Uniformly Weighted Star-Factors of Graphs *

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Abstract

A *star-factor* of a graph G is a spanning subgraph of G such that each component is a star. An *edge-weighting* of G is a function $w : E(G) \longrightarrow \mathbb{N}^+$, where $\mathbb{N}^+$ is the set of positive integers. Let Ω be the family of all graphs G such that every star-factor of G has the same weight under some fixed edge-weighting w. The open problem of characterizing the class Ω, posed by Hartnell and Rall, is motivated by the minimum cost spanning tree and the optimal assignment problems. In this paper, we present a simple structural characterization of the graphs in Ω that have girth at least five.

Key words: star-factor; girth; edge-weighting; uniform star-factor.

1 Introduction

Throughout this paper, all graphs considered are simple. We refer the reader to [2] for standard graph theoretic terms not defined in this paper.

Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. If G is not a forest, then the length of the shortest cycle in G is called the *girth* of G, denoted by $g(G)$ and a forest is considered to have infinite girth. If $S \subset V(G)$, then $G - S = G[V - S]$ is the subgraph of G obtained by deleting the vertices in S and all edges incident with them. Similarly, if $E' \subset E(G)$, then $G - E' = (V(G), E(G) - E')$. We denote the degree of a vertex x in G by $d_G(x)$, and the set of vertices adjacent to x in G by $N_G(x)$. The minimum degree of vertices in G is denoted by $\delta(G)$. A cycle

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(or path) with n vertices is denoted by C_n (or P_n). If vertices u and v are connected in G, the *distance* between u and v in G, $d_G(u,v)$, is the length of a shortest (u,v)-path in G. The *diameter* of G is the maximum distance over all pairs of vertices in G. A *leaf* is a vertex of degree one and a *stem* is a vertex which has at least one leaf as its neighbor. A *star* is a tree isomorphic to $K_{1,n}$ for some $n \geq 1$, and the vertex of degree n is called the *center* of the star. A *star-factor* of a graph G is a spanning subgraph of G such that each component is a star. Clearly a graph with isolated vertices has no star-factors. On the other hand, it is not hard to see that every graph without isolated vertices admits a star-factor. If one limits the size of the stars used, the existence of such a star-factor is non-trivial. In [1], Amahashi and Kano presented a criterion for the existence of a star-factor, i.e., $\{K_{1,1}, \cdots, K_{1,n}\}$-factor.

An *edge-weighting* of a graph G is a function $w : E(G) \longrightarrow \mathbb{N}^+$, where $\mathbb{N}^+$ is the set of positive integers. For a subgraph H, the *weight* of H under w is the sum of all the weight values for edges belonging to H, i.e., $w(H) = \Sigma_{e \in E(H)} w(e)$. An edge-weighting of G is *good* if every star-factor of G has the same weight under w. Motivated by the minimum cost spanning tree and the optimal assignment problems, Hartnell and Rall posed an interesting general question: for a given graph, does there exist an edge-weighting function w such that a certain type of spanning subgraph always has the same weight? In particular, they investigated the following narrow version of the problem in which the spanning subgraph is a star-factor.

Star-Weighting Problem (Hartnell and Rall [3]): Characterizing all graphs which admit a *good* edge-weighting?

To start the investigation, one may consider the special case where w is a *constant* edge-weighting function, i.e., all edges in G are assigned the same weight. In this case, two star-factors of G have the same weight if and only if they both have the same number of edges. For simplicity, we assume that all edges are assigned weight one.

Let $\mathscr{U}$ be the family of all graphs G such that every star-factor of G has the same number of edges. Clearly, two star-factors of G have the same number of edges if and only if they have the same number of components. Hartnell and Rall [3] classified the family of graphs in $\mathscr{U}$ that have girth at least five. In [6], the authors characterized the family of graphs in $\mathscr{U}$ whose members all have girth three and minimum degree at least two.

We say that a vertex subset S *dominates* a vertex v of G if $v \in S \cup N_G(S)$, and that S is a *dominating set* of G if every vertex of G is dominated by S. The cardinality of a smallest dominating set is called the *domination number* of G and denoted by $\gamma(G)$. A *matching* M of G is a set of independent edges of G. The number of edges in M is called the *size* of M and denoted by $|M|$. If G has no matching M' such that $|M'| > |M|$, then M is called a *maximum matching*, and the size of a maximum matching is called the *matching number* of G and denoted by $\nu(G)$. Recently, Kano, Wu and Yu [4] show a relationship between the family $\mathscr{U}$ and parameters $\nu(G)$, $\gamma(G)$.

Theorem 1. ([4]) *A connected graph $G \in \mathscr{U}$ if and only if $\gamma(G) = \nu(G)$.*

In [5], Randerath and Volkmann characterized all the graphs with equal domination number and matching number. With Theorem 1, the family $\mathscr{U}$ are largely characterized. However, there is no progress reported for uniformly weighted star-factors with *non-constant* function.

Denote by Ω the class of all graphs that admit a good edge-weighting. From the definition of edge-weighting, we see that $w(e) \geq 1$ for every edge e of G. In this paper, we attempt to initiate the study of uniform star-factors with a general (i.e., non-constant) edge-weighting function and obtain a structural characterization of the graphs in Ω that have girth at least five.

2 Main Results

Our attack on this problem is very closely followed that of the original problem in [3], with the appropriate modifications to the more general problem. It is interesting to note that the characterization for *any* edge-weighting function (Theorem 6) largely overlaps with that for a *constant* edge-weighting function, with only one additional case. We start with a few easy observations and lemmas.

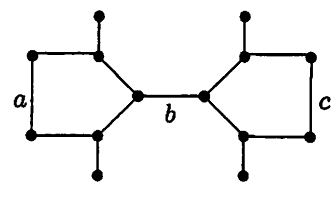


Figure 1. An example

Clearly, if $G \in \mathscr{U}$, then $G \in \Omega$. But the converse is not true. Figure 1 is an example which is star-factor uniform but requiring a *non-constant* edge-weighting function. If we let all the edges have the same weight, then we can find two star-factors with 10 edges and 7 edges, respectively. So $G \notin \mathscr{U}$. But if assigning a non-constant edge-weighting function w as follows:

$$w(e) = \begin{cases} 2k & e \in \{a,b,c\} \\ k & e \in E(G) - \{a,b,c\} \end{cases}$$

then it is not hard to verify that all the star-factors of G have the same weight under w, where $k \in \mathbb{N}^+$.

Note that if H is a spanning subgraph of G, then any star-factor of H is also a star-factor of G. This fact implies the following lemma which will be used frequently in reducing the problem of determining membership in Ω to its spanning subgraphs.

Lemma 1. *Let F be a subset of $E(G)$ such that $G - F$ has no isolated vertices. If one of the components in $G - F$ is not in Ω, then $G \notin \Omega$.*

The above lemma implies that if G is in Ω, then so is $G - F$. The basic idea to show that a graph does not belong to Ω is to decompose G into several components without isolated vertices and then simply find one of them not belonging to Ω.

Observation 1. *Let $P_6 = v_1v_2v_3v_4v_5v_6$ be a component of G and $G \in \Omega$. Then $w(v_3v_4) = w(v_2v_3) + w(v_4v_5)$ for any good edge-weighting w of G.*

Proof. Let S be a star-factor of $G - P_6$ with weight x. Then $S \cup \{v_1v_2, v_3v_4, v_5v_6\}$ and $S \cup \{v_1v_2v_3, v_4v_5v_6\}$ are two star-factors of G. Since $G \in \Omega$, $x + w(v_1v_2) + w(v_3v_4) + w(v_5v_6) = x + w(v_1v_2) + w(v_2v_3) + w(v_4v_5) + w(v_5v_6)$, i.e., $w(v_3v_4) = w(v_2v_3) + w(v_4v_5)$. □

In fact, Observation 1 holds even if one of v_2 and v_5 has more than one leaf as its neighbors. Using Observation 1 and Lemma 1, we have the following observation.

Observation 2. *Let C be a cycle. If $|C| = 6$ or $|C| \geq 8$, then C is not in Ω.*

We investigate the graphs, in Ω, with girth at least five but without leaves first.

Theorem 2. *If G is a graph with minimum degree at least two and girth six, then $G \notin \Omega$.*

Proof. Let C be a cycle of order six in G and $F = \{u_1u_2 \mid u_1 \in V(C), u_2 \in V(G) - V(C)\}$. Since the girth of G is six and $\delta(G) \geq 2$, there are no isolated vertices in $G - F$. Hence $G \notin \Omega$ by Lemma 1 and Observation 2. □

Using a similar technique, it is easy to see the following theorem.

Theorem 3. *If G is a graph with minimum degree at least two and girth at least eight, then $G \notin \Omega$.*

Theorem 4. *If G is a graph with minimum degree at least two and girth seven, then $G \in \Omega$ if and only if G is a 7-cycle.*

Proof. Let G be a 7-cycle, and $w(e) = k$ for each $e \in E$, where $k \in \mathbb{N}^+$. Then it is easy to check that $G \in \Omega$.

On the other hand, assume $G \in \Omega$ but G is not a 7-cycle. Let $C = v_1v_2v_3v_4v_5v_6v_7$ be a cycle in G. Without loss of generality, assume that v_7 has a neighbor u not on C. Let $F_1 = \{u_1u_2 \mid u_1 \in V(C), u_2 \in V(G) - V(C)\}$. Since the girth of G is seven and $\delta(G) \geq 2$, there are no isolated vertices in $G - F_1$. We see that C, as a component in $G - F_1$, is a cycle of length seven. Since $G \in \Omega$, then all edges in the cycle C must have the same weight. Let $F_2 = \{u_1u_2 \mid u_1 \in \{v_1, v_2, v_3, v_4, v_5, v_6\}, u_2 \in V(G) - \{v_1, v_2, v_3, v_4, v_5, v_6\}$. Then there are no isolated vertices in $G - F_2$ again. But $P_6 = v_1v_2v_3v_4v_5v_6$, as a component in $G - F_2$, is a path of order six. Since $G \in \Omega$, $w(v_3v_4) = w(v_2v_3) + w(v_4v_5)$ by Observation 1. Hence all the weights of edges in the cycle C must be 0, a contradiction. □

Lemma 2. *Let G be a graph with an induced cycle of order five such that four of the vertices are of degree two and the fifth is a stem. Then G does not belong to Ω.*

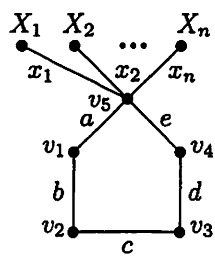


Figure 2. The component H

Proof. Suppose G belongs to Ω, and w is a good edge-weighting function. Let $C = v_1v_2v_3v_4v_5$ be a 5-cycle in G with a stem v_5. Let $X = \{X_1, X_2, \cdots, X_n\}$ be the set of leaves adjacent to v_5 and $F = \{u_1v_5 \mid u_1 \in V(G) - (X \cup \{v_1, v_4\})\}$. A component H of $G - F$ is isomorphic to the graph shown in Figure 2 (weights are indicated). Since $G - F$ has no isolated vertices, $H \in \Omega$. Thus all star-factors of H have the same weight under w and we have

$$\begin{aligned} & x_1 + x_2 + \cdots + x_n + a + c + e \\ = & \; x_1 + x_2 + \cdots + x_n + a + c + d \\ = & \; x_1 + x_2 + \cdots + x_n + b + c + e \\ = & \; x_1 + x_2 + \cdots + x_n + b + d \end{aligned}$$

From the above equation, we have $c = 0$, a contradiction. □

Lemma 3. *Let G be a graph in Ω with an induced 5-cycle. If exactly one of the vertices on this 5-cycle has degree at least three, then all of its neighbors not belonging to this 5-cycle must be stems.*

Proof. Let v be a vertex of degree at least three on the 5-cycle. Assume v has a neighbor x not on the 5-cycle and x is not a stem. By Lemma 2, x is not a leaf. Let F be the set of edges incident with x except vx. The graph $G - F$ has no isolated vertices, and the vertex v is a stem belonging to an induced 5-cycle that satisfies the hypothesis of Lemma 2. Thus G does not belong to Ω, a contradiction. □

Theorem 5. *If G is a graph with girth five and minimum degree at least two, then $G \in \Omega$ if and only if G is a 5-cycle.*

Proof. If G is a 5-cycle, clearly $G \in \Omega$ under a constant weight function.

Next consider a graph G, in Ω, with girth five and minimum degree at least two but G is not isomorphic to C_5. Let $C = v_1v_2v_3v_4v_5$ be a cycle in G. Assume, without loss of generality, that v_5 has a neighbor u not on C. Let $F = \{u_1u_2 \mid u_1 \in \{v_1, v_2, v_3, v_4, v_5\}, u_2 \in V(G) - (V(C) \cup u)\}$. If we delete all edges in F from G, then no isolated vertices are created in $G - F$ since the girth of G is five, so $G - F \in \Omega$ and u is a stem in $G - F$ by Lemma 3.

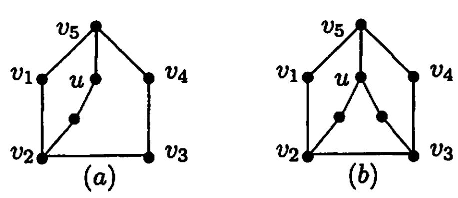


Figure 3.

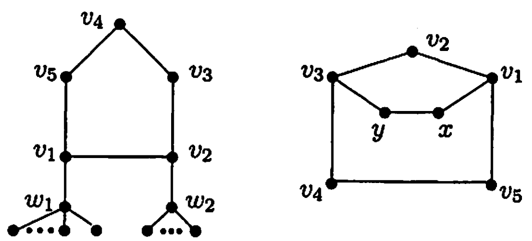
Moreover, u has at most two leaves as its neighbors are in $G - F$, and they are adjacent to v_2 or (and) v_3 in G. Without loss of generality, assume that all neighbors of u except v_5 in $G - F$ are leaves. Let H be the component containing the cycle C in $G - F$, and H' be the induced subgraph of G with $V(H)$. Then $G - V(H')$ and H' have no isolated vertices, and so $H' \in \Omega$ by Lemma 1. If there is exactly one leaf as a neighbor of u in H, then H' is shown in Figure 3(a). Otherwise, H' is isomorphic to the graph shown in Figure 3(b). It is not hard to check that both graphs are not in Ω, a contradiction to $G \in \Omega$. □

From the four theorems above, we obtain the following corollary.

Corollary 1. *If G is a graph with minimum degree at least two and girth at least five, then $G \in \Omega$ if and only if G is a 5-cycle or 7-cycle. Moreover, all edges of G must have the same weight under every good edge-weighting of G.*

Next, we attempt to determine all members in Ω which have girth at least five and with *leaves*. To derive our main theorem, we need the following lemmas.

Lemma 4. *Let G be a graph of girth five. Suppose that G contain a 5-cycle C that has no stem of G and there exist two adjacent vertices of degree at least three. Then G does not belong to Ω.*



(a) (b)

Figure 4.

Proof. Assume $G \in \Omega$ and let $C = v_1v_2v_3v_4v_5$ be the 5-cycle in G such that v_1 and v_2 both have degree at least three. Let F_1 be the set of all edges not on C but incident with one of v_3, v_4 and v_5. Since the girth of G is five, then no pair of vertices on C have a common neighbor not on C, and thus $G' = G - F_1$ has no isolated vertices. Furthermore, neither v_1 nor v_2 is a stem in G'. Let G'' be the graph obtained by deleting all edges incident with v_1 but not on C from G'. Then none of the stems created in G'' is a neighbor of v_2. However, by Lemma 1, G'' belongs to Ω and so by Lemma 3 all neighbors of v_2 not on C, in G'', must be stems. Thus all neighbors of v_2 not on C in G' must be stems. A similar argument yields that all neighbors of v_1 not on C in G' must be stems. Let w_1 and w_2 be stems adjacent to v_1 and v_2, respectively, in G'. Then there exists at most one common neighbor of degree two between w_1 and w_2 since the girth of G is five. Let X be the set of leaves adjacent to w_1 or w_2 in G' and $F_2 = \{u_1u_2 \mid u_1 \in \{v_1, v_2, w_1, w_2\}, u_2 \in V(G') - (\{v_1, v_2, w_1, w_2, v_3, v_5\} \cup X)\}$. Then $G' - F_2$ (or $(G' - F_2) \cup uw_1$, if there exists a common neighbor u of degree two between w_1 and w_2) has no isolated vertices and a component H in $G' - F_2$ (or $(G' - F_2) \cup uw_1$) is isomorphic to the graph shown in Figure 4(a). By Lemma 1, H belongs to Ω. It is easy to show that $H \notin \Omega$ by Observation 1, a contradiction. □

Lemma 5. *Let G be a graph of girth five. Suppose that G contain a 5-cycle $C = v_1v_2v_3v_4v_5$ in which no vertex is a stem and two nonadjacent vertices v_1 and v_3 of C have degree at least three. If G belongs to Ω, then all neighbors of v_1 and v_3 not belonging to C must be stems.*

Proof. Assume that v_1 has a neighbor x not on C but x is not a stem. By Lemma 4, v_2, v_4 and v_5 must all have degree two in G. Let F be the set of all edges other than xv_1, that are incident with x, and let $G_1 = G - F$. Since x is not a stem in G, $G_1 \in \Omega$. If v_3 is not a stem in G_1, let F_1 be the set of edges, not in C, but incident with v_3. Then $G_1 - F_1$ is in Ω, but it contains a 5-cycle satisfying the conditions of Lemma 2. Therefore v_3 is a stem in G_1. Moreover, we note that v_3 has exactly one leaf, say y, as its neighbor in G_1 since $g(G) = 5$. Thus, we derive an induced subgraph H with vertices $v_1, v_2, v_3, v_4, v_5, x$ and y shown in Figure 4(b). Let $F_2 = \{u_1u_2 \mid u_1 \in V(H), u_2 \in V(G) - V(H)\}$. Then $G - F_2$ has no isolated vertices and H is a component. Thus $H \in \Omega$ by Lemma 1. But by Theorem 5, $H \notin \Omega$, a contradiction. □

Lemma 6. *Let G be a graph with an induced cycle C of order i (where $i = 6$ or 7) such that all the vertices on C are of degree two except one which is a stem. Then $G \notin \Omega$.*

Proof. Similar to the proof of Lemma 2. □

Lemma 7. *Let G be a graph with an induced cycle C of order i (where $i = 6$ or 7) such that all the vertices on C are of degree two except one of degree at least three. Then $G \notin \Omega$.*

Proof. Let v_1 be a vertex on the cycle C of degree at least three. If v_1 is a stem, then by Lemma 6, G does not belong to Ω. Otherwise, let F_1 be the set of edges incident with v_1 but not on the cycle C. Then the graph $G - F_1$ has no isolated vertices and has the cycle C as a component.

Case 1. C is a 6-cycle. Then $G \notin \Omega$ by Observation 2.

Case 2. C is a 7-cycle. If $G \in \Omega$, then all the edges on C must have the same weight by Corollary 1. We delete edges v_1v_2 and v_1v_7 from G, and obtain $P_6 = v_2v_3v_4v_5v_6v_7$ as a component of $G - \{v_1v_2, v_1v_7\}$. Moreover, $G - \{v_1v_2, v_1v_7\}$ contains no isolated vertices since v_1 is not a stem. Hence $w(v_4v_5) = w(v_3v_4) + w(v_5v_6)$ by Observation 1. But $w(v_4v_5) = w(v_3v_4) = w(v_5v_6)$, so $w(v_4v_5) = w(v_3v_4) = w(v_5v_6) = 0$, a contradiction. □

Now we present and prove our main result.

Theorem 6. *Let G be a connected graph of girth at least five. Then $G \in \Omega$ if and only if G is*

1) *a C_5, or*

2) *a C_7, or*

3) *$\delta(G) = 1$ and each component of the graph obtained by removing the leaves and stems from G is one of the following:*

3a) *a 5-cycle with at most two vertices of degree three or more in G. Furthermore, if there are two such vertices, then they are non-adjacent on the 5-cycle;*

3b) *a star $K_{1,m}$ $(m \geq 2)$. Moreover, the center of $K_{1,m}$ has degree m in G;*

3c) *an edge; or*

3d) *an isolated vertex.*

Proof. If G has girth at least five and $\delta(G) \geq 2$, then the theorem follows from Corollary 1.

Next, suppose G contains at least one leaf *and* $G \in \Omega$. Let L be the set of leaves of G and $S = N(L)$ the set of stems.

If $G - (L \cup S)$ has a component which is a 5-cycle C, then, by Lemmas 4 and 5, there are at most two vertices of C with degree greater than two in G (and if there are two such vertices, they are nonadjacent on C) and all of their neighbors not on C must be stems.

Hence we now consider those components of $G - (L \cup S)$ that have no 5-cycles. If H is such a component, then $g(H) \geq 6$. We shall show that *the diameter of H is at most two.*

Case 1. The diameter of H is three.

It is easy to show that H is not a tree. Since $g(H) \geq 6$, there exists a path $P = abcd$ in H such that a is adjacent to a stem s of G and d is not a leaf in H.

Claim 1. d is adjacent to at least one stem in G.

Otherwise, all neighbors of d, in G, are not stems. But d is not a leaf in H, so there is another path of length 3 or 4, disjoint from P in H, joining a and d because $g(H) \geq 6$ and the diameter of H is three.

Therefore the two paths form a 6-cycle or 7-cycle, denote it by C. Let $F_1 = \{u_1u_2 \mid u_1 \in V(C), u_2 \in V(G) - (V(C) \cup s)\}$. Since $g(H) \geq 6$, then $G_1 = G - F_1$ induces no isolated vertices and has a component satisfying the hypothesis of Lemma 7, a contradiction to $G \in \Omega$ by Lemma 1. Hence d is adjacent to at least one stem s' in G.

Claim 2. There exists no common neighbor of degree two between s and d, or s' and a, or a and d in G.

Since $g(G) \geq 5$, there is at most one common neighbor u between s and d. If $d_G(u) = 2$, then $d_H(a, u) = 4$ since $g(H) \geq 6$, a contradiction to the fact that the diameter of H is three. The same arguments can be applied to vertices s' and a. Moreover, $g(H) \geq 6$ implies that there exists no common neighbor of degree two between a and d in G either.

Claim 3. There exists no common neighbor of degree two between s and c, or s' and b in G.

Let X and Y be the sets of leaves adjacent to vertices s and s' in G, respectively. Suppose there is a common neighbor u of degree two between s and c. Let $F_2 = \{u_1u_2 \mid u_1 \in \{a, b, c, u, s\}, u_2 \in V(G) - (\{a, b, c, u, s\} \cup X)\}$. Then the graph $G_2 = G - F_2$ has no isolated vertices and has a component satisfying the hypothesis of Lemma 2, so $G \notin \Omega$, a contradiction. Similarly, we can show that there exists no common neighbor of degree two between s' and b in G, either.

Clearly, there is at most one common vertex between s and s' since $g(G) \geq 5$. If there is no common neighbor between s and s' or the common neighbor is of degree at least three in G, let $F_3 = \{u_1u_2 \mid u_1 \in \{a, b, c, d, s, s'\}, u_2 \in V(G) - (\{a, b, c, d, s, s'\} \cup X \cup Y)\}$ and $G_3 = G - F_3$; otherwise (i.e., there exists a common neighbor v of degree two between s and s' in G), let $G_3 = G - (F_3 \cup sv)$. By Claims 2 and 3, we see that G_3 has no isolated vertices but has a component isomorphic to the graph shown in Figure 5(a). Applying Observation 1, we have

$$w(ab) = w(as) + w(bc). \tag{1}$$

$$w(cd) = w(ds') + w(bc). \tag{2}$$

Since $G \in \Omega$, we also have

$$w(as) + w(ds') + w(bc) = w(ab) + w(cd). \tag{3}$$

Combining (1), (2) and (3), we have $w(bc) = 0$, a contradiction. Thus the diameter of H can not be three.

Case 2. The diameter of H is at least four.

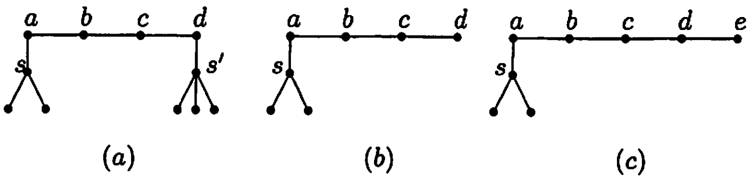


Figure 5.

Since $g(H) \geq 6$, there exists a path *abcde* in H so that a is adjacent to a stem s of G.

Claim 4. There are no common neighbors of degree two between a and d, or a and e, or b and e in G.

Clearly there are no common neighbors of degree two in G between a and d, or b and e since $g(H) \geq 6$.

Say there is a common neighbor u between vertices a and e. Let F_4 be the set of edges incident to the cycle $C_6 = abcdeu$ but not on it. Then $G_4 = G - F_4$ has no isolated vertices but has a 6-cycle as a component. Hence $G \notin \Omega$ by Observation 2, a contradiction.

Claim 5. There are no common neighbors of degree two between s and c, or d, or e in G.

Applying the same technique as in the proof of Claim 3, we can see that there is no common neighbor of degree two between s and c.

Suppose u is a common neighbor of degree two between s and d. Let Z be the set of leaves adjacent to vertex s in G and $F_5 = \{u_1u_2 \mid u_1 \in \{a, b, c, d, s, u\}, u_2 \in V(G) - (\{a, b, c, d, s, u\} \cup Z)\}$. Then $G_5 = G - F_5$ has no isolated vertices but has a component satisfying the hypothesis of Lemma 6, a contradiction to $G \in \Omega$ by Lemma 1.

Say there is a common neighbor v of degree two between vertices s and e. Let $F_6 = \{u_1u_2 \mid u_1 \in \{a, b, c, d, e, s, v\}, u_2 \in V(G) - (\{a, b, c, d, e, s, v\} \cup Z)\}$. Then, by Claim 4 and the fact that there are no common neighbors of degree two between s and c, or s and d, $G_6 = G - F_6$ has no isolated vertices but contains a component satisfying the hypothesis of Lemma 6, a contradiction to $G \in \Omega$.

Claims 4 and 5 yield that there are no common neighbors of degree two between any two vertices of $\{s, a, b, c, d, e\}$. Let $F_7 = \{u_1u_2 \mid u_1 \in \{a, b, c, d, s\}, u_2 \in V(G) - (\{a, b, c, d, s\} \cup Z)\}$. Then $G_7 = G - F_7$ has no isolated vertices and has a component H' isomorphic to the graph shown in Figure 5(b). By Observation 1, we see

$$w(ab) = w(as) + w(bc). \tag{4}$$

On the other hand, let $F_8 = \{u_1u_2 \mid u_1 \in \{a,b,c,d,e,s\}, u_2 \in V(G) - (\{a,b,c,d,e,s\} \cup Z)\}$. Then $G_8 = G - F_8$ has no isolated vertices and has a component H'' isomorphic to the graph shown in Figure 5(c). Then $H'' \in \Omega$ and so

$$w(ab) = w(as), w(bc) = w(cd). \tag{5}$$

Equations (4) and (5) imply that $w(bc) = w(cd) = 0$, a contradiction. Thus the diameter of H can not be more than three.

Hence the diameter of H is at most two, i.e., H is either an isolated vertex or isomorphic to a star, say $K_{1,m}$. For $m \geq 2$, let the vertices of H be $c, b_1, b_2, \cdots, b_m$ where c has degree m in H. For each $1 \leq i \leq m$, let s_i be a stem of G adjacent to b_i.

Claim 6. If $m \geq 2$, then c does not have a neighbor, in G, which is a stem.

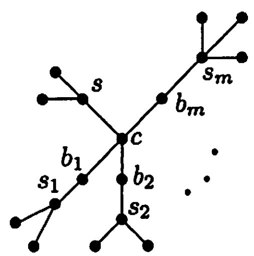


Figure 6.

Otherwise, let s be one such neighbor and let L_i and L_s be the sets of leaves adjacent to s_i and s in G, respectively. If there exists a vertex u adjacent only to vertices in $\{s_1, s_2, \ldots, s_m, b_1, b_2, \ldots, b_m\}$ and to at least one vertex of $\{s_1, s_2, \ldots, s_m\}$, then we can delete all edges which are adjacent to u except one s_ku (for some $1 \leq k \leq m$). Thus we obtain a spanning subgraph of G without isolated vertices, and u is a leaf adjacent to s_k. Hence we may assume that there are no vertices which are only adjacent to vertices $\{s_1, s_2, \ldots, s_m, b_1, b_2, \ldots, b_m\}$. For the same reason, we may assume no vertices only adjacent to vertices s and $s_i, 1 \leq i \leq m$. Let $F_9 = \{u_1u_2 \mid u_1 \in \{c, b_1, b_2, \cdots, b_m, s_1, s_2, \cdots, s_m, s\}, u_2 \in V(G) - (\{c, b_1, b_2, \cdots, b_m, s_1, s_2, \cdots, s_m, s\} \cup L_1 \cup L_2 \cup \cdots \cup L_m \cup L_s)\}$. Then $G_9 = G - F_9$ has no isolated vertices, since $g(G) \geq 5$, but has a component H''' isomorphic to the graph shown in Figure 6. Let the total weight of all edges incident with the leaves in $L_1 \cup L_2 \cup \cdots \cup L_m \cup L_s$ be w'. Since $H''' \in \Omega$, then by Observation 1, we have

$$w(cb_i) = w(b_i s_i) + w(cs), \; 1 \le i \le m. \tag{6}$$

Since $G \in \Omega$, we also have

$$w' + w(cb_1) + w(cb_2) + \cdots + w(cb_m) = w' + w(cs) + w(b_1 s_1) + \cdots + w(b_m s_m). \tag{7}$$

Equations (6) and (7) imply $m = 1$, a contradiction to $m \ge 2$.

From Claim 6, we conclude that the center c of the star $H = K_{1,m}$ ($m \ge 2$) has degree m in G.

Therefore, if $\delta(G) = 1$ and $G \in \Omega$, then every component of $G - (L \cup S)$ is one of 3a) - 3d).

Conversely, assume G has the specified structure. In the following, we construct a good edge-weighting function.

Case 1. No component of $G - (L \cup S)$ is $K_{1,1}$.

In this case, all edges of G are given the same weight and it follows from the characterization in [3] that all star-factors have the same weight.

Case 2. $K_{1,1}$ appears as a component of $G - (L \cup S)$.

For each $K_{1,1} = uv$ of $G - (L \cup S)$, assign edge-weights for the edges incident to $N_G(u) \cup N_G(v)$ as follows:

$$w(e) = \begin{cases} a & e \in \{ux \mid x \in N(u)\}, e \ne uv \\ b & e \in \{vx \mid x \in N(v)\}, e \ne uv \\ a+b & e = uv \end{cases}$$

where $a, b > 0$. All other edges are assigned the same weight.

Let T be any star-factor of G. Then T contains exactly one edge incident to each leaf of G. For each component which is a 5-cycle or an isolated vertex, it can be dealt with as in Case 1. For $H = uv$, T contains an edge uv or an edge joining a stem s_1 to u and an edge joining another stem s_2 to v. Since $w(uv) = w(us_1) + w(vs_2)$, we conclude that every star-factor of G has the same weight.

This completes the proof. □

Remark. The main theorem has classified all graphs in Ω with girth at least five. The families remaining to be determined are graphs of girth three or four. It seems that the structure of these families are much more complicated, but it would be an interesting topic to investigate.

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