Cubic edge-transitive graphs of order $12p$ or $12p^2$

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Abstract

A graph is said to be *edge-transitive* if its automorphism group acts transitively on its edge set. In this paper, all connected cubic edge-transitive graphs of order $12p$ or $12p^2$ are classified.

Key words: Edge-transitive graphs, Symmetric graphs, Semisymmetric graphs.

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1 Introduction

Throughout this paper a *graph* means a finite, connected, simple and undirected graph. For a graph X, denote by $V(X)$, $E(X)$ and $\mathrm{Aut}(X)$ the vertex set, the edge set and the automorphism group of X, respectively. For $u, v \in V(X)$, denote by $\{u, v\}$ or uv the edge incident to u and v in X. An *s-arc* in a graph X is an ordered $(s+1)$-tuple $(v_0, v_1, \cdots, v_{s-1}, v_s)$ of vertices of X such that v_{i-1} is adjacent to v_i for $1 \le i \le s$ and $v_{i-1} \ne v_{i+1}$ for $1 \le i \le s-1$. A graph X is said to be *s-arc-transitive* if $\mathrm{Aut}(X)$ is transitive on the set of s-arcs in X. In particular, 0-arc-transitive means *vertex-transitive*, and 1-arc-transitive means *arc-transitive* or *symmetric*. A subgroup of $\mathrm{Aut}(X)$ is *s-regular* if the subgroup acts regularly on the set of s-arcs in X, and X is said to be *s-regular* if $\mathrm{Aut}(X)$ is s-regular. A graph X is *edge-transitive* if $\mathrm{Aut}(X)$ acts transitively on $E(X)$, and *semisymmetric* provided that X has regular valency and is edge- but not vertex-transitive.

In his classical work [41, 42], Tutte showed that every cubic symmetric graph is s-regular for some $s \le 5$. Following this pioneering work, cubic graphs with high levels of symmetry have been extensively studied over 6 decades by many authors. For example, Djoković and Miller [13] proved that there are seven types of arc-transitive group action on finite cubic graphs, characterized by the stabilizers of a vertex and an edge. Conder and Nedela [11] gave a more detailed classification of finite cubic symmetric graphs, by determining exactly which combinations of types are realizable for arc-transitive subgroups of the full automorphism group. Goldschmit [25] extended Djoković and Miller's work to all cubic edge-transitive graphs. Foster [4] produced a list of cubic symmetric graphs on up to 512 vertices. Based on Djoković and Miller's classification, an exhaustive computer search by Conder and Dobcsányi [6] resulted in a complete list of cubic symmetric graphs on up to 768 vertices. Recently, a similar method based on Goldschmidt's classification was used to compile a list of all cubic semisymmetric graphs on up to 768 vertices [7]. For more results regarding cubic graphs with high levels of symmetry, we refer the reader to [9, 10, 15, 16, 22, 21, 28, 29, 30, 31, 32, 33, 35, 36, 37, 38, 39, 40].

This paper is devoted to the classification of cubic edge-transitive graphs with given orders. Let p be a prime. From Cheng and Oxley [5] we can obtain a classification of cubic edge-transitive graphs of order $2p$. Feng et al. [17, 18, 20] classified all cubic symmetric graphs of order kp or kp^2 with $4 \leq k \leq 10$, and they [19] also classified cubic symmetric graphs of order $2p^2$. Folkman [23] proved that there is no cubic semisymmetric graph of order $2p$ or $2p^2$, while Malnic et al. [34] classified cubic semisymmetric graphs of order $2p^3$. From Du and Xu [14] we can see that there is no cubic semisymmetric graph of order $6p$ or $10p$. The classification of cubic semisymmetric graphs of order $6p^2$ was given by Lu et al. [29], and Alaeiyan et al. [1, 2] proved that there are no cubic semisymmetric graphs of order $8p$ or $8p^2$. Recently, Hua and Feng [26] classified cubic semisymmetric graphs of order $8p^3$. In this paper, we classify all cubic edge-transitive graphs of order $12p$ or $12p^2$. The main result is the following theorem.

Theorem 1.1 *Let p be a prime and let X be a connected cubic edge-transitive graphs of order $12p$ or $12p^2$. Then X is isomorphic either to the 2-regular graphs* F024, F048, F060, F084 *or* F108, *or to the 4-regular graph* F204.

2 Preliminaries

In this section, we describe some preliminary results which will be used later in the paper. Throughout this paper we denote by $\mathbb{Z}_n$ the cyclic group of order n as well as the ring of integers modulo n, by $\mathbb{Z}_n^*$ the multiplicative group of $\mathbb{Z}_n$ consisting of numbers coprime to n, respectively. For two groups M and N, $N \leq M$ means that N is a subgroup of M, and $N < M$ means that N is a proper subgroup of M, and $N \rtimes M$ denotes a semidirect product of N by M. For a subgroup H of a group G, denote by $C_G(H)$ the centralizer of H in G and by $N_G(H)$ the normalizer of H in G.

Let X be a cubic graph and let $G \leq \mathrm{Aut}(X)$ act transitively on the edges of X. Let N be a normal subgroup of G. The *quotient graph* X_N of X relative to N is defined as the graph with vertices the orbits of N

in $V(X)$ and with two orbits adjacent if there is an edge in X between those two orbits. We introduce two propositions, of which the first one is a special case of [27, Theorem 9].

Proposition 2.1 *Let G be transitive on $V(X)$. Then G is an s-regular subgroup of* $\mathrm{Aut}(X)$ *for some integer s. If N has more than two orbits in $V(X)$, then N is semiregular on $V(X)$, X_N is a cubic symmetric graph with G/N as an s-regular group of automorphisms, and X is a regular N-cover of X_N.*

The next proposition is a special case of [29, Lemma 3.2].

Proposition 2.2 *Let G be intransitive on $V(X)$. Then X is a bipartite graph with two partition sets, say V_0 and V_1. If N is intransitive on the bipartition sets, then N is semiregular on both V_0 and V_1, X_N is a cubic graph with G/N as an edge- but not vertex-transitive group of automorphisms and X is a regular N-cover of X_N.*

By [18, Theorem 6.2], we have the following proposition.

Proposition 2.3 [18, Theorem 6.2] *Let X be a connected cubic symmetric graph of order $4p$ or $4p^2$ for a prime p. Then X is isomorphic to the 2-regular hypercube Q_3 of order 8, the 2-regular generalized Petersen graphs $P(8,3)$ or $P(10,7)$ of order 16 or 20 respectively, the 3-regular Dodecahedron of order 20 or the 3-regular Coxeter graph of order 28.*

3 Proof of Theorem 1.1

Lemma 3.1 *Let $p > 7$ be a prime and n a positive integer. Then there exists no connected cubic edge-transitive graphs of order $4p^n$ with $n \leq 2$.*

Proof. Suppose to the contrary that X is a connected cubic edge-transitive graph of order $4p^n$. By Proposition 2.3, X is not arc-transitive. It follows that X is semisymmetric and hence it is bipartite. Let $A = \mathrm{Aut}(X)$. By [41, 42] and [34, Proposition 2.4], $|A| \mid 2^9 \cdot 3 \cdot p^t$ with $t = 1$ or 2. Assume that A is non-solvable. Then A has a non-abelian simple composite factor T_1/T_2. Since $|T_1/T_2| \mid 2^9 \cdot 3 \cdot p^\ell$ and $p \geq 11$, by [24, pp.12–14], $T_1/T_2 \cong A_5$ or $\mathrm{PSL}(2,7)$, forcing $p \leq 7$, a contradiction. Thus, A is solvable. Let N be a minimal normal subgroup of A. Then N is an elementary abelian 2- or p-group. Clearly, N is intransitive on each partition set of X. By Proposition 2.2, N is semiregular on $V(X)$, implying that $|N| \mid 2p^t$ with $t = 1$ or 2. Therefore, $N \cong \mathbb{Z}_2$ or $\mathbb{Z}_p^t$ with $t = 1$ or 2. Assume that $N \cong \mathbb{Z}_2$. Let M be a maximal normal 2-subgroup of A. Then M is intransitive on each partition set of X. By Proposition 2.2, M is semiregular on $V(X)$, implying $M = N \cong \mathbb{Z}_2$. Let T/M be a minimal normal subgroup of A/M. By the maximality of M, T/M is an elementary abelian p-group. Let P_1 be a Sylow p-subgroup of T. Since $M \cong \mathbb{Z}_2$, one has $T = P_1 \times M$ and P_1 is characteristic in T. Then $P_1 \trianglelefteq A$ since $T \trianglelefteq A$. Thus, A always has a minimal normal p-subgroup. Without loss of the generality, assume that $N \cong \mathbb{Z}_p$ or $\mathbb{Z}_p^2$. If N is a Sylow p-subgroup of A, then the quotient graph X_N of X relative to N is a cubic edge-transitive graph of order 4. It follows that $X_N \cong K_4$, contradicting that X is bipartite. As a result, one may conclude that there exists no connected cubic edge-transitive graphs of order $4p$. Further, $N \cong \mathbb{Z}_p$ is the maximal normal p-subgroup of A and $|X| = 4p^2$. Then the quotient graph X_N of X relative to N is a cubic edge-transitive graph of order $4p$. A contradiction occurs again. □

Lemma 3.2 *Let $p > 7$ be a prime and X a connected cubic graph of order $12p$ or $12p^2$. If $A \leq \mathrm{Aut}(X)$ acts transitively on the edge-set of X, then A is non-solvable.*

Proof. Suppose to the contrary that A is solvable. Let H be a minimal normal subgroup of A. Then H must have more than two orbits on $V(X)$ and the quotient graph X_H of X relative to H is still a cubic graph with A/H as an edge-transitive group of automorphisms. It follows that $H \cong \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_p$ or $\mathbb{Z}_p \times \mathbb{Z}_p$. If $H \cong \mathbb{Z}_3$, then the quotient graph X_H has order $4p$ or $4p^2$, which is impossible by Lemma 3.1. Assume that $H \cong \mathbb{Z}_2$. Let T be a maximal normal 2-subgroup of A. Then T is intransitive on each partition set of X. By Proposition 2.2 and 2.1, T is semiregular on $V(X)$, implying $T = H \cong \mathbb{Z}_2$. Let K/T be a minimal normal subgroup of A/T. By the maximality of T, K/T is an elementary abelian q-group with $q = 3$ or p. Let Q be a Sylow q-subgroup of K. Since $T \cong \mathbb{Z}_2$, one has $K = Q \times T$ and Q is characteristic in K. Then $Q \trianglelefteq A$ because $K \trianglelefteq A$. If $q = 3$ then Q is a normal 3-subgroup of A. This is impossible. Thus, Q is a p-subgroup, and hence A always has a minimal normal p-subgroup. Without loss of the generality, assume that $H \cong \mathbb{Z}_p$ or $\mathbb{Z}_p^2$. If H is a Sylow p-subgroup of A, then the quotient graph X_N of X relative to N is a cubic edge-transitive graph of order 12. However, from [6, 7] we know that there are no cubic edge-transitive graphs of order 12, a contradiction. As a result, one may conclude that there exists no connected cubic graphs of order $12p$ with a solvable edge-transitive automorphism group. Further, $H \cong \mathbb{Z}_p$ is the maximal normal p-subgroup of A and $|X| = 12p^2$. Then the quotient graph X_H is a cubic graph of order $12p$ with A/H as a solvable edge-transitive group of automorphisms. This is a contradiction. □

Let G be a non-abelian simple group and Z an abelian group. We call an extension E of Z by G a *central extension* of G if $Z \leq Z(E)$. If E is perfect, that is, the derived group $E' = E$, we call E is *covering group* of G. Schur proved that for every simple group G there is a unique maximal covering group M such that every covering group of G is a factor group of M. This group M is called the *full covering group* of G, and the center of M is called the *Schur multiplier* of G, denoted by Mult(G).

Proof of Theorem 1.1 Let $p \leq 7$. Then X has order $24, 36, 48, 60, 84, 108$, 300 or 588. By [6, 7], X is isomorphic to the 2-regular graphs F024, F048, F060, F084 or F108. (The notations are from [6].)

Let $p > 7$. Let $A = \text{Aut}(X)$. By [41, 42] and [34, Proposition 2.4], $|A| \mid 2^9 \cdot 3^2 \cdot p^\ell$ with $\ell = 1$ or 2. By Lemma 3.2, A is non-solvable. Then A has a non-abelian main factor M/N. Since $|M/N| \mid 2^9 \cdot 3^2 \cdot p^\ell$ and $p \geq 11$, by [24, pp.12–14], M/N is a simple group, and $M/N \cong \text{PSL}(2, 17)$ which has order $2^4 \cdot 3^2 \cdot 17$. It follows that $p = 17$ and $3 \nmid |N|$. If $\ell = 1$, then by [6, 7], X is isomorphic to the 4-regular F204. Let $\ell = 2$. Then $|V(X)| = 12 \cdot 17^2$.

In this case, we first prove the following claim.

Claim: A has no normal subgroups with order $2^r \cdot 3^2 \cdot 17$ for some integer r.

Suppose to the contrary that T is a normal subgroup of A with order $2^r \cdot 3^2 \cdot 17$ for some integer r. Since $17^2 \nmid |T|$, T has more than two orbits in $V(X)$. By Propositions 2.2 and 2.1, T is semiregular and hence $|T| \mid |V(X)|$, that is, $2^r \cdot 3^2 \cdot 17 \mid 12 \cdot 17^2$, a contradiction.

Thus, the claim is true. As a result, $17 \mid |N|$. Since if not, then M has order $2^r \cdot 3^2 \cdot 17$, which is impossible. Since $3 \nmid |N|$, N has more than two orbits in $V(X)$. By Propositions 2.2 and 2.1, N is semiregular and the quotient graph X_N of X relative to N is still a cubic edge-transitive graph. It follows that $|N| \mid 34$, and hence $N \cong \mathbb{Z}_{17}, \mathbb{Z}_{34}$ or D_{34}. It is easily seen that Aut(N) is solvable. Set $C = C_M(N)$. Then M/C is isomorphic to a subgroup of Aut(N), implying that M/C is solvable. Let $N \cong D_{34}$. Then $C \cap N = 1$ and hence $C \cong CN/N$. Since M/N is simple, $CN/N \trianglelefteq M/N$ implies that $CN/N = 1$ or M/N, that is, $C = 1$ or PSL$(2,17)$. Since M/C is solvable, $C \cong$ PSL$(2,17)$ and hence $M = N \times C$. Clearly, C is characteristic in M. Then $C \trianglelefteq A$ because $M \trianglelefteq A$. This is contrary to the Claim. Let $N \cong \mathbb{Z}_{17}$ or $\mathbb{Z}_{34}$. Then $N \leq C$. Since M/C is solvable and $M/N \cong$ PSL$(2,17)$, one has $1 \neq C/N \trianglelefteq M/N$, implying $M = C$. It follows that N is in the center of M. Let M' be the derived subgroup of M. Since M/N is non-abelian simple, $M'N/N = M/N$, implying $M'/(M' \cap N) \cong M/N \cong$ PSL$(2,17)$. If $N \leq M'$, then $M' = M$, and hence M is a covering group of PSL$(2,17)$, implying $|N| \mid |PSL(2,17)|$. However, from [12] we know Mult(PSL$(2,17)) \cong \mathbb{Z}_2$, a contradiction. Thus, $N \not\leq M'$, and if $N \cong \mathbb{Z}_{17}$ then $M' \cap N = 1$, if $N \cong \mathbb{Z}_{34}$, then $M' \cap N = 1$, $\mathbb{Z}_2$ or $\mathbb{Z}_{17}$. If $M' \cap N = 1$ or $\mathbb{Z}_2$ then M' has order $2^4 \cdot 3^2 \cdot 17$ or $2^5 \cdot 3^2 \cdot 17$, contrary to the Claim. Let $M' \cap N = \mathbb{Z}_{17}$. In this case, let M'' be the derived subgroup of M' and set $L = M' \cap N$. Since N is in the center of M, L is in the center of M'. Since $M'/L \cong$ PSL$(2,17)$, one has $M''L/L = M'/L$, namely, $M' = M''L$. If $L \leq M''$ then $M'' = M'$, and hence M' is a covering group of PSL$(2,17)$, implying $L \leq$Mult(PSL$(2,17)) \cong \mathbb{Z}_2$, a contradiction. Thus, $L \not\leq M''$ and hence $L \cap M'' = 1$ because $L \cong \mathbb{Z}_{17}$. It follows that

$M'' \cong M''L/L = M'/L \cong \mathrm{PSL}(2,17)$, and hence $M' = L \times M''$. Since M' is characteristic in M and M'' is characteristic in M', M'' is characteristic in M. Then $M'' \trianglelefteq A$ because $M \trianglelefteq A$, contrary to the Claim. □

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