A note on bounds for the maximum traceable number of a graph

FUTABA FUJIE
Graduate School of Mathematics
Nagoya University
Nagoya, 464-8602, Japan.

Abstract

For a connected graph G of order $n \geq 2$ and a linear ordering $s : v_1, v_2, \ldots, v_n$ of $V(G)$, $d(s) = \sum_{i=1}^{n-1} d(v_i, v_{i+1})$, where $d(v_i, v_{i+1})$ is the distance between v_i and v_{i+1}. The traceable number $t(G)$ and upper traceable number $t^+(G)$ of G are defined by $t(G) = \min\{d(s)\}$ and $t^+(G) = \max\{d(s)\}$, respectively, where the minimum and maximum are taken over all linear orderings s of $V(G)$. The traceable number $t(v)$ of a vertex v in G is defined by $t(v) = \min\{d(s)\}$, where the minimum is taken over all linear orderings s of $V(G)$ whose first term is v. The maximum traceable number $t^*(G)$ of G is then defined by $t^*(G) = \max\{t(v) : v \in V(G)\}$. Therefore, $t(G) \leq t^*(G) \leq t^+(G)$ for every nontrivial connected graph G. We show that $t^*(G) \leq \left\lfloor \frac{t(G)+t^+(G)+1}{2} \right\rfloor$ for every nontrivial connected graph G and that this bound is sharp. Furthermore, it is shown that for positive integers a and b, there exists a nontrivial connected graph G with $t(G) = a$ and $t^*(G) = b$ if and only if $a \leq b \leq \lfloor \frac{3a}{2} \rfloor$.

Keywords: traceable number of a graph, maximum traceable number of a graph, upper traceable number of a graph.

AMS subject classification: 05C12, 05C45.

1 Introduction

We refer to the book [2] for graph-theoretical notation and terminology not described in this paper. In [4, 5] Goodman and Hedetniemi introduced the concept of a *Hamiltonian walk* in a connected graph G, defined as a closed spanning walk of minimum length in G. In [3] this concept was studied from a different point of view. For a connected graph G of order $n \geq 3$ and a *cyclic ordering* $s : v_1, v_2, \ldots, v_n, v_{n+1} = v_1$ of vertices of G, the number

$d(s)$ is defined as $\sum_{i=1}^{n} d(v_i, v_{i+1})$, where $d(v_i, v_{i+1})$ is the distance between v_i and v_{i+1} (the length of a shortest v_i - v_{i+1} path in G). The *Hamiltonian number* $h(G)$ and *upper Hamiltonian number* $h^+(G)$ of G are defined in [3] by $h(G) = \min\{d(s)\}$ and $h^+(G) = \max\{d(s)\}$, respectively, where the minimum and maximum are taken over all cyclic orderings s of vertices of G. It is shown that $h(G)$ is, in fact, the length of a Hamiltonian walk in G.

For a nontrivial connected graph G of order n and a *linear ordering* $s: v_1, v_2, \ldots, v_n$ of vertices of G, the number $d(s)$ is defined in [7] as $\sum_{i=1}^{n-1} d(v_i, v_{i+1})$. The *traceable number* $t(G)$ of G is then defined by

$$t(G) = \min\{d(s)\},$$

where the minimum is taken over all linear orderings s of vertices of G. Thus if G is a connected graph of order $n \geq 2$, then $t(G) \geq n-1$. Furthermore, $t(G) = n-1$ if and only if G is traceable (that is, G contains a Hamiltonian path). In fact, the traceable number of a connected graph G is the minimum length of a spanning walk in G. The *upper traceable number* $t^+(G)$ of G is defined in [8] by

$$t^+(G) = \max\{d(s)\},$$

where the maximum is taken over all linear orderings s of vertices of G. Another related measure of traversability of a graph was introduced in [7]. For a vertex v in G, the *traceable number* $t(v)$ *of* v is defined by $t(v) = \min\{d(s)\}$, where the minimum is taken over all linear orderings s of vertices of G *whose first term is* v. Observe that the traceable number $t(G)$ of a connected graph G can be alternatively defined as

$$t(G) = \min\{t(v) : v \in V(G)\}.$$

On the other hand, the *maximum traceable number* $t^*(G)$ of G is defined in [6] by

$$t^*(G) = \max\{t(v) : v \in V(G)\}.$$

By the definitions of $t(G)$, $t^*(G)$, and $t^+(G)$, we have the following observation.

Observation 1.1 *For every nontrivial connected graph G,*

$$t(G) \leq t^*(G) \leq t^+(G). \tag{1}$$

Initially, it may appear that $t^*(G) = t^+(G)$ for every nontrivial connected graph G but this is not the case. For example, for the graph G of Figure 1, $t(G) = 4$, $t^*(G) = 5$, and $t^+(G) = 9$. In Figure 1 each vertex of G is labeled by its traceable number.

For the chromatic number $\chi(G)$ of a graph G, there are a number of instances when a lower bound $f(G)$ and an upper bound $g(G)$ are given for

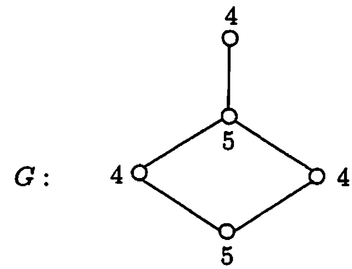

Figure 1: A graph G with $t(G) = 4$, $t^*(G) = 5$, and $t^+(G) = 9$

$\chi(G)$ and it is shown that $\chi(G)$ can never be closer to $g(G)$ than to $f(G)$, that is,

$$\chi(G) \leq \left\lfloor \frac{f(G) + g(G)}{2} \right\rfloor.$$

For example, it is well known that the chromatic number $\chi(G)$ of a graph G of order n is at least as large as its clique number and at most n, that is,

$$\omega(G) \leq \chi(G) \leq n.$$

Reed [9] showed that $\chi(G)$ can never be closer to n than to $\omega(G)$. Thus

$$\chi(G) \leq \left\lfloor \frac{\omega(G) + n}{2} \right\rfloor$$

for every graph G of order n. It is also well known that $\chi(G) \leq \Delta(G)+1$ for every graph G. Reed conjectured that $\chi(G)$ can never be closer to $\Delta(G)+1$ than to $\omega(G)$. That is, Reed conjectured that

$$\chi(G) \leq \left\lfloor \frac{\omega(G) + \Delta(G) + 1}{2} \right\rfloor$$

for every graph G. Also, it is well known that

$$\omega(G) \leq \chi(G) \leq n + 1 - \beta(G)$$

for every graph G of order n, where $\beta(G)$ denotes the independence number of G. Brigham and Dutton [1] showed that $\chi(G)$ can never be closer to $n + 1 - \beta(G)$ than to $\omega(G)$. Thus

$$\chi(G) \leq \left\lfloor \frac{\omega(G) + n + 1 - \beta(G)}{2} \right\rfloor$$

for every graph G of order n. For the bounds for $t^*(G)$ given in (1), we show that $t^*(G)$ can never be considerably closer to $t^+(G)$ than to $t(G)$.

2 An improved upper bound for the maximum traceable number of a graph

We noted in (1) that for every nontrivial connected graph G,

$$t(G) \le t^*(G) \le t^+(G).$$

We now show that in most instances there is an improved upper bound for $t^*(G)$. First, we establish an additional definition. The *eccentricity* $e(v)$ of a vertex v in a connected graph G is the maximum distance from v to a vertex of G.

Theorem 2.1 *For every nontrivial connected graph G,*

$$t^*(G) \le \left\lfloor \frac{t(G) + t^+(G) + 1}{2} \right\rfloor .$$

Proof. We first show that

$$t^*(G) - t(G) \le t^+(G) - t^*(G) + 1. \qquad (2)$$

Since the result follows immediately if $t(G) = t^*(G)$, let us assume that $t^*(G) \ge t(G) + 1$. Let x be a vertex in G such that $t(x) = t^*(G)$. We will show that

$$t^*(G) - t(G) \le e(x) \le t^+(G) - t^*(G) + 1.$$

Since $t(G) < t^*(G)$, suppose that G is a graph of order $n \ge 3$ and let $s_0 : v_1, v_2, \ldots, v_n$ be a linear ordering of vertices of G such that $d(s_0) = t(G)$. Then $x = v_i$ for some integer i $(2 \le i \le n)$. If $x = v_n$, then let s_0^{-1} be the reversal of s_0 and observe that

$$t^*(G) = t(x) \le d(s_0^{-1}) = d(s_0) = t(G),$$

which is a contradiction. Hence assume that $2 \le i \le n-1$ and consider the linear ordering

$$s_1 : \; v_i = x, v_1, v_2, \ldots, v_{i-1}, v_{i+1}, v_{i+2}, \ldots, v_n.$$

Observe that

$$\begin{aligned} t^*(G) &= t(x) \le d(s_1) \\ &= d(s_0) + d(x, v_1) + d(v_{i-1}, v_{i+1}) - [d(v_{i-1}, v_i) + d(v_i, v_{i+1})] \\ &\le d(s_0) + d(x, v_1) \le t(G) + e(x). \end{aligned}$$

Therefore, $t^*(G) - t(G) \le e(x)$.

To show that $t^+(G) - t^*(G) + 1 \geq e(x)$, observe that there are distinct vertices $y, z \in V(G) - \{x\}$ such that $d(x, y) = 1$ and $d(x, z) = e(x)$. Let

$$s_2 : \ w_1 = x, w_2 = y, w_3, \ldots, w_n = z$$

be a linear ordering whose initial, second, and terminal vertices are x, y, and z, respectively, and consider the linear ordering $s_3 : w_2, w_3, \ldots, w_n, w_1$. Observe that

$$\begin{aligned} t^+(G) &\geq d(s_3) = d(s_2) + d(x, z) - d(x, y) \\ &\geq t(x) + e(x) - 1 = t^*(G) + e(x) - 1, \end{aligned}$$

that is, $t^+(G) - t^*(G) + 1 \geq e(x)$. Thus (2) holds, as claimed. Adding $t^*(G) + t(G)$ to both sides of (2), we obtain

$$t^*(G) \leq \frac{t(G) + t^+(G) + 1}{2}$$

and so the result follows. ■

If G is a star of order $n \geq 3$, then $t(G) = 2n - 4$ and $t^*(G) = t^+(G) = 2n - 3$, that is, $t^*(G) = \frac{t(G)+t^+(G)+1}{2}$. Therefore, the upper bound in Theorem 2.1 is sharp.

3 A realization result

We now investigate the sharpness of the lower bound for $t^*(G)$ given in (1), that is, $t(G) \leq t^*(G)$. Indeed, we determine all pairs a, b of positive integers for which there exists a nontrivial connected graph G with $t(G) = a$ and $t^*(G) = b$. We begin with a lemma.

Lemma 3.1 *For a pair n, r of integers with $3 \leq r \leq n - 1$, let $G_{n,r}$ be the graph of order n obtained from a complete graph K_r of order r and a path P of order $n - r + 1$ by identifying one of the r vertices of K_r and one of the two end-vertices of P. Then $G_{n,r}$ is traceable and*

$$t^*(G_{n,r}) = \left\lfloor \frac{3n-r-1}{2} \right\rfloor.$$

Proof. Let $V(K_r) = \{u_1, u_2, \ldots, u_r\}$ and $P : v_0, v_1, \ldots, v_q = u_1$, where $q = n - r \geq 1$. First observe that $t(v_0) = t(u_i) = n - 1$ for $2 \leq i \leq r$. Also, $t(v_q) = n$. On the other hand, observe that for each i $(1 \leq i \leq q - 1)$, either of the two linear orderings

$$\begin{aligned} s_{i_0} &: \ v_i, v_0, v_1, \ldots, v_{i-1}, v_{i+1}, v_{i+2}, \ldots, v_q, u_2, u_3, \ldots, u_r \\ s_{i_1} &: \ v_i, v_{i+1}, \ldots, v_q, u_2, u_3, \ldots, u_r, v_{i-1}, v_{i-2} \ldots, v_0 \end{aligned}$$

gives us $t(v_i)$. Hence

$$t(v_i) = \min\{n - 1 + i, n + q - i\}.$$

If q is odd, then observe that

$$\max\{t(v_i):\ 0 \le i \le q\} = n - 1 + \tfrac{1}{2}(q+1) = \tfrac{1}{2}(3n - r - 1).$$

If q is even, then

$$\max\{t(v_i):\ 0 \le i \le q\} = n - 1 + \tfrac{q}{2} = \tfrac{1}{2}(3n - r - 2).$$

Therefore, the result follows in each case. ∎

We are now prepared to present the desired result.

Theorem 3.2 *Let a, b be a pair of positive integers. Then there exists a nontrivial connected graph G with $t(G) = a$ and $t^*(G) = b$ if and only if $a \le b \le \lfloor \frac{3a}{2} \rfloor$.*

Proof. Since the statement clearly holds for $1 \le a \le 3$, we assume that $a \ge 4$. Suppose that G is a graph of order n such that $t(G) = a$ and $t^*(G) = b$. Hence there exists a spanning walk $W_0 : v_0, v_1, \ldots, v_a$ of length a and for each vertex $v \in V(G)$, $v = v_i$ for some i $(0 \le i \le a)$. By symmetry, we may further assume that $0 \le i \le \lfloor a/2 \rfloor$. If $v = v_0$, then clearly $t(v_0) = a$. Otherwise, consider the spanning walk W_i given by v followed by W_0. Let $\ell(W)$ denote the length of a walk W and observe that

$$t(v) \le \ell(W_i) = d(v, v_0) + \ell(W_0) \le \lfloor \tfrac{a}{2} \rfloor + a = \lfloor \tfrac{3a}{2} \rfloor.$$

Therefore, $b = t^*(G) \le \lfloor 3a/2 \rfloor$.

For the converse, let a, b be a pair of integers with $4 \le a \le b \le \lfloor 3a/2 \rfloor$. We construct a traceable graph G (of order $n = a + 1$) such that $t(G) = a$ and $t^*(G) = b$. If $b = a$, then consider $G = K_{a+1}$. If $b = \lfloor 3a/2 \rfloor$, then consider $G = P_{a+1}$. Otherwise, observe that $4 \le 3a - 2b + 2 \le a = n - 1$. Let $G_{a+1,3a-2b+2}$ be the traceable graph of order $a+1$ described in Lemma 3.1. Then $t(G_{a+1,3a-2b+2}) = a$ and

$$t^*(G_{a+1,3a-2b+2}) = \left\lfloor \frac{3(a+1) - (3a - 2b + 2) - 1}{2} \right\rfloor = b,$$

which is the desired result. ∎

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