GROUPS WITH MAXIMAL IRREDUNDANT COVERS AND MINIMAL BLOCKING SETS

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ABSTRACT. Let n be a positive integer. Denote by $\mathrm{PG}(n,q)$ the n-dimensional projective space over the finite field $\mathbb{F}_q$ of order q. A blocking set in $\mathrm{PG}(n,q)$ is a set of points that has non-empty intersection with every hyperplane of $\mathrm{PG}(n,q)$. A blocking set is called minimal if none of its proper subsets are blocking sets. In this note we prove that if $\mathrm{PG}(n_i,q)$ contains a minimal blocking set of size k_i for $i \in \{1,2\}$, then $\mathrm{PG}(n_1+n_2+1,q)$ contains a minimal blocking set of size k_1+k_2-1. This result is proved by a result on groups with maximal irredundant covers.

1. Introduction and Results

Let G be a group. A set $\mathcal{S}$ of proper subgroups of G is called a cover for G whenever $G = \bigcup_{H\in\mathcal{S}} H$. The cover $\mathcal{S}$ is called irredundant if no proper sub-collection of $\mathcal{S}$ is a cover for G. The cover $\mathcal{S}$ is called an n-cover if $|\mathcal{S}| = n$. The cover $\mathcal{S}$ is called maximal if each member of $\mathcal{S}$ is a maximal subgroup of the group G. The cover $\mathcal{S}$ is called core-free if the core of $D = \bigcap_{H\in\mathcal{S}} H$ in G is trivial, i.e. $D_G = \bigcap_{g\in G} g^{-1}Dg$ is the trivial subgroup of G. The cover $\mathcal{S}$ for a group is called a $\mathfrak{C}_n$-cover if $\mathcal{S}$ is a maximal irredundant core-free n-cover.

Let n be a positive integer. Denote by $\mathrm{PG}(n,q)$ the n-dimensional projective space over the finite field $\mathbb{F}_q$ of order q. A blocking set in $\mathrm{PG}(n,q)$ is a set of points that has non-empty intersection with every hyperplane of $\mathrm{PG}(n,q)$. A blocking set is called minimal if none of its proper subsets are blocking sets.

There is a well-known relationship between minimal blocking sets in $\mathrm{PG}(n,q)$ and irredundant covers of the abelian group $\mathbb{F}_q^{n+1}$, the direct product of $n+1$ copies of $\mathbb{F}_q$ (see e.g., [1]). In fact a minimal blocking set of size k exits in $\mathrm{PG}(n,q)$ if and only if an irredundant k-cover for $\mathbb{F}_q^{n+1}$ whose all members are hyperplane exists (see e.g., Proposition 2.2 of [1]).

The main results of this note are the following. Theorem 1.1 is inspired by the proof of Lemma 3.1 of [3].

Theorem 1.1. *Let $C_1 = \{M_1,\dots,M_m\}$ and $C_2 = \{N_1,\dots,N_n\}$ be irredundant m- and n-covers for two groups G_1 and G_2, respectively. Suppose that M_1 and N_1 are normal subgroups of G_1 and G_2, respectively; and assume that a and b are elements of $G_1\backslash M_1$ and $G_2\backslash N_1$, respectively, such that M_1a and N_1b have the same order p for some prime number p. Then*

$$C = \{(M_1\times N_1)\langle(a,b)\rangle, M_i\times G_2, G_1\times N_j \mid i=2,\dots,m;\ j=2,\dots,n\}$$

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is an irredundant $(n+m-1)$*-cover for* $G_1 \times G_2$ *with intersection* $D_1 \times D_2$*, where* $D_1 = \cap_{i=1}^m M_i$*,* $D_2 = \cap_{j=1}^n N_i$ *and* $(M_1 \times N_1)\langle(a,b)\rangle$ *is the subgroup of* $G_1 \times G_2$ *generated by* $M_1 \times N_1$ *and the element* (a,b)*. In particular, if both* C_1 *and* C_2 *are maximal, then* C *is a maximal cover for* $G_1 \times G_2$*.*

Corollary 1.2. *Let* $\mathrm{PG}(n_i,q)$ *contain a minimal blocking set of size* k_i *for* $i \in \{1,2\}$*. Then* $\mathrm{PG}(n_1+n_2+1,q)$ *contains a minimal blocking set of size* k_1+k_2-1*.*

2. Proofs

Proof of Theorem 1.1. It is clear that C is an $(m+n-1)$-cover for $G_1 \times G_2$. We now prove that C is irredundant. First, $(M_1 \times N_1)\langle(a,b)\rangle$ is an irredundant member of C: for C_1 and C_2 are both irredundant, there exist elements $x \in M_1 \setminus \bigcup_{i=2}^m M_i$ and $y \in N_1 \setminus \bigcup_{j=2}^n N_i$. It follows that the element (x,y), among the members of C, belongs only to $(M_1 \times N_1)\langle(a,b)\rangle$ and so $(M_1 \times N_1)\langle(a,b)\rangle$ is an irredundant member. Now we show that each $M_i \times G_2$ is an irredundant member for the cover (for each $i = 2,\dots,m$). Without loss of generality and for convenience, we prove only the latter statement for $i=2$, the other cases are similar. Let $x_0 \in M_2 \setminus \bigcup_{\substack{i=1\\i\neq 2}}^m M_i$ and $y_0 \in N_1 \setminus \bigcup_{j=2}^n N_j$. Then (x_0,y_0), among the members of C, could only possibly belong to $(M_1 \times N_1)\langle(a,b)\rangle$. If it is possible, then $x_0 \in M_1 a^\ell$ and $y_0 \in N_1 b^\ell$ for some integer ℓ. Since $x_0 \notin M_1$, $a^\ell \notin M_1$ and so p does not divide ℓ. This implies that $b^\ell \notin N_1$. Thus $y_0 \in N_1 \cap N_1 b^\ell = \emptyset$, a contradiction. Thus $M_2 \times G_1$ is an irredundant member of C. By a similar argument, one may prove that each $G_1 \times N_j$ is an irredundant member for the cover (for each $j = 2,\dots,n$). This completes the proof of irredundancy of the cover C.
Now by Lemma 2.2(b) of [2], $D_1 = \bigcap_{i=2}^m M_i$, $D_2 = \bigcap_{j=2}^n N_j$ and

$$\bigcap_{S\in C} S = \Big(\bigcap_{i=2}^{m} M_i \times G_2\Big) \bigcap \Big(\bigcap_{j=2}^{n} G_1 \times N_j\Big).$$

It follows that the intersection of the cover C is $D_1 \times D_2$.
For the last statement, note that it follows from the hypothesis that $|G_1 : M_1| = |G_2 : N_1| = p$, as we are assuming both M_1 and N_1 are maximal. Thus $(M_1 \times N_1)\langle(a,b)\rangle$ has prime index p in $G_1 \times G_2$ and so it is a maximal subgroup of $G_1 \times G_2$. The other members of the cover C are clearly maximal subgroups of $G_1 \times G_2$. This completes the proof of the last statement. □

Corollary 2.1. *Let* $V(n_i,q)$ $(i=1,2)$ *be the* n_i*-dimensional vector space over the finite field of order* q*. If* $C_1 = \{M_1,\dots,M_m\}$ *and* $C_2 = \{N_1,\dots,N_n\}$ *be irredundant* m*- and* n*-covers for the abelian groups* $V(n_1,q)$ *and* $V(n_2,q)$*, respectively; and assume that* a *and* b *are arbitrary elements of* $G_1 \setminus M_1$ *and* $G_2 \setminus N_1$*, respectively. Then*

$$C = \{(M_1 \times N_1)\langle(a,b)\rangle, M_i \times V(n_2,q), V(n_1,q) \times N_j \mid i = 2,\dots,m;\ j = 2,\dots,n\}$$

is an irredundant $(n+m-1)$*-cover for* $V(n_1,q) \times V(n_2,q)$ *with intersection* $D_1 \times D_2$*, where* $D_1 = \cap_{i=1}^m M_i$ *and* $D_2 = \cap_{j=1}^n N_i$*. In particular, if all members of* C_1 *and* C_2 *are hyperplanes of the corresponding spaces, then all members of* C *are hyperplanes of* $V(n_1,q) \times V(n_2,q)$*.*

Proof of Corollary 1.2. It follows from Proposition 2.2 of [1] and Corollary 2.1.
□

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