On the maximum packing problem of $MP_\lambda(3, K_4^{(3)} - e, v)$

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Abstract: In this paper we deal with a special kind of hypergraph decomposition. We show that there exists a decomposition of the 3-uniform hypergraph $\lambda K_v^{(3)}$ into a special kind of hypergraph $K_4^{(3)} - e$ whose leave has at most two edges for any positive integers $v \geq 4$ and λ.

Keywords: hypergraph decomposition; t-(v, Γ, λ) packing; leave; packing number; group divisible (Γ, t)-design; candelabra (Γ, t)-system

1 Introduction

A *hypergraph* H is a pair (V, E), where V is a finite set of vertices, E is a family of subsets of V (called *hyperedges* or *edges*). A hypergraph is called *simple* if E has no repeated edges. A *sub-hypergraph* $H' = (V', E')$ of $H = (V, E)$ is a hypergraph satisfying $V' \subseteq V$ and $E' \subseteq E$.

A hypergraph is said to be *t-uniform* if each of its edges contains exactly t vertices. In particular a 2-uniform hypergraph is just a graph. $\lambda K_v^{(t)}$ is a t-uniform hypergraph on the point-set V in which the edge-set E contains each t-subset of V exactly λ times, where $v = |V|$ and v is called the order of the t-uniform hypergraph. If $\lambda = 1$, this hypergraph is said to be *complete*. For an edge $e \in E$, $K_v^{(t)} - e$ denotes the hypergraph obtained from $K_v^{(t)}$ by deleting the edge e. From [1], we can know more information about hypergraphs.

Let H be a t-uniform hypergraph, and Γ be a set of t-uniform hypergraphs. A *decomposition* of H into hypergraphs of Γ is a partition of the edges of H into sub-hypergraphs, each of which is isomorphic to one of Γ. Such a decomposition of H into Γ is denoted by (H, Γ)-design. Hypergraph decompositions have an interesting application in secret sharing schemes [3]. Clearly, hypergraph decomposition is a generalization of graph decomposition. For a more comprehensive introduction on graph decompositions, the interested reader may refer to [10].

In what follows we always assume that K is a set of positive integers, Γ is a set of simple t-uniform hypergraphs, and Ω is a set of complete t-uniform hypergraphs, where the order of each element of Ω is from K.

A t-(v, Γ, λ) *packing* is a pair $(X, \mathcal{B})$, where X is a set of v points, $\mathcal{B}$ is a collection of hypergraphs (called *blocks*) on the subsets of X, such that for each $B \in \mathcal{B}$, B is isomorphic to one of Γ, and every t-subset of X is contained in at most λ blocks. We denote it by $P_\lambda(t, \Gamma, v)$. If $\lambda = 1$, we often write $P(t, \Gamma, v)$ for $P_1(t, \Gamma, v)$. If Γ only contains one hypergraph J, we write $P_\lambda(t, \{J\}, v)$ simply as $P_\lambda(t, J, v)$.

A $P_\lambda(t, \Gamma, v)$ $(X, \mathcal{A})$ is called *maximum* (denoted by $MP_\lambda(t, \Gamma, v)$) if there does not exist any $P_\lambda(t, \Gamma, v)$ $(X, \mathcal{B})$ with $|\mathcal{A}| < |\mathcal{B}|$. The *packing number* is the number of blocks in an $MP_\lambda(t, \Gamma, v)$ and denoted by $D_\lambda(t, \Gamma, v)$. If $\lambda = 1$, we often write $D(t, \Gamma, v)$ for $D_1(t, \Gamma, v)$.

For any t-subset $e = \{x_1, x_2, \ldots, x_t\}$ of X, let $m(e)$ be the number of blocks containing e. The *leave* of a packing $P_\lambda(t, \Gamma, v)$ is the t-uniform hypergraph spanned by all t-subsets e of X with multiplicity $\lambda - m(e)$.

If the leave of a packing $P_\lambda(t, \Gamma, v)$ is null, such a packing is called a *t-wise balanced Γ design with index* λ and denoted by $S_\lambda(t, \Gamma, v)$. Obviously, if Γ only contains one hypergraph J, then an $S_\lambda(t, J, v)$ is an $MP_\lambda(t, J, v)$. If we replace Γ by Ω, then an $S_\lambda(t, \Omega, v)$ is denoted by $S_\lambda(t, K, v)$, which corresponds to the traditional concept of t-wise balanced design (t-BD) [2]. Therefore the t-wise balanced Γ design can be regarded as a generalization of the t-wise balanced design.

The purpose of this paper is to determine the packing number of $MP_\lambda(3, K_4^{(3)} - e, v)$. A simple counting argument shows that $D_\lambda(3, K_4^{(3)} - e, v) \leq \lfloor \lambda v(v-1)(v-2)/18 \rfloor$. For brevity, let $d_\lambda(3, K_4^{(3)} - e, v) = \lfloor \lambda v(v-1)(v-2)/18 \rfloor$. Since there are three edges in a hypergraph $K_4^{(3)} - e$, then if $D_\lambda(3, K_4^{(3)} - e, v) = d_\lambda(3, K_4^{(3)} - e, v)$, that means the leave of an $MP_\lambda(3, K_4^{(3)} - e, v)$ has at most 2 edges and vice versa. Feng and Chang [5] gave the following result.

Lemma 1.1 ([5]) *An $S(3, K_4^{(3)} - e, v)$ exists if and only if* $v \equiv 0, 1, 2$ (mod

9) *and* $v \geq 9$.

Lemma 1.1 implies that $D(3, K_4^{(3)} - e, v) = d(3, K_4^{(3)} - e, v)$ when $v \equiv 0, 1, 2 \pmod 9$ and $v \geq 9$. In this paper, direct and recursive constructions are discussed for these designs and we shall establish that

Theorem 1.2 *An* $MP_\lambda(3, K_4^{(3)} - e, v)$ *with* $d_\lambda(3, K_4^{(3)} - e, v)$ *blocks exists for any positive integers* $v \geq 4$ *and* λ*, its leave has at most* 2 *edges.*

2 Recursive constructions

In this section we shall introduce some auxiliary designs and establish some fundamental results which will be used later. The reader is referred to [2] for more information on design theory.

Let n and t be positive integers. Suppose that X is a set of points, $\mathcal{B}$ is a collection of hypergraphs on the subsets of X (called *blocks*), and $\mathcal{G}$ is a partition of X into n non-empty subsets (called *groups* or *holes*). A *group divisible* (Γ, t)*-design* is a triple $(X, \mathcal{G}, \mathcal{B})$, where for each $B \in \mathcal{B}$, B is isomorphic to one of Γ, such that each block intersects any given group in at most one point, and each t-subset of X from t distinct groups is contained in a unique block.

The type of group divisible (Γ, t)-design is defined as usual. Then type $g_1^{a_1} g_2^{a_2} \cdots g_m^{a_m}$ denotes that there are a_i groups of size g_i, $1 \leq i \leq m$. For brevity, a group divisible (Γ, t)-design of type $g_1^{a_1} g_2^{a_2} \cdots g_m^{a_m}$ can be denoted by GDD(t, Γ, v) of type $g_1^{a_1} g_2^{a_2} \cdots g_m^{a_m}$, where $v = \sum_{i=1}^{m} a_i g_i$. If Γ contains only one hypergraph J, we write GDD$(t, \{J\}, v)$ simply as GDD(t, J, v).

If we replace Γ by Ω, then a GDD(t, Ω, v) is denoted by GDD(t, K, v), which corresponds to the traditional concept of group divisible t-design (t-GDD) [13]. Therefore the group divisible (Γ, t)-design is a generalization of the group divisible t-design. A 2-GDD is simply called a GDD. Furthermore, if all the n groups have the same size g, the GDD(t, K, v) is called an H design, denoted by $H(n, g, K, t)$. The reader may refer to [2, 8, 12, 13] for more information on these designs.

The following results are useful in the following sections.

Lemma 2.1 ([7]) *For* $n > 2$*, a GDD*$(2, 3, gn)$ *of type* g^n *exists if and only if* $(n-1)g \equiv 0 \pmod 2$ *and* $n(n-1)g^2 \equiv 0 \pmod 3$.

Lemma 2.2 ([4]) *Let g, u, and m be nonnegative integers. There exists a $GDD(2,3,v)$ of type $g^u m^1$ if and only if the following conditions are all satisfied:*

(1) *if $g > 0$, then $u \geq 3$, or $u = 2$ and $m = g$, or $u = 1$ and $m = 0$, or $u = 0$;*

(2) $m \leq g(u-1)$ *or* $gu = 0$;

(3) $g(u-1) + m \equiv 0 \pmod 2$ *or* $gu = 0$;

(4) $gu \equiv 0 \pmod 2$ *or* $m = 0$;

(5) $\frac{1}{2}g^2u(u-1) + gum \equiv 0 \pmod 3$.

Let v, m, λ and t be positive integers, and s be a nonnegative integer. Suppose that X is a set of $v = s + \sum_{1 \leq i \leq m} a_i g_i$ points, S is a subset of X of size s (called *stem*), $\mathcal{T}$ is a partition of $X \backslash S$ of type $g_1^{a_1} \cdots g_m^{a_m}$ (called *groups* or *branches*), G_i is a group in $\mathcal{T}$, $1 \leq i \leq \Sigma_{j=1}^{m} a_j$, and $\mathcal{A}$ is a collection of hypergraphs on the subsets of X (called *blocks*). A *candelabra* (Γ, t)-*system with index* λ of type $(g_1^{a_1} \cdots g_m^{a_m} : s)$ is a quadruple $(X, S, \mathcal{T}, \mathcal{A})$, where for each $A \in \mathcal{A}$, A is isomorphic to one of Γ, such that every t-subset $T \subset X$ with $|T \cap (S \cup G_i)| < t$ for all i is contained in λ blocks and no t-subset of $S \cup G_i$ is contained in any block. Such a system is denoted by $CS_\lambda(t, \Gamma, v)$ of type $(g_1^{a_1} \cdots g_m^{a_m} : s)$. If $\lambda = 1$, we often write $CS(t, \Gamma, v)$ for $CS_1(t, \Gamma, v)$. If Γ contains only one hypergraph J, we write $CS_\lambda(t, \{J\}, v)$ simply as $CS_\lambda(t, J, v)$.

If we replace Γ by Ω, then a $CS_\lambda(t, \Gamma, v)$ is denoted by $CS_\lambda(t, K, v)$, which corresponds to the traditional concept of candelabra t-design [13]. Thus the candelabra (Γ, t)-system is a generalization of the candelabra t-system. The reader is referred to [6, 8, 9, 11, 13] for more information on this design.

By the definition of the candelabra (Γ, t)-system, we know that if $(X, \mathcal{B})$ is an $S_\lambda(t, \Gamma, v)$, then the quadruple $(X, \emptyset, \{\{x\} : x \in X\}, \mathcal{B})$ is a $CS_\lambda(t, \Gamma, v)$ of type $(1^v : 0)$, thus by Lemma 1.1, we have

Corollary 2.3 *There exists a $CS(3, K_4^{(3)} - e, v)$ of type $(1^v : 0)$ for any positive integer $v \equiv 0, 1, 2 \pmod 9$ and $v \geq 9$.*

Furthermore, if $v \equiv 0 \pmod 2$ we can also have the following corollary:

Corollary 2.4 *There exists a $CS(3, K_4^{(3)} - e, v)$ of type $(2^{\frac{v}{2}} : 0)$ for any positive integer $v \equiv 0, 2, 10 \pmod{18}$ and $v \geq 10$.*

Let v and s be nonnegative integers such that $s < v$. A t-(v,Γ,λ) *packing with a hole of size* s is a triple $(X,S,\mathcal{B})$, where X is a set of v points, S is a subset of X of size s (called *hole*), $\mathcal{B}$ is a collection of hypergraphs (called *blocks*) on the subsets of X, such that for each $B \in \mathcal{B}$, B is isomorphic to one of Γ, and every t-subset $T \subset X$ with $|T \cap S| < t$ is contained in at most λ blocks and no t-subset of S is contained in any block. We denote it by $HP_\lambda(t,\Gamma,v,s)$. If $\lambda = 1$, we often write $HP(t,\Gamma,v,s)$ for $HP_1(t,\Gamma,v,s)$. If Γ only contains one hypergraph J, we write $HP_\lambda(t,\{J\},v,s)$ simply as $HP_\lambda(t,J,v,s)$.

An $HP_\lambda(t,\Gamma,v,s)$ $(X,S,\mathcal{A})$ is called *maximum* (denoted by $HMP_\lambda(t,\Gamma,v,s)$) if there does not exist any $HP_\lambda(t,\Gamma,v,s)$ $(X,S,\mathcal{B})$ with $|\mathcal{A}| < |\mathcal{B}|$. If in an $HP_\lambda(t,\Gamma,v,s)$, every t-subset $T \subset X$ with $|T \cap S| < t$ is contained in exactly λ blocks, then we denote it by $HS_\lambda(t,\Gamma,v,s)$. Clearly, if $s < t$, then an $HMP_\lambda(t,\Gamma,v,s)$ is also an $MP_\lambda(t,\Gamma,v)$.

The following three constructions are simple but very useful.

Construction 2.5 *If there exists an* $HS(t,\Gamma,v,s)$ ($S(t,\Gamma,v)$ *or* $CS(t,\Gamma,v)$), *then there also exists an* $HS_\lambda(t,\Gamma,v,s)$ ($S_\lambda(t,\Gamma,v)$ *or* $CS_\lambda(t,\Gamma,v)$) *for each positive integer* λ.

Construction 2.6 *If there exist an* $HS_\lambda(t,\Gamma,v,s)$ *and an* $MP_\lambda(t,\Gamma,s)$, *then there exists a* $P_\lambda(t,\Gamma,v)$ *whose leave has the same number of edges as that of* $MP_\lambda(t,\Gamma,s)$.

Construction 2.7 *Suppose that there exists a* $CS_\lambda(t,\Gamma,v)$ *of type* $(g_1^{a_1} \cdots g_{m-1}^{a_{m-1}} g_m^1 : s)$. *If there exist an* $HS_\lambda(t,\Gamma,g_i+s,s)$ *for each* $1 \le i \le m-1$ *and an* $MP_\lambda(t,\Gamma,g_m+s)$, *then there exists a* $P_\lambda(t,\Gamma,v)$ *whose leave has the same number of edges as that of* $MP_\lambda(t,\Gamma,g_m+s)$.

Construction 2.7 shows that it is useful to find some $CS_\lambda(t,\Gamma,v)$s with a stem of size s. For the purpose of determining the packing number of $MP_\lambda(3,K_4^{(3)}-e,v)$, in what follows we always assume that $t=3$ and $\Gamma = K_4^{(3)}-e$. For brevity, in the following if we refer to $MP_\lambda(3,K_4^{(3)}-e,v)$, that always means the $MP_\lambda(3,K_4^{(3)}-e,v)$ with $d_\lambda(3,K_4^{(3)}-e,v)$ blocks. In this case, the leave of an $MP_\lambda(3,K_4^{(3)}-e,v)$ has at most 2 edges. Conversely, if the leave of a $P_\lambda(3,K_4^{(3)}-e,v)$ has at most 2 edges, then it must be an $MP_\lambda(3,K_4^{(3)}-e,v)$.

A simple computation shows that the number of blocks in an $HMP_\lambda(3,K_4^{(3)}-e,v,s)$ is at most $\lfloor \lambda[v(v-1)(v-2)-s(s-1)(s-2)]/18 \rfloor$. In this paper, for brevity, in the following if we refer to $HMP_\lambda(3, K_4^{(3)} - e, v, s)$, that always means the $HMP_\lambda(3, K_4^{(3)} - e, v, s)$ with $\lfloor \lambda[v(v-1)(v-2) - s(s-1)(s-2)]/18 \rfloor$ blocks. In this case, the leave of an $HMP_\lambda(3, K_4^{(3)} - e, v, s)$ has at most 2 edges.

In the following we denote the copy of $K_4^{(3)} - e$ with vertices a, b, c, d and edges $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$ by (a, b, c, d). The notation $\lambda(a, b, c, d)$ means that (a, b, c, d) replicate λ times.

Construction 2.8 *Suppose that there exists a $CS_\lambda(3, K_4^{(3)} - e, v)$ of type $(g_1^{a_1} \cdots g_m^{a_m} : 0)$. If there is a $GDD(2, 3, v)$ of type $g_1^{a_1} \cdots g_m^{a_m}$, then there is a $CS_\lambda(3, K_4^{(3)} - e, v + s)$ of type $(g_1^{a_1} \cdots g_m^{a_m} : s)$ for each nonnegative integer s.*

Proof Suppose that $(X, \emptyset, \mathcal{G}, \mathcal{A})$ is a $CS_\lambda(3, K_4^{(3)} - e, v)$ of type $(g_1^{a_1} \cdots g_m^{a_m} : 0)$. Let $S = \{\infty_1, \infty_2, \ldots, \infty_s\}$ and $X \cap S = \emptyset$. We can construct a $GDD(2, 3, v)$ $(X, \mathcal{G}, \mathcal{B})$ of type $g_1^{a_1} \cdots g_m^{a_m}$ by the assumption. Define the multisets $\mathcal{A}_{\infty_i} = \{\lambda(\infty_i, a, b, c) : \{a, b, c\} \in \mathcal{B}\}$ for $1 \le i \le s$ and $\mathcal{C} = (\cup_{1 \le i \le s} \mathcal{A}_{\infty_i}) \cup \mathcal{A}$. It is readily checked that $(X \cup S, S, \mathcal{G}, \mathcal{C})$ is a $CS_\lambda(3, K_4^{(3)} - e, v + s)$ of type $(g_1^{a_1} \cdots g_m^{a_m} : s)$. □

Construction 2.9 *Suppose that there exist an $MP_\lambda(3, K_4^{(3)} - e, v)$ and a $GDD(2, 3, v)$ of type $g_1^{a_1} \cdots g_{m-1}^{a_{m-1}} g_m^1$, $g_i < 3$ for each $1 \le i \le m - 1$. If there exist an $HS_\lambda(3, K_4^{(3)} - e, g_i + s, s)$ for each $1 \le i \le m - 1$ and an $HMP_\lambda(3, K_4^{(3)} - e, g_m + s, g_m)$, then there is a $P_\lambda(3, K_4^{(3)} - e, v + s)$ whose leave is the union of the leaves of the $MP_\lambda(3, K_4^{(3)} - e, v)$ and the $HMP_\lambda(3, K_4^{(3)} - e, g_m + s, g_m)$. If the number of the edges in the union of leaves is at most 2, then the resultant packing is an $MP_\lambda(3, K_4^{(3)} - e, v + s)$.*

Proof Suppose that $(X, \mathcal{A})$ is an $MP_\lambda(3, K_4^{(3)} - e, v)$. Let $S = \{\infty_1, \infty_2, \ldots, \infty_s\}$ and $X \cap S = \emptyset$. By the assumption, we can construct a $GDD(2, 3, v)$ $(X, \mathcal{G}, \mathcal{B})$ of type $g_1^{a_1} \cdots g_{m-1}^{a_{m-1}} g_m^1$ with $G_0 \in \mathcal{G}$ and $|G_0| = g_m$. Define the multiset $\mathcal{A}_{\infty_i} = \{\lambda(\infty_i, a, b, c) : \{a, b, c\} \in \mathcal{B}\}$ where $1 \le i \le s$. For each group $G \in \mathcal{G}$ and $G \ne G_0$, let $(G \cup S, S, \mathcal{A}_G)$ be an $HS_\lambda(3, K_4^{(3)} - e, g_i + s, s)$. Let $(G_0 \cup S, G_0, \mathcal{A}_{G_0})$ be an $HMP_\lambda(3, K_4^{(3)} - e, g_m + s, g_m)$. Define the multiset $\mathcal{C} = (\cup_{1 \le i \le s} \mathcal{A}_{\infty_i}) \cup (\cup_{G \in \mathcal{G}} \mathcal{A}_G) \cup \mathcal{A}$. Then it is readily checked that $(X \cup S, \mathcal{C})$ is a $P_\lambda(3, K_4^{(3)} - e, v + s)$, whose leave is the union of the leaves of the $MP_\lambda(3, K_4^{(3)} - e, v)$ and the $HMP_\lambda(3, K_4^{(3)} - e, g_m + s, g_m)$. Clearly, if the number of the edges in the union of leaves is at most 2, then the resultant packing is an $MP_\lambda(3, K_4^{(3)} - e, v + s)$. □

3 Constructions for small orders

In this section we give some designs with small orders, which can be used for the master designs and ingredient designs in Constructions 2.5-2.9. For positive integer n, let $I_n = \{0, 1, \ldots, n-1\}$.

Lemma 3.1 *There exist an $MP(3, K_4^{(3)}-e, 5)$ and an $HS(3, K_4^{(3)}-e, 5, 3)$.*

Proof Let $X = I_5$. The blocks for this design are $(2, 0, 1, 3)$, $(3, 0, 1, 4)$ and $(4, 0, 1, 2)$. Clearly, this design is also an $HS(3, K_4^{(3)} - e, 5, 3)$ with the hole $\{2, 3, 4\}$. □

Lemma 3.2 *There exists an $MP_\lambda(3, K_4^{(3)} - e, 6)$, $\lambda \in \{1, 2, 3\}$.*

Proof Let $X = I_6$. The blocks for these designs are given below.

$\lambda = 1:$	$(0,1,2,3)$,	$(0,1,4,5)$,	$(1,2,3,4)$,	$(2,0,4,5)$,
	$(3,0,4,5)$,	$(5,1,2,3)$.		
$\lambda = 2:$	$2(0,1,2,3)$,	$2(0,1,4,5)$,	$2(1,2,3,4)$,	$(2,0,4,5)$,
	$(3,0,4,5)$,	$(4,0,2,3)$,	$(5,0,2,3)$,	$(5,1,2,3)$,
	$(5,1,2,4)$,	$(5,1,3,4)$.		
$\lambda = 3:$	$3(0,1,2,3)$,	$3(0,1,4,5)$,	$2(1,2,3,4)$,	$(1,2,3,5)$,
	$(1,2,4,5)$,	$(2,0,4,5)$,	$(2,3,4,5)$,	$(3,0,4,5)$,
	$(3,1,4,5)$,	$2(4,0,2,3)$,	$2(5,0,2,3)$,	$(5,1,2,4)$,
	$(5,1,3,4)$.			

□

Lemma 3.3 *There exists an $HMP(3, K_4^{(3)} - e, 7, 4)$.*

Proof Let $X = I_7$ and $S = \{0, 1, 2, 3\}$. The blocks for this design are given below:

$$(0,1,4,5),\quad (0,2,4,6),\quad (1,2,4,6),\quad (2,3,4,6),\quad (3,0,4,6),$$
$$(3,1,5,6),\quad (4,1,3,5),\quad (5,0,2,3),\quad (5,2,4,6),\quad (6,0,1,5).$$

It is easy to check that the leave of this design has one edge $\{1, 2, 5\}$. □

Lemma 3.4 *There exists an $HS(3, K_4^{(3)} - e, v, v-2)$, $v \in \{11, 17\}$.*

Proof Let $X = Z_{v-2} \cup \{\infty_1, \infty_2\}$. Base blocks for these designs are given below. All other blocks are obtained by developing these base blocks by $+1$ modulo $(v-2)$, where $\infty_i + 1 = \infty_i$ for $i = 1, 2$.

$v = 11$: $(\infty_1, 0, 1, 3)$, $(\infty_2, 0, 1, 3)$, $(0, \infty_1, \infty_2, 4)$.
$v = 17$: $(\infty_1, 0, 1, 4)$, $(\infty_1, 0, 2, 7)$, $(\infty_2, 0, 1, 4)$, $(\infty_2, 0, 2, 7)$, $(0, \infty_1, \infty_2, 6)$. □

Lemma 3.5 *There exists an* $MP(3, K_4^{(3)} - e, 7)$.

Proof By Lemma 3.3, there exists an $HMP(3, K_4^{(3)} - e, 7, 4)$ with the hole $S = \{0, 1, 2, 3\}$. Then we can get the required design by adding the block $(0, 1, 2, 3)$ to the block set of the $HMP(3, K_4^{(3)} - e, 7, 4)$. It is easy to check that the leave of this design has two edges $\{1, 2, 3\}$ and $\{1, 2, 5\}$. □

Lemma 3.6 *There exists an* $MP_\lambda(3, K_4^{(3)} - e, 15)$, $\lambda \in \{2, 3\}$.

Proof By Lemmas 1.1 and 3.4, there exist an $S(3, K_4^{(3)} - e, 11)$ and an $HS(3, K_4^{(3)} - e, 11, 9)$. Applying Construction 2.5, we have an $S_\lambda(3, K_4^{(3)} - e, 11)$ and an $HS_\lambda(3, K_4^{(3)} - e, 11, 9)$. By Lemma 3.2 there exists an $MP_\lambda(3, K_4^{(3)} - e, 6)$ for each $\lambda \in \{2, 3\}$. By Lemma 2.1 there exists a GDD$(2, 3, 6)$ of type 2^3. Applying Construction 2.9 with $s = 9$ we can get a $P_\lambda(3, K_4^{(3)} - e, 15)$. Clearly the needed $HMP_\lambda(3, K_4^{(3)} - e, 11, 2)$ is $S_\lambda(3, K_4^{(3)} - e, 11)$. Checking the leave of the resultant $P_\lambda(3, K_4^{(3)} - e, 15)$, we can find that what we have constructed is an $MP_\lambda(3, K_4^{(3)} - e, 15)$. □

Lemma 3.7 *There exists an* $HS(3, K_4^{(3)} - e, s+1, s)$ *for each* $s \equiv 1, 3 \pmod 6$ *and* $s \geq 3$.

Proof Suppose that $(X, \{\{x\} : x \in X\}, \mathcal{B})$ is a GDD$(2, 3, s)$ of type 1^s by Lemma 2.1. Let $\mathcal{A} = \{(\infty, a, b, c) : \{a, b, c\} \in \mathcal{B}\}$. It is readily checked that $(X \cup \{\infty\}, X, \mathcal{A})$ is an $HS(3, K_4^{(3)} - e, s+1, s)$. □

Lemma 3.8 *There exists an* $MP(3, K_4^{(3)} - e, 8)$.

Proof By Lemma 3.7, there exists an $HS(3, K_4^{(3)} - e, 8, 7)$. Applying Construction 2.6, we have an $MP(3, K_4^{(3)} - e, 8)$, where the needed $MP(3, K_4^{(3)} - e, 7)$ is from Lemma 3.5. □

4 The case of $\lambda = 1$

In this section we shall prove that the packing number of an $MP(3, K_4^{(3)} - e, v)$ is $d(3, K_4^{(3)} - e, v)$ for any positive integer $v \geq 4$. A simple computation give the following lemma.

Lemma 4.1 *If there exists an $MP(3, K_4^{(3)} - e, v)$ with $d(3, K_4^{(3)} - e, v)$ blocks, then the number of edges in the leave is*

$$b = \begin{cases} 0 & v \equiv 0, 1, 2 \ (mod\ 9), \\ 1 & v \equiv 3, 4, 5 \ (mod\ 9), \\ 2 & v \equiv 6, 7, 8 \ (mod\ 9). \end{cases}$$

Lemma 4.2 *There exists an $MP(3, K_4^{(3)} - e, v)$ for positive integer $v \equiv 13$* (mod 18).

Proof By Lemma 1.1, there exists an $S(3, K_4^{(3)} - e, v-3)$ for each positive integer $v \equiv 13$ (mod 18). By Lemma 2.2, there exists a GDD$(2, 3, v-3)$ of type $2^{\frac{v-7}{2}} 4^1$. Applying Construction 2.9 with $s = 3$, we can get a $P(3, K_4^{(3)} - e, v)$ whose leave is just the leave of the $HMP(3, K_4^{(3)} - e, 7, 4)$ which has unique edge. The needed $HS(3, K_4^{(3)} - e, 5, 3)$ and $HMP(3, K_4^{(3)} - e, 7, 4)$ come from Lemmas 3.1 and 3.3. □

Lemma 4.3 *There exists an $MP(3, K_4^{(3)} - e, 14)$.*

Proof By Lemma 3.7, there exists an $HS(3, K_4^{(3)} - e, 14, 13)$. Applying Construction 2.6, we have an $MP(3, K_4^{(3)} - e, 14)$, where the needed $MP(3, K_4^{(3)} - e, 13)$ is from Lemma 4.2. □

Lemma 4.4 *There exists an $MP(3, K_4^{(3)} - e, v)$ for positive integer $v \equiv 4$,* 12, 14 (mod 18).

Proof By Corollary 2.3, there exists a $CS(3, K_4^{(3)} - e, m)$ of type $(1^m : 0)$ for each $m \equiv 1, 9$ (mod 18) and $m \geq 9$. By Lemma 2.1, there exists a GDD$(2, 3, m)$ of type 1^m. Applying Construction 2.8, we have a $CS(3, K_4^{(3)} - e, m + s)$ of type $(1^m : s)$ for each nonnegative integer s. Applying Construction 2.7 with $s = 3, 13$, we have an $MP(3, K_4^{(3)} - e, v)$

for $v = m + s \equiv 4, 12, 14 \pmod{18}$. The needed $HS(3, K_4^{(3)} - e, s+1, s)$ and $MP(3, K_4^{(3)} - e, 14)$ are from Lemmas 3.7 and 4.3. Clearly, the needed $MP(3, K_4^{(3)} - e, 4)$ exists, it has unique block. This completes the proof. □

Lemma 4.5 *There exists an $MP(3, K_4^{(3)} - e, v)$ for positive integer $v \equiv 7 \pmod{18}$.*

Proof The conclusion follows when $v = 7$ by Lemma 3.5. For $v \equiv 7 \pmod{18}$ and $v \geq 25$, there exist an $MP(3, K_4^{(3)} - e, v-3)$ and a GDD$(2, 3, v-3)$ of type $2^{\frac{v-7}{2}} 4^1$ by Lemma 4.4 and Lemma 2.2, respectively. Applying Construction 2.9 with $s = 3$, we can get a $P(3, K_4^{(3)} - e, v)$. The needed $HS(3, K_4^{(3)} - e, 5, 3)$ and $HMP(3, K_4^{(3)} - e, 7, 4)$ are from Lemmas 3.1 and 3.3. Checking the leave of the resulting $P(3, K_4^{(3)} - e, v)$, we know that its leave is just the union of the leaves of the $HMP(3, K_4^{(3)} - e, 7, 4)$ and the $MP(3, K_4^{(3)} - e, v-3)$, it has totally two edges. Hence the resultant packing is an $MP(3, K_4^{(3)} - e, v)$ for positive integer $v \equiv 7 \pmod{18}$ and $v \geq 25$. □

Lemma 4.6 *There exists an $MP(3, K_4^{(3)} - e, v)$ for positive integer $v \equiv 3, 5, 15, 17 \pmod{18}$, $v \geq 5$.*

Proof The conclusion follows when $v = 5$ by Lemma 3.1. For $v \equiv 3, 5, 15, 17 \pmod{18}$ and $v \geq 15$, there exists an $MP(3, K_4^{(3)} - e, v-3)$ by Lemmas 1.1 and 4.4. By Lemma 2.1 there exists a GDD$(2, 3, v-3)$ of type $2^{\frac{v-3}{2}}$. Applying Construction 2.9 with $s = 3$, we can get a $P(3, K_4^{(3)} - e, v)$. The needed $HS(3, K_4^{(3)} - e, 5, 3)$ and $HMP(3, K_4^{(3)} - e, 5, 2)$ are from Lemma 3.1. Checking the leave of the resulting $P(3, K_4^{(3)} - e, v)$, we know that its leave is just the union of the leaves of the $HMP(3, K_4^{(3)} - e, 5, 2)$ and the $MP(3, K_4^{(3)} - e, v-3)$, it has totally at most two edges. Hence the resultant packing is an $MP(3, K_4^{(3)} - e, v)$ for positive integer $v \equiv 3, 5, 15, 17 \pmod{18}$ and $v \geq 15$. □

Lemma 4.7 *There exists an $MP(3, K_4^{(3)} - e, v)$ for positive integer $v \equiv 6, 8, 16 \pmod{18}$.*

Proof The conclusion follows when $v = 6$ by Lemma 3.2. By Corollary 2.3 and Lemma 2.1, there exist a $CS(3, K_4^{(3)} - e, m)$ of type $(1^m : 0)$ and a

GDD$(2,3,m)$ of type 1^m for each $m \equiv 1,9 \pmod{18}$ and $m \geq 9$. Applying Construction 2.8, we have a $CS(3, K_4^{(3)} - e, m+s)$ of type $(1^m : s)$ for each nonnegative integer s. Applying Construction 2.7 with $s = 7$, we have an $MP(3, K_4^{(3)} - e, v)$ for $v = m + 7 \equiv 8, 16 \pmod{18}$ and $v \geq 16$. The needed $HS(3, K_4^{(3)} - e, 8, 7)$ and $MP(3, K_4^{(3)} - e, 8)$ come from Lemmas 3.7 and 3.8. Again, applying Construction 2.7 with $s = 15$, we have an $MP(3, K_4^{(3)} - e, v)$ for $v = m + 15 \equiv 6 \pmod{18}$ and $v \geq 24$. The needed $HS(3, K_4^{(3)} - e, 16, 15)$ and $MP(3, K_4^{(3)} - e, 16)$ are from Lemma 3.7 and as above, respectively. □

Combining Lemma 1.1, Lemma 4.2 and Lemmas 4.4-4.7, we have the following theorem.

Theorem 4.8 *An $MP(3, K_4^{(3)} - e, v)$ with $d(3, K_4^{(3)} - e, v)$ blocks exists for any positive integer $v \geq 4$, its leave has at most 2 edges.*

5 The case of $\lambda = 2, 3$

In this section we shall prove that the packing number of $MP_\lambda(3, K_4^{(3)} - e, v)$ is $d_\lambda(3, K_4^{(3)} - e, v)$ for any positive integer $v \geq 4$, $\lambda \in \{2, 3\}$. For convenience, we use the notation $\dot{\cup}$ to denote the union of multisets. A simple counting gives the following lemma.

Lemma 5.1 (1). *If there exists an $MP_2(3, K_4^{(3)} - e, v)$ with $d_2(3, K_4^{(3)} - e, v)$ blocks, then the number of edges in the leave is*

$$b = \begin{cases} 0 & v \equiv 0,1,2 \ (mod\ 9), \\ 2 & v \equiv 3,4,5 \ (mod\ 9), \\ 1 & v \equiv 6,7,8 \ (mod\ 9). \end{cases}$$

(2). *If there exists an $MP_3(3, K_4^{(3)} - e, v)$ with $d_3(3, K_4^{(3)} - e, v)$ blocks, then the number of edges in the leave is 0.*

Lemma 5.2 *There exists an $MP_\lambda(3, K_4^{(3)} - e, v)$ with $d_\lambda(3, K_4^{(3)} - e, v)$ blocks for positive integer $v \equiv 0, 1, 2, 3, 4, 5 \pmod 9$, $\lambda \in \{2, 3\}$.*

Proof For $v \equiv 0, 1, 2 \pmod 9$, the conclusion follows by Lemma 1.1 and Construction 2.5. Next we deal with the case of $v \equiv 3, 4, 5 \pmod 9$. By

Theorem 4.8 there exists an $MP(3, K_4^{(3)} - e, v)$ $(X, \mathcal{B})$ whose leave has unique edge. Then $(X, \mathcal{B}\dot{\cup}\mathcal{B})$ is an $MP_2(3, K_4^{(3)} - e, v)$ with $d_2(3, K_4^{(3)} - e, v)$ blocks. Again, by Theorem 4.8, let $(X, \mathcal{B}_1)$, $(X, \mathcal{B}_2)$ and $(X, \mathcal{B}_3)$ be three $MP(3, K_4^{(3)} - e, v)$s for $v \equiv 3, 4, 5 \pmod 9$ and $v \geq 4$. Without loss of generality, suppose their leaves are $\{a, b, c\}$, $\{a, b, d\}$ and $\{a, c, d\}$ respectively, where $\{a, b, c, d\} \subseteq X$. Then $(X, \mathcal{B}_1\dot{\cup}\mathcal{B}_2\dot{\cup}\mathcal{B}_3\dot{\cup}\{(a, b, c, d)\})$ is an $MP_3(3, K_4^{(3)} - e, v)$ with $d_3(3, K_4^{(3)} - e, v)$ blocks. □

Lemma 5.3 *There exists an* $MP_\lambda(3, K_4^{(3)} - e, v)$ *for positive integer* $v \equiv 7 \pmod{18}$, $\lambda \in \{2, 3\}$.

Proof Let $X = I_{v-3}$, $S = \{x, y, z\}$ and $X \cap S = \emptyset$. By Theorem 4.8 there exists an $MP(3, K_4^{(3)} - e, v - 3)$ $(X, \mathcal{A})$ with the unique edge $\{1, 2, 3\}$ in its leave. There exists a GDD$(2, 3, v - 3)$ of type $2^{\frac{v-7}{2}}4^1$ by Lemma 2.2 where the group of size 4 is $\{0, 1, 2, 3\}$. From the proof of Construction 2.9 with $s = 3$, we can get an $MP(3, K_4^{(3)} - e, v)$ on $X \cup S$, whose leave is just the union of the leaves of the $HMP(3, K_4^{(3)} - e, 7, 4)$ on $S \cup \{0, 1, 2, 3\}$ with hole $\{0, 1, 2, 3\}$ and the $MP(3, K_4^{(3)} - e, v - 3)$ $(X, \mathcal{A})$. Hence, without loss of generality, we have an $MP(3, K_4^{(3)} - e, v)$ $(X \cup S, \mathcal{C})$ whose leave has only two edges $\{1, 2, 3\}$ and $\{1, 2, x\}$ for any positive integer $v \equiv 7 \pmod{18}$. Let $(X \cup S, \mathcal{D})$ be an $MP(3, K_4^{(3)} - e, v)$ whose leave has only two edges, one of which is supposed to be $\{1, 3, x\}$ by a suitable permutation on $X \cup S$. Then $(X \cup S, \mathcal{C}\dot{\cup}\mathcal{D}\dot{\cup}\{(1, 2, 3, x)\})$ is an $MP_2(3, K_4^{(3)} - e, v)$ whose leave has unique edge.

By the arguments as above, suppose that there exists an $MP_2(3, K_4^{(3)} - e, v)$ $(X \cup S, \mathcal{E})$ whose leave being $\{1, 3, x\}$ by a suitable permutation on $X \cup S$. Then $(X \cup S, \mathcal{C}\dot{\cup}\mathcal{E}\dot{\cup}\{(1, 2, 3, x)\})$ is an $MP_3(3, K_4^{(3)} - e, v)$ whose leave is null. □

Lemma 5.4 *There exists an* $MP_\lambda(3, K_4^{(3)} - e, v)$ *for positive integer* $v \equiv 6, 8, 16 \pmod{18}$, $\lambda \in \{2, 3\}$.

Proof By Construction 2.5 and Lemma 3.7 there exist an $HS_\lambda(3, K_4^{(3)} - e, 8, 7)$ and an $HS_\lambda(3, K_4^{(3)} - e, 16, 15)$. Applying Construction 2.6 with $s \in \{7, 15\}$ to get an $MP_\lambda(3, K_4^{(3)} - e, 8)$ and an $MP_\lambda(3, K_4^{(3)} - e, 16)$, where the needed $MP_\lambda(3, K_4^{(3)} - e, 7)$ and $MP_\lambda(3, K_4^{(3)} - e, 15)$ are from Lemmas 5.3 and 3.6. By the proof of Lemma 4.7 and Construction 2.5, there exists a $CS_\lambda(3, K_4^{(3)} - e, m + s)$ of type $(1^m : s)$ for $m \equiv 1, 9 \pmod{18}$, $m \geq 9$ and $s \in \{7, 15\}$. Applying Construction 2.7 with $s = 7, 15$, we have an $MP_\lambda(3, K_4^{(3)} - e, v)$ for $v \equiv 6, 8, 16 \pmod{18}$, $v \geq 8$. The needed $HS_\lambda(3, K_4^{(3)} - e, s + 1, s)$ and $MP_\lambda(3, K_4^{(3)} - e, s + 1)$ are constructed as above, where $s = 7, 15$. By Lemma 3.2 an $MP_\lambda(3, K_4^{(3)} - e, 6)$ exists. The assertion then follows. □

Lemma 5.5 *There exists an $MP_\lambda(3, K_4^{(3)} - e, v)$ for positive integer $v \equiv 15, 17 \pmod{18}$, $\lambda \in \{2, 3\}$.*

Proof By Construction 2.5 and Lemma 3.4 there exists an $HS_\lambda(3, K_4^{(3)} - e, 17, 15)$. Applying Construction 2.6 with $s = 15$, we have an $MP_\lambda(3, K_4^{(3)} - e, 17)$ where the needed $MP_\lambda(3, K_4^{(3)} - e, 15)$ is from Lemma 3.6.

By Corollary 2.4 there exists a $CS(3, K_4^{(3)} - e, v - 15)$ of type $(2^{\frac{v-15}{2}} : 0)$ for each $v \equiv 15, 17 \pmod{18}$ and $v \geq 33$. By Lemma 2.1 there exists a GDD$(2, 3, v - 15)$ of type $2^{(v-15)/2}$. Applying Construction 2.8, we have a $CS(3, K_4^{(3)} - e, v + s - 15)$ of type $(2^{\frac{v-15}{2}} : s)$ for each $v \equiv 15, 17 \pmod{18}$ and $v \geq 33$. Applying Construction 2.5, we have a $CS_\lambda(3, K_4^{(3)} - e, v + s - 15)$ of type $(2^{\frac{v-15}{2}} : s)$. Applying Construction 2.7 with $s = 15$ to get an $MP_\lambda(3, K_4^{(3)} - e, v)$ for $v \equiv 15, 17 \pmod{18}$, $v \geq 33$. The needed $HS_\lambda(3, K_4^{(3)} - e, 17, 15)$ and $MP_\lambda(3, K_4^{(3)} - e, 17)$ is constructed as above. This completes the proof. □

Combining Lemmas 5.2-5.5, we have the following result.

Theorem 5.6 *An $MP_\lambda(3, K_4^{(3)} - e, v)$ with $d_\lambda(3, K_4^{(3)} - e, v)$ blocks exists for any positive integer $v \geq 4$ where $\lambda \in \{2, 3\}$, its leave has at most 2 edges.*

6 The proof of Theorem 1.2

For any positive integer λ, let $\lambda = 3k + i$ where k is nonnegative integer and $i = 0, 1, 2$. We distinguish two cases as follows:

Case 1: $i = 0$, i.e., $\lambda = 3k$. By Theorem 5.6 there exists an $MP_3(3, K_4^{(3)} - e, v)$ $(X, \mathcal{A})$ with $d_3(3, K_4^{(3)} - e, v)$ blocks and its leave being null. Then $(X, \mathcal{A}\dot{\cup}\mathcal{A}\dot{\cup}\cdots\dot{\cup}\mathcal{A})$ is an $MP_\lambda(3, K_4^{(3)} - e, v)$ with $d_\lambda(3, K_4^{(3)} - e, v)$

blocks and its leave being null, where $\mathcal{A}\dot{\cup}\mathcal{A}\dot{\cup}\cdots\dot{\cup}\mathcal{A}$ is the union of k multisets $\mathcal{A}$.

Case 2: $i = 1, 2$, i.e., $\lambda = 3k + i$. By Case 1 there exists an $MP_{3k}(3, K_4^{(3)} - e, v)$ $(X, \mathcal{A})$ with $d_{3k}(3, K_4^{(3)} - e, v)$ blocks and its leave being null. By Theorems 4.8 and 5.6 there exists an $MP_i(3, K_4^{(3)} - e, v)$ $(X, \mathcal{B})$ with $d_i(3, K_4^{(3)} - e, v)$ blocks. Then $(X, \mathcal{A}\dot{\cup}\mathcal{B})$ is an $MP_\lambda(3, K_4^{(3)} - e, v)$ with $d_\lambda(3, K_4^{(3)} - e, v)$ blocks.

This completes the proof of Theorem 1.2.

Remark: At present, much work has been done on graph decompositions, however, less is know on hypergraph decompositions. An interesting open problem is to consider the decompositions of the 3-uniform hypergraphs $\lambda K_v^{(3)}$ into other types of hypergraphs such as hypercycle and hyperpath.

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