8,\cdots,n-2,1,2\}$, $N^-_{T_n}(4) = \{7,9,\cdots,n-3,n-1,2,3\}$, $N^-_{T_n}(5) = \{8,10,\cdots,n-2,1,3,4\}$, $N^-_{T_n}(7) = \{10,12,\cdots,n-2,1,3,5,6\}$.

Hence $V(T_{n-1}) = N^-_{T_n}(3) \cup N^-_{T_n}(4) \cup N^-_{T_n}(5) \cup N^-_{T_n}(7) \subseteq \bigcup_{t \in N^-_{T_n}(n)} N^-_{T_n}(t)$.

In conclusion, there exists a path of length 2 for each pair of distinct vertices i, j in T_n. ■

Corollary 2.1. *If* $n \geq 7$ *and* $k = 2,3$, $F(T_n, k) = 2$.

Proof. By Theorem 2.1, we have $F(T_n, k) \leq 2$. By [4, Lemma 3.4.] we have $F(T_n, k) \geq 2$. Hence $F(T_n, k) = 2$. ■

Base on the conclusion above, we can have the following theorem.

Theorem 2.2. *For* $n \geq 7$ *and* $2 \leq k \leq n-1$,

$$E(n,k) = \begin{cases} \{2,3,4,\cdots,n+1\} & k = 2, \\ \{2,3,4,\cdots,n-k+2\} & 3 \leq k \leq [n/2]+1, \\ \{1,2,3,4,\cdots,n-k+2\} & [n/2]+2 \leq k \leq n-1. \end{cases}$$

Proof. For $n \geq 7$ and $k = 2,3$, by Corollary 2.1 we have $2 = F(T_n, k) \in E(n,k)$. And by Theorem 1.1 we know that $\{3,4,\cdots,n+1\} \subseteq E(n,2)$ and $\{3,4,\cdots,n-1\} \subseteq E(n,3)$. By combining Theorem 1.1 we complete the proof. ■

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