Coupled choosability of near-outerplane graphs

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Abstract

It is proved that if G is a plane embedding of a K_4-minor-free graph, then G is coupled 5-choosable; that is, if every vertex and every face of G is given a list of 5 colours, then each of these elements can be given a colour from its list such that no two adjacent or incident elements are given the same colour. Using this result it is proved also that if G is a plane embedding of a $K_{2,3}$-minor-free graph or a $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graph, then G is coupled 5-choosable. All results here are sharp, even for outerplane graphs.

Keywords: Series-parallel graph; Minor-free graph; Outerplanar graph.

1 Introduction

All graphs considered in this paper are simple; that is, they do not contain loops or multiple edges. *List-colourings*, in which each element is coloured from its own list of colours, were introduced independently by Vizing [15] and by Erdős, Rubin and Taylor [4]. Formally, let $G = (V, E, F)$ be a plane graph. A *list-assignment* L to the elements of G is the assignment of an unordered list $L(z)$ of colours to each element z of G. If G has a list-assignment L, then a *coupled list-colouring* is an assignment of a colour to every vertex v and every face f from its own list $L(v)$ or $L(f)$ of colours. A coupled list-colouring is *proper* if no two adjacent or incident elements are given the same colour. Furthermore, G is *coupled k-choosable* if G has a proper coupled list-colouring from all possible lists $L(z)$ with $|L(z)| \geq k$ for every element $z \in V \cup F$. The smallest integer k such that G is coupled k-choosable is the *coupled list-chromatic number* or *coupled choosability* $\mathrm{ch_{vf}}(G)$ of G. It is clear that $\mathrm{ch_{vf}}(G) \geq \chi_{\mathrm{vf}}(G)$, where $\chi_{\mathrm{vf}}(G)$ is the coupled chromatic number. The (vertex) choosability $\mathrm{ch}(G)$ and edge-choosability $\mathrm{ch}'(G)$ of G are defined in a similar manner.

It is well known that a graph is outerplanar if and only if it is both K_4-minor-free and $K_{2,3}$-minor-free. We will call a graph *near-outerplane* if it is a plane embedding of a K_4-minor-free graph or a $K_{2,3}$-minor-free graph. In fact, in Theorem 1.1 we will replace the class of $K_{2,3}$-minor-free graphs by the slightly larger class of $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graphs.

By an abuse of terminology we will call two elements *neighbours* if they are adjacent or incident, since no two such elements can be given the same colour. All other terminology is standard, as defined in the references: for example [2, 19].

Ringel [12] introduced *simultaneous colourings*, in which more than one type of element is coloured, and he conjectured that the vertices and faces of a plane graph can be coloured with six colours, which was proved by Borodin [1]. Wang and Liu [17] proved that the vertices and faces of an outerplane graph can be coloured with five colours. (Examples that attain this bound are given immediately after Theorem 1.1.) Simultaneous list-colourings are considered in [5]. In this paper we will prove that if G is a near-outerplane graph, then G is coupled 5-choosable.

Melnikov [11] conjectured that if G is a plane graph with maximum degree Δ, then the number of colours needed for an edge-face colouring of G is at most $\Delta+3$. This was proved independently by Sanders and Zhao [13] and by Waller [16]. Kronk and Mitchem [9] proposed the *Entire Colouring Conjecture*, which states that if G is a plane graph, then the number of colours needed to colour the vertices, edges and faces of G is at most $\Delta+4$. This is still an open problem for graphs with $\Delta = 4$ or 5; see [10] for a proof when $\Delta \leq 3$ and [14] for a proof when $\Delta \geq 6$. Edge-face choosability and entire choosability of near-outerplane graphs are considered in [6, 7], respectively.

Theorem 1.1. *Let G be a near-outerplane graph. Then* $\mathrm{ch}_{\mathrm{vf}}(G) \leq 5$. *In particular,*

(*i*) *if* $\Delta = 0$, *then* $\mathrm{ch}_{\mathrm{vf}}(G) = 2$;

(*ii*) *if* $\Delta = 1$, *then* $\mathrm{ch}_{\mathrm{vf}}(G) = 3$;

(*iii*) *if* $\Delta = 2$, *then*

$$\mathrm{ch}_{\mathrm{vf}}(G) = \begin{cases} 5 & \textit{if } G \textit{ contains an odd cycle;} \\ 4 & \textit{if } G \textit{ contains an even cycle but no odd cycle;} \\ 3 & \textit{if } G \textit{ is cycle-free.} \end{cases}$$

Note that Theorem 1.1 is sharp, even for $\chi_{\mathrm{vf}}(G)$ and even for the smaller class of outerplane graphs, since any graph with an odd cycle as a block attains the upper bound. We will make use of the following two theorems. Theorem 1.2 is a slight extension of a theorem of Dirac [3]. Theorem 1.3 summarises the results for edge choosability of near-outerplanar graphs. In particular we will make use of the well-known result [4, 15] that $\mathrm{ch}(C_4) = \mathrm{ch}'(C_4) = 2$, which is included in Theorem 1.3 since choosability and edge-choosability are equivalent when $\Delta = 2$.

Theorem 1.2. [18] *A K_4-minor-free graph G with $|V(G)| \geq 4$ has at least two nonadjacent vertices with degree at most 2.*

Theorem 1.3. [8] *If G is a near-outerplanar graph with maximum degree Δ, then $\mathrm{ch}'(G) = \chi'(G) = \Delta$ with the exception that if $\Delta = 2$ and G has a component that is an odd cycle, then $\mathrm{ch}'(G) = \chi'(G) = 3 = \Delta + 1$.*

Clearly, Theorem 1.1 holds when $\Delta \leq 2$; see [5]. We will now prove Theorem 1.1 in general. In Section 2 we will prove Theorem 1.1 for plane embeddings of K_4-minor-free graphs, which is restated in Theorem 2.1. In Section 3 we will use Theorem 2.1 to prove Theorem 1.1 for plane embeddings of $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graphs, which is restated in Theorem 3.1. This will complete the proof of Theorem 1.1.

2 Results for K_4-minor-free graphs

Let the *bounding cycle* of a 2-connected block B of a plane graph G be the cycle of B that has the largest area inside it; that is, in a plane embedding of B the bounding cycle forms the boundary of the outer face of B.

Lemma 2.1. [5, 6, 7] *Every component C of a plane graph with $|V(C)| \geq 3$ is either 2-connected or has an end-block B such that no interior face of B has a block of C embedded in it.*

Proof. The proof is left as an exercise for the reader. □

Let C be a component of a plane embedding of a K_4-minor-free graph G such that no interior face of C has another component of G embedded in it. If C is 2-connected, then let $B = C$ and let z_0 be any vertex of maximum degree in C; otherwise, by Lemma 2.1, let B be an end-block of C with cut-vertex z_0 such that no interior face of B has a block of C embedded in it.

If B contains a vertex with degree at least 3 in G, then let B_1 be the graph whose vertices are the vertices of B that have degree at least 3 in G, where two vertices are adjacent in B_1 if and only if they are connected in

G by an edge or by a path whose interior vertices have degree 2. It is clear that B_1 is a minor of B.

If $u, x \in V(B)$, then let P_{ux} be the set of paths in B of length 1 or 2 between u and x that contain no interior vertex of degree at least 3; that is, if $uvx \in P_{ux}$ then $d_G(v) = 2$. Also, let p_{ux} be the number of paths in P_{ux}.

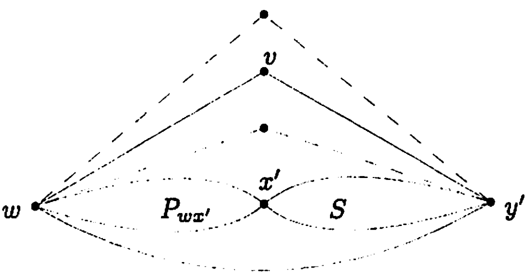

Figure 1

Lemma 2.2. *Suppose that B does not contain a vertex of degree 1 or two adjacent vertices of degree 2 in G. Then the graph B_1 exists and does not contain a vertex of degree 0. If B_1 does not contain a vertex of degree 1, then B_1 contains a vertex u of degree 2 that is adjacent in B_1 to x and y say, where $p_{ux} + p_{uy} = d_G(u) \geq 3$, and where $p_{uy} \geq 2$. Moreover, no two paths in P_{uy} bound a region that has a path not in P_{uy} embedded in it, and if $p_{ux} \geq 2$, then no two paths in P_{ux} bound a region that has a path not in P_{ux} embedded in it also.*

Proof. If B does not contain a vertex of degree 1, then $B \not\cong K_2$, and if B does not contain two adjacent vertices of degree 2 in G, then B is not a cycle. So B has at least two vertices with degree at least 3, and so it follows that B_1 exists and does not contain a vertex of degree 0. Since B_1 is a minor of B, it follows that B_1 is K_4-minor-free.

Suppose that B_1 does not contain a vertex of degree 1. Then either $B_1 \cong K_3$, or, by Theorem 1.2, B_1 has at least two nonadjacent vertices with degree exactly 2.

Let w be a vertex of degree 2 in B_1 that is adjacent in B_1 to x' and y'. Then, by the definition of B_1 and since B does not contain two adjacent vertices of degree 2 in G, it follows that $p_{wx'}, p_{wy'} \geq 1$ and $p_{wx'} + p_{wy'} = d_G(w) \geq 3$. Furthermore, since $d_G(w) \geq 3$, we may assume without loss of generality that $p_{wy'} \geq 2$.

If no two paths in $P_{wy'}$ bound a region that has a path not in $P_{wy'}$ embedded in it and no two paths in $P_{wx'}$ bound a region that has a path not in $P_{wx'}$ embedded in it, then we may set $w = u$, $x' = x$ and $y' = y$ and the proof is complete. So we may assume without loss of generality that there is a region R bounded by two paths in $P_{wy'}$ that has a path $w \dots y'$ not in $P_{wy'}$ embedded in it. Since $p_{wx'} + p_{wy'} = d_G(w)$ it follows that every such path in R must contain x'. So x' is embedded in R and it follows that $B \setminus P_{wy'}$ is embedded in R also, and so the bounding cycle of B consists of two paths in $P_{wy'}$. Let S be the subgraph of B obtained by deleting w and all its neighbours of degree 2 in B. An example is shown in Figure 1, where $R = wvy'w$, where the dashed edges may or may not be present, and if B is an end-block, then $y' = z_0$.

Since w is adjacent in B_1 to y', and since $B_1 \cong K_3$ or has at least two nonadjacent vertices with degree exactly 2, then there is a vertex $u \neq y'$ in S such that $d_{B_1}(u) = 2$, where $u = x'$ if $d_{B_1}(x') = d_{B_1}(y') = 2$. Let u be adjacent in B_1 to x and y. Then, by what we have proved about w, the result follows since every region bounded by paths in P_{ux} or P_{uy} is inside the bounding cycle of B. This completes the proof of Lemma 2.2. □

We will now prove Theorem 1.1 for plane embeddings of K_4-minor-free graphs, which is restated in the following theorem.

Theorem 2.1. *Let G be a plane embedding of a K_4-minor-free graph. Then* $\mathrm{ch_{vf}}(G) \leq 5$.

Proof. Suppose, if possible, that G is a plane embedding of a K_4-minor-free graph with the smallest number of vertices such that $\mathrm{ch_{vf}}(G) > 5$. Assume that every vertex v and every face f of G is given a list $L(v)$ or $L(f)$ of five colours such that G has no proper coupled colouring from these lists. Clearly G has neither a trivial component nor a K_2 component; so every component C of G has at least three vertices. Let C and B be as defined before Lemma 2.2.

Claim 2.1. *G does not contain a vertex of degree* 1.

Proof. Suppose that u is a vertex of degree 1 in G. Let $H = G - u$. By hypothesis H has a proper coupled colouring from its lists. Since u has two coloured neighbours and a list of five colours, it follows that u can be coloured from its list. This contradiction proves Claim 2.1. □

Claim 2.2. *If G contains a vertex u of degree* 2, *then the neighbours of u are adjacent.*

Proof. Suppose that G contains a vertex u of degree 2 that is adjacent to x and y, where $xy \notin E(G)$. Let H be obtained from G by replacing the path

xuy with the edge xy. By hypothesis H has a proper coupled colouring from its lists. We can now colour u since it has at most four coloured neighbours and a list of five colours. This contradiction completes the proof of Claim 2.2. □

Claim 2.3. *B does not contain a triangle $xuyx$, where $xuyx$ bounds a face in G and where u has degree 2 in G.*

Proof. Suppose that B contains a triangle $xuyx$, where $xuyx$ bounds a face f in G and where u has degree 2 in G. Let $H = G - u$ where the face in H in which u was embedded is given the same list as the face in G that has xuy as part of its boundary and is different from f. By hypothesis H has a proper coupled colouring from its lists. We can now colour f and then u since each has at most four coloured neighbours at the time of its colouring. This contradiction completes the proof of Claim 2.3. □

Claim 2.1 implies that $B \not\cong K_2$ and Claims 2.2 and 2.3 imply that B is not a cycle; so B has at least two vertices with degree at least 3 and $d_G(z_0) \geq 3$. Let B_1 be as defined before Lemma 2.2.

Claim 2.4. *B_1 is not K_4-minor-free.*

Proof. Since B has at least two vertices with degree at least 3, it follows that B_1 has no vertex of degree 0. Suppose that x is a vertex of degree 1 in B_1. Then x is adjacent in B_1 to z_0. By the definition of B_1 and by Claims 2.2 and 2.3, it follows that $d_G(x) \geq 3$, so that $p_{xz_0} \geq 3$, and that $xz_0 \in E(G)$ and every path in B between x and z_0 is in P_{xz_0}. So, by the definition of B, it follows that B contains a face that is bounded by a triangle xvz_0x, where $d_G(v) = 2$. However, Claim 2.3 shows that this is impossible. So B_1 has no vertex of degree 1.

In view of Claims 2.1 and 2.2, it follows from Lemma 2.2 that B_1 contains a vertex u of degree 2 that is adjacent in B_1 to x and y say, such that there are two paths in P_{uy} that bound a face in B that is a triangle $uvyu$, where $d_G(v) = 2$. However, Claim 2.3 shows that this is impossible. This contradiction completes the proof of Claim 2.4. □

Since B_1 is a minor of G, Claim 2.4 implies that G is not K_4-minor-free. This contradiction completes the proof of Theorem 2.1. □

3 Results for $(\bar{K}_2+(K_1 \cup K_2))$-minor-free graphs

We will make use of Theorem 2.1. For each uncoloured element z in G, let $L'(z)$ denote the list of usable colours for z; that is, $L'(z)$ denotes $L(z)$ minus any colours already used on neighbours of z in G.

Let C be a component of a plane embedding of a $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graph G such that no interior face of C has another component of G embedded in it. If C is 2-connected, then let $B = C$ and let z_0 be any vertex of maximum degree in C; otherwise, by Lemma 2.1, let B be an end-block of C with cut-vertex z_0 such that no interior face of B has a block of C embedded in it.

Lemma 3.1. *Let G be a $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graph. Then each block of G is either K_4-minor-free or else isomorphic to K_4.*

Proof. Suppose that B is a block of G that has a K_4 minor. Since $\Delta(K_4) = 3$, it follows that B has a subgraph B' that is homeomorphic to K_4. If an edge of K_4 is subdivided, or if a path is added joining two vertices of K_4, then a $\bar{K}_2 + (K_1 \cup K_2)$ minor is formed. So $B' \cong K_4$ and $B = K_4$. □

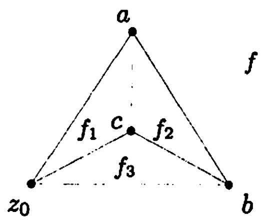

Figure 2

Lemma 3.2. *Let G be a plane embedding of K_4, as shown in Figure 2. If both f and z_0 are precoloured, and each of the elements a, b, c, f_1, f_2, f_3 has a list of at least 3, 3, 4, 3, 4, 3 usable colours respectively, then any given colouring of f and z_0 can be extended to the remaining vertices and faces of G.*

Proof. Each of the remaining elements

$$a, b, c, f_3, f_1, f_2 \tag{1}$$

has a list of at least 3, 3, 4, 3, 3, 4 usable colours respectively. Note that these elements are equivalent to a 4-cycle abf_3f_1a where c and f_2 are the interior and exterior faces.

If possible, give b and f_1 the same colour. At this point, each of the remaining elements

$$a, f_3, c, f_2 \tag{2}$$

has a list L'' of at least 2, 2, 3, 3 usable colours respectively. If possible, give a and f_3 the same colour. The remaining elements can now be coloured in the order (2). So we may assume that $L''(a) \cap L''(f_3) = \emptyset$ so that $|L''(a) \cup L''(f_3)| \geq 4$. Now either $|L''(f_2)| \geq 4$, or else a or f_3 can be given a colour that is not in $L''(f_2)$. In each case the remaining elements can be coloured in the order (2). So we may assume that this is not possible so that $L'(b) \cap L'(f_1) = \emptyset$, and, by symmetry, that $L'(a) \cap L'(f_3) = \emptyset$.

If $L'(f_1) = L'(f_3)$, then either $|L'(f_1)| \geq 4$, or else f_2 can be given a colour that is not in $L'(f_1)$. In each case colour f_2. Since $L'(a) \cap L'(f_1) = \emptyset$ the remaining elements can now be coloured in the order (1). So we may assume that $L'(f_1) \neq L'(f_3)$, and similarly that $L'(f_1) \neq L'(a)$, $L'(b) \neq L'(a)$, and $L'(b) \neq L'(f_3)$.

So give colours to c and f_2. The remaining elements are equivalent to a 4-cycle. Since $L'(a) \cap L'(f_3) = \emptyset$ and $L'(b) \cap L'(f_1) = \emptyset$, it follows that any colour given to either c or f_2 is in at most two of $L'(a)$, $L'(b)$, $L'(f_1)$, $L'(f_3)$. If each remaining element has a list of at least two usable colours, then the result follows from Theorem 1.3. So we may assume that at least one remaining element has only one usable colour in its list. This means that each of the colours on c and f_2 was in the list of usable colours of one remaining element.

Suppose that exactly one remaining element, say f_1, has only one usable colour in its list. Then each of a, b, f_3 has at least 2, 3, 2 usable colours in its list respectively, and so the remaining elements can be coloured in the order f_1, a, f_3, b. So we may assume that there are two remaining elements each of which has only one usable colour in its list. Since these elements are adjacent, then, by symmetry, we may assume that these elements are f_1 and f_3. Since $L'(f_1) \neq L'(f_3)$, and since each of a and b has at least three usable colours in its list, it follows that the remaining elements can be coloured in the order f_1, f_3, a, b. In every case the colouring can be completed, which proves Lemma 3.2. □

We will now prove Theorem 1.1 for plane embeddings of $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graphs, which is restated in the following theorem.

Theorem 3.1. *Let G be a plane embedding of a $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graph. Then* $\mathrm{ch}_{\mathrm{vf}}(G) \leq 5$.

Proof. Suppose, if possible, that G is a plane embedding of a $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graph with the smallest number of vertices such that $\mathrm{ch}_{\mathrm{vf}}(G) > 5$. Assume that every vertex v and every face f of G is given a list $L(v)$ or $L(f)$ of five colours such that G has no proper coupled colouring from these lists. Clearly G has neither a trivial component nor a K_2 component; so every component C of G has at least three vertices. Let C and B be as defined before Lemma 3.1.

Claim 3.1. $B \not\cong K_4$.

Proof. Suppose that $B \cong K_4$ and let the elements of B be labelled as in Figure 2. Then, by hypothesis, $G-(B-z_0)$ has a proper coupled colouring from its lists in which both f and z_0 are coloured. So each of the remaining elements a, b, c, f_1, f_2, f_3 has a list of at least 3, 3, 4, 3, 4, 3 usable colours respectively, and so it follows from Lemma 3.2 that G can be coloured from its lists. This contradiction proves Claim 3.1. □

By Lemma 3.1 and Claim 3.1, it follows that B is K_4-minor-free. Claim 2.1 implies that $B \not\cong K_2$ and Claims 2.2 and 2.3 imply that B is not a cycle; so B has at least two vertices with degree at least 3 and $d_G(z_0) \geq 3$. Let B_1 be as defined before Lemma 2.2. By Claim 2.4, B_1 is not K_4-minor-free. However, since B_1 is a minor of B this implies that B is not K_4-minor-free. This contradiction completes the proof of Theorem 3.1. □

Since we have now proved Theorems 2.1 and 3.1, this completes the proof of Theorem 1.1.

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