Bounds of eigenvalues of a nontrivial bipartite graph

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Abstract: Let G be a simple graph with n vertices and m edges, and let λ_1 and λ_2 denote the largest and second largest eigenvalues of G. For a nontrivial bipartite graph G, we prove that,

(i) $\lambda_1 \leq \sqrt{m - \frac{3-\sqrt{5}}{2}}$, where equality holds if and only if $G \cong P_4$;

(ii) If $G \ncong P_n$, then $\lambda_1 \leq \sqrt{m - \frac{5-\sqrt{17}}{2}}$, where equality holds if and only if $G \cong K_{2,3} - e$;

(iii) If G is connected, then $\lambda_2 \leq \sqrt{m - 4\cos^2\left(\frac{\pi}{n+1}\right)}$, where equality holds if and only if $G \cong P_n$, $2 \leq n \leq 5$;

(iv) $\lambda_2 \geq \frac{\sqrt{5}-1}{2}$, where equality holds if and only if $G \cong P_4$;

(v) If G is connected and $G \ncong P_n$, then $\lambda_2 \geq \sqrt{\frac{5-\sqrt{17}}{2}}$, where equality holds if and only if $G \cong K_{2,3} - e$.

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1 Introduction

Graphs considered in this note are finite and simple. Undefined notation and terminology will follow those in [1]. Throughout this note, G denotes a simple graph with n vertices and m edges, $\Delta(G)$ denotes the maximum degree of G and $A(G)$ denotes the adjacency matrix of G. The eigenvalues of $A(G)$ are called the eigenvalues of G. Since $A(G)$ is symmetric, all of its eigenvalues are real. We assume, without loss of generality, that they are ordered in decreasing order, i.e., $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$. When the graph G is understood in the context, we may omit G and simply use $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ to denote the eigenvalues of G. For a graph G that is not a forest, the *girth* of G is the length of the shortest cycle of G.

A bipartite graph without isolated vertices is *nontrivial* if it is not isomorphic to a complete bipartite graph. In 1970, Nosal [7] proved that if $\lambda_1 > \sqrt{m}$, then G contains a triangle. This yields an upper bound for λ_1, among bipartite graphs, as stated in Theorem 1.1. We list several other bounds on the eigenvalues of a bipartite graphs below.

Theorem 1.1 *(Nosal [7], also Theorem 3.9 of [3]) If G is a bipartite graph, then $\lambda_1 \leq \sqrt{m}$.*

Theorem 1.2 *(Collatz and Sinogowitz, [2]) If G is a tree, then $\lambda_1 \leq \sqrt{n-1}$.*

Theorem 1.3 *(Peterović, [8]) A connected bipartite graph G has the property $\lambda_2 \leq 1$ if and only if G is an induced subgraph of any of the graphs $G_1 - G_7$ (see Fig. 1 of [8]).*

Aside from the bounds above, little is known on the second and the third largest eigenvalues of a nontrivial bipartite graph. In this note, we shall investigate new bounds for the first, second, and third largest eigenvalues of a nontrivial bipartite graph.

2 Lower Bounds

In this section, we present some lemmas needed in the proofs of our main results, and several lower bounds of the second eigenvalue of a bipartite graph.

We denote a path of order n by P_n, a cycle of order n by C_n, the vertex disjoint union of two graphs G and H by $G \cup H$, the disjoint union of k copies of the same graph H by kH. Let $K'_{1,3}$ be the tree of order 5 with exactly three vertices of degree 1. The *joint* of two vertex disjoint graphs G and H, denoted by $G \vee H$, is the graph obtained from $G \cup H$ by adding all the edges joining a vertex in $V(G)$ to a vertex in $V(H)$. For a graph G, $\overline{G}$ denotes the *complement* of G. If $X \subseteq V(G)$, we write $G[X]$ for the subgraph of G induced by the vertices in X.

Theorem 2.1 *(Collatz and Sinogowitz, [2]) Let G be a connected graph of order n. Then $\lambda_1 \geq 2\cos\left(\frac{\pi}{n+1}\right)$, where equality holds if and only if $G \cong P_n$.*

Theorem 2.2 *(Interlacing Theorem, Theorem 0.10 of [3]) For $1 \leq i \leq n-k$, $V' \subseteq V(G)$ with $|V'| = k$,*

$$\lambda_i(G) \geq \lambda_i(G - V') \geq \lambda_{i+k}(G).$$

Lemma 2.3 *Let G be a connected bipartite simple graph of order n. Then one of the following holds.*
(i) $G \in \{K_1, K_{1,n-1}, K_{2,n-2}, P_n\}$.
(ii) G contains one member in $\mathcal{F}$ as an induced subgraph, where $\mathcal{F} = \{C_6, K_2 \cup P_4, K_2 \cup K_2 \cup P_3, K_{2,3} - e, K'_{1,3}, K_{3,3}\}$.

Proof. We assume that (i) fails to establish (ii). If $n \leq 4$, then since G is connected, simple and bipartite, G must be one of the graphs listed in (i). Therefore we assume $n \geq 5$.

Suppose first that G is a tree. Since (i) fails, G is not a path and so $\Delta(G) \geq 3$. Let $v_0 \in V(G)$ with $d(v_0) = t = \Delta(G)$ and let $v_1, v_2, \cdots, v_t$ be the vertices adjacent to v_0. If $V(G) = \{v_0, v_1, v_2, \cdots, v_t\}$, then $G = K_{1,n-1}$ and so (i) must hold. Thus there exists $u \in V(G) -$

$\{v_0, v_1, v_2, \cdots, v_t\}$ such that $uv_i \in E(G)$ for some $i \in \{1, 2, \cdots, t\}$. This implies that $K'_{1,3}$ is an induced subgraph of G, and so (ii) holds.

Now assume that G is not a tree. Let $g(G)$ be the girth of G. Since G is a connected simple bipartite and not a tree, $g(G) \in \{4, 6, 8, 10, \cdots\}$. If $g(G) \geq 10$, then $K_2 \cup K_2 \cup P_3$ is an induced subgraph. If $g(G) = 8$, then $K_2 \cup P_4$ is an induced subgraph. If $g(G) = 6$, then C_6 is an induced subgraph. If $g(G) = 4$, then G contains a proper subgraph isomorphic to a $K_{2,p}$ for some $p \geq 2$.

Let H be a maximal $K_{2,p}$ subgraph of G (that is, $H \cong K_{2,p}$, but G does not have a subgraph isomorphic to $K_{2,p+1}$ which properly contains H as a subgraph). Let v_0, v'_0 be the two nonadjacent vertices of degree p in H and let $v_1, v_2, \cdots, v_p$ be the vertices of degree 2, which are adjacent to v_0, v'_0 in H. By the assumption that (i) fails, $G \not\cong K_{2,n-2}$, and so there must be a vertex $u \in V(G) \backslash V(H)$. If uv_0 (or uv'_0) $\in E(G)$, then by the maximality of H, uv'_0 (respectively, uv_0) $\notin E(G)$, and so $K_{2,3} - e$ is an induced subgraph of G. Now suppose that $uv_0 \notin E(G)$ and $uv'_0 \notin E(G)$. Since G is connected, we may assume that $uv_p \in E(G)$. If $uv_1, uv_2 \notin E(G)$, then $G[\{v_0, v_1, v_2, v_p, u\}] \cong K'_{1,3}$ is an induced subgraph of G. If $uv_1 \notin E(G)$ but $uv_i \in E(G)$, for each $i = 2, 3, \cdots, p$, then $G[\{v_0, v'_0, v_1, v_p, u\}] \cong K_{2,3} - e$ is an induced subgraph of G. If $uv_i \in E(G)$ for $i = 1, 2, 3, \cdots, p$, then when $p \geq 3$, $G[\{v_0, v'_0, v_1, v_2, v_p, u\}] \cong K_{3,3}$ is an induced subgraph of G; and when $p = 2$, $G[\{v_0, v_1, v_2, v'_0, u\}] \cong K_{2,3}$, contrary to the maximality of $H = K_{2,p} = K_{2,2} \subsetneq K_{2,3}$. ■

Corollary 2.4 *Let G be a connected nontrivial bipartite graph. If $G \not\cong P_n$, then G has an induced subgraph in $\mathcal{F}$.*

Proof. This follows directly from Lemma 2.3.■

Lemma 2.5 *Let G be a graph with $n \geq 5$ vertices and with $m \geq 1$ edges. The following are equivalent.*
(i) G is a complete bipartite graph.
(ii) If H is an induced subgraph of G with 5 vertices, then $H \in \{\overline{K_5}, K_{1,4}, K_{2,3}\}$.

Proof. It suffices to prove that (ii) implies (i), and so we assume that (ii) holds. First, we shall show that G must be bipartite.

Define $g_1(G)$ to be the length of the shortest odd cycle of G, if G has an odd cycle; and $g_1(G) = 1$ if G does not have an odd cycle. Note that $g_1(G) = 1$ if and only if G is bipartite. We shall prove that $g_1(G) = 1$.

Suppose that $g_1(G) > 1$. If $g_1(G) \leq 5$, then let $X \subseteq V(G)$ be a set of 5 vertices that contains the vertices of a 3-cycle or a 5-cycle of G. Then $H = G[X]$ has an odd cycle, and cannot be any of the three graphs listed in Lemma 2.5 (ii). Thus $g_1(G) = 2k + 1 \geq 7$, for some integer $k \geq 3$. Let $C = v_1v_2 \cdots v_{2k}v_{2k+1}v_1$ be an odd cycle of G with length $2k + 1$. Let $H_1 = G[\{v_1, v_2, v_3, v_4, v_5\}]$. Then by Lemma 2.5(ii), $H_1 \cong K_{2,3}$. Since the path $v_1v_2v_3v_4v_5$ is a spanning path in H_1, $v_1v_4, v_2v_5 \in E(G)$. Therefore, G contains an odd cycle $v_1v_4v_5 \cdots v_{2k+1}v_1$ with length $2k-1$, contrary to the assumption that $g_1(G) = 2k + 1$. This proves that we must have $g_1(G) = 1$, that is, G must be bipartite.

Now let $V(G) = V_1 \cup V_2$, where every edge of G has one end in V_1 and the other end in V_2. Without loss of generality, we assume that $|V_1| \leq |V_2|$. We shall show that G is a complete bipartite graph.

By contradiction, suppose that G is not complete. If $|V_1| = 1$, then Lemma 2.5(ii) trivially implies that $G \cong K_{1,n+1}$, contrary to the assumption that G is not complete. Hence we assume that $|V_1| \geq 2$. Since G is not a complete bipartite graph, we may assume that there are $x \in V_1$ and $y \in V_2$ such that $xy \notin E(G)$. Since $m \geq 1$, we can find $x' \in V_1$ and $y' \in V_2$ such that $x'y' \in E(G)$. Note that it is possible that $x = x'$ or $y = y'$, but they cannot occur simultaneously.

Assume first that $x = x'$. Then $y \neq y'$. Since $|V_2| \geq |V_1| \geq 2$ and $n \geq 5$, we can find $x'' \in V_1 - \{x\}$ and $y'' \in V_2 - \{y, y'\}$. Let $H_2 = G[\{x, x'', y, y', y''\}]$. Since $E(H_2) \neq \emptyset, H_2 \not\cong \bar{K}_5$. Since G is a bipartite graph with bipartition (V_1, V_2) and $|V(H_2) \cap V_1| = 2$ and $|V(H_2) \cap V_2| = 3$, $H_2 \not\cong K_{1,4}$. Since $xy \notin E(G)$, $H_2 \not\cong K_{2,3}$. Therefore, $x \neq x'$. Similarly, $y \neq y'$. Hence we may assume that $x \neq x'$ and $y \neq y'$. Choose y'' and argue as above with x' replacing x''. We also get a contradiction. Thus G must be a complete bipartite graph.■

Theorem 2.6 *If G is a nontrivial bipartite graph with order $n \geq 4$, then $\lambda_2(G) \geq \dfrac{\sqrt{5}-1}{2}$, where equality holds if and only if $G \cong P_4$.*

Proof. Since G is a nontrivial bipartite graph, G has no isolated vertices. Thus the degree of each vertex of G is at least one and G has at least two edges by the fact that $n \geq 4$. So G contains an induced subgraph isomorphic to $2K_2$ or P_4 because G is not a complete bipartite graph. By Theorem 2.2, $\lambda_2(G) \geq \min\{\lambda_2(2K_2), \lambda_2(P_4)\} = \frac{\sqrt{5}-1}{2}$.

If $n = 4$ and if G is not isomorphic to P_4, it is routine to check that $\lambda_2(G) > \frac{\sqrt{5}-1}{2}$. If $n \geq 5$, then by Lemma 2.5, G has an induced subgraph H of order 5 which is not isomorphic to $K_{2,3}$. It follows from the appendix in [3] that $\lambda_2(H) > \frac{\sqrt{5}-1}{2}$, and so by Theorem 2.2, $\lambda_2(G) \geq \lambda_2(H) > \frac{\sqrt{5}-1}{2}$. This proves that the equality holds if and only if $G \cong P_4$. ■

Theorem 2.7 *Let G be a connected nontrivial bipartite graph with order n. If $G \ncong P_n$, then $\lambda_2 \geq \sqrt{\frac{5-\sqrt{17}}{2}}$, where equality holds if and only if $G \cong K_{2,3} - e$.*

Proof. By Corollary 2.4, G contains an induced subgraph in $\mathcal{F}$. By Theorem 2.2,

$$\begin{aligned}\lambda_2 &\geq \min\{\lambda_2(C_6), \lambda_2(K_2 \cup P_4), \lambda_2(K_2 \cup K_2 \cup P_4), \lambda_2(K_{2,3}-e), \lambda_2(K'_{1,3})\\ &= \lambda_2(K_{2,3}-e) = \sqrt{\frac{5-\sqrt{17}}{2}}.\end{aligned}$$

Suppose $G \ncong K_{2,3} - e$. Then $n \geq 5$ and by the appendix in [4], $\lambda_2 \geq \sqrt{\frac{5-\sqrt{17}}{2}}$. By the appendix in [4] again, $\lambda_2 > \sqrt{\frac{5-\sqrt{17}}{2}}$ for all connected nontrivial bipartite graphs of order 6. Therefore, $\lambda_2 > \sqrt{\frac{5-\sqrt{17}}{2}}$ for $n > 5$. Hence the equality holds if and only if $G \cong K_{2,3} - e$. ■

3 Upper Bounds

In this section, we will consider upper bounds of λ_1, λ_2 and λ_3 for a bipartite graph G with reasonably large number of vertices.

Lemma 3.1 *(Theorem 3.4 of [3]) If G is a bipartite graph, then for $i = 1, 2, \cdots, \lceil n/2 \rceil$,*

$$\lambda_i(G) = -\lambda_{n+1-i}(G).$$

It follows from Lemma 3.1 that if G is a bipartite graph with n vertices and m edges, then

$$2m = \sum_{i=1}^{n} \lambda_i^2 = 2\lambda_1^2 + 2\sum_{i=2}^{[\frac{n}{2}]} \lambda_i^2. \tag{1}$$

By Theorem 1.3 and (1), the following corollary follows.

Corollary 3.2 *If a connected bipartite graph G is not an induced subgraph of any of the graphs $G_1 - G_7$ in [8] (see Fig. 1 of [8]), then $\lambda_1 < \sqrt{m-1}$.*

Theorem 3.3 *Let G be a nontrivial bipartite graph with n vertices and m edges. Each of the following holds.*

(i) $\lambda_1(G) \leq \sqrt{m - \frac{3-\sqrt{5}}{2}}$, where equality holds if and only if $G \cong P_4$.

(ii) If $G \ncong P_n$, then $\lambda_1(G) \leq \sqrt{m - \frac{5-\sqrt{17}}{2}}$, where equality holds if and only if $G \cong K_{2,3} - e$.

Proof. We argue by contradiction and assume that

$$\lambda_1 > \sqrt{m - \frac{3-\sqrt{5}}{2}}. \tag{2}$$

Since G is a bipartite graph, it follows by (1) and by (2) that

$$\lambda_2^2 \leq \sum_{i=2}^{[\frac{n}{2}]} \lambda_i^2 = m - \lambda_1^2 < \frac{3-\sqrt{5}}{2},$$

and so $\lambda_2 < \frac{\sqrt{5}-1}{2}$, contrary to Theorem 2.6. Therefore, the inequality of Theorem 3.3(i) must hold.

To prove the case when equality holds, we first note that $\lambda_1(P_4) = \sqrt{m - \frac{3-\sqrt{5}}{2}}$. If G is not isomorphic to P_4, then by Theorem 2.6, $\lambda_2 > \frac{\sqrt{5}-1}{2}$. It follows that $m - \lambda_1^2 \geq \lambda_2^2 > \frac{3-\sqrt{5}}{2}$, and so $\lambda_1 < \sqrt{m - \frac{3-\sqrt{5}}{2}}$. This proves Theorem 3.3(i).

The proof for Theorem 3.3(ii) is similar, using Theorem 2.7 instead of Theorem 2.6.■

Corollary 3.4 and Corollary 3.5 below immediately follow from Theorem 3.3. They improve Theorem 1.2 for trees which are not isomorphic to stars.

Corollary 3.4 *If G is a forest of order n that is not a star, then $\lambda_1 \leq \sqrt{n-1-\frac{3-\sqrt{5}}{2}}$, where equality occurs if and only if G is a path of order 4.*

Corollary 3.5 *If G is a forest of order n that is neither a path P_n nor a star, then $\lambda_1 < \sqrt{n-1-\frac{5-\sqrt{17}}{2}}$.*

We now consider the upper bounds of the second largest eigenvalue of a bipartite graph. As far as we know, little has been done in this direction. We need one more lemma.

Lemma 3.6 *Let G be a graph with n vertices and m edges. Let $a > 0$ and $b > 0$ be two numbers, and let $\lambda_1 = \lambda_1(G), \lambda_2 = \lambda_2(G)$ and $\lambda_3 = \lambda_3(G)$. Then each of the following holds.*
(i) If $n \geq 2$ and if $\lambda_1 \geq a$, then $\lambda_2 \leq \sqrt{m - a^2}$.
(ii) If $n \geq 3, \lambda_1 \geq a$ and $\lambda_2 \geq b$, then $\lambda_3 \leq \sqrt{m - a^2 - b^2}$.

Proof. By (1), we have

$$\lambda_2^2 \leq m - \lambda_1^2, \text{ and } \lambda_3^2 \leq m - \lambda_1^2 - \lambda_2^2. \tag{3}$$

Thus Lemma 3.6 follows directly from (3). ■

Corollary 3.7 *Let G be a graph with n vertices and m edges. Let $\Delta(G)$ denote the maximum degree of G, $d_1, d_2, \cdots, d_n$ the degree sequence of G, and $\chi(G)$ the chromatic number of G. Then each of the following holds.*
(i) $\lambda_2 \leq \sqrt{m - \Delta(G)}$.
(ii) $\lambda_2 \leq \sqrt{m - \frac{1}{n}\sum_{i=1}^{n} d_i^2}$.
(iii) $\lambda_2 \leq \sqrt{m - (\chi(G) - 1)^2}$.

Proof. Homfmeister [5] showed that $\lambda_1(G) \geq \sqrt{\frac{1}{n}\sum_{i=1}^{n} d_i^2}$; Nosal [7], Lovás and Pelikán [6] proved that $\lambda_1(G) \geq \sqrt{\Delta(G)}$; and Wilf [9] showed that $\lambda_1(G) \geq \chi(G) - 1$. Thus Corollary 3.7 follows from Lemma 3.6(i) and these prior results. ■

Theorem 3.8 *Let G be a connected bipartite graph with n vertices and m edges. Then*

$$\lambda_2 \leq \sqrt{m - 4\cos^2\left(\frac{\pi}{n+1}\right)},$$

where equality holds if and only if $G \cong P_n$ where $2 \leq n \leq 5$.

Proof. By Theorem 2.1, $\lambda_1 \geq 2\cos\left(\frac{\pi}{n+1}\right)$. Thus the inequality of Theorem 3.8 is proved by applying Lemma 3.6(i) with $a = 2\cos\left(\frac{\pi}{n+1}\right)$.

To prove equality part of the theorem, we first observe that if G is not isomorphic to P_n, then by Theorem 2.1, $\lambda_1 > 2\cos\left(\frac{\pi}{n+1}\right)$. It follows by Lemma 3.6(i) again that $\lambda_2 < \sqrt{m - 4\cos^2\frac{\pi}{n+1}}$. Direct computation of $\lambda_2(P_n)$ shows that $\lambda_2(P_n) = \sqrt{m - 4\cos^2\left(\frac{\pi}{n+1}\right)}$ if and only if $2 \leq n \leq 5$. This completes the proof. ■

As when $n \geq 2$, $4\cos^2\left(\frac{\pi}{n+1}\right) > 1$, it follows that the bound of Theorem 3.8 is better than $\lambda_2 \leq \lambda_1 \leq \sqrt{m}$ from Theorem 1.1.

Turning to upper bounds of $\lambda_3(G)$ for a bipartite graph, we have the following result.

Theorem 3.9 *Let G be a nontrivial bipartite graph with $n \geq 6$ vertices and m edges. Then*

$$\lambda_3(G) \leq \sqrt{m - \frac{3-\sqrt{5}}{2} - 4\cos^2\frac{\pi}{n+1}}.$$

Proof. By Theorem 2.1, $\lambda_1 \geq 2\cos\left(\frac{\pi}{n+1}\right)$; by Theorem 2.6, $\lambda_2 \geq \frac{\sqrt{5}-1}{2}$. Therefore Theorem 3.9 follows from Lemma 3.6(ii) with $a = 2\cos\left(\frac{\pi}{n+1}\right)$ and $b = \frac{\sqrt{5}-1}{2}$. ■

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