On constructing graphs with the same status sequence

Jen-Ling Shang *
Department of Banking and Finance, Kainan University
Tao-Yuan, Taiwan 33857, R.O.C.
jlshang@mail.knu.edu.tw

Abstract

The status of a vertex v in a graph is the sum of the distances between v and all vertices. The status sequence of a graph is the list of the statuses of all vertices arranged in nondecreasing order. It is well known that non-isomorphic graphs may have the same status sequence. This paper gives a sufficient condition for a graph G with the property that there exists another graph G' such that G' and G have the same status sequence and G' is not isomorphic to G.

1 Introduction

All graphs considered in this paper are finite, simple and loopless. Let G be a connected graph. For a given vertex v of G, the *status* of v, denoted by $s_G(v)$, is defined by $s_G(v) = \sum_{u \in V(G)} d_G(v,u)$, where $d_G(v,u)$ is the distance between v and u. The *status sequence* of G is the list of the statuses of all the vertices of G arranged in nondecreasing order. It is well known that non-isomorphic graphs may have the same status sequence. Example of non-isomorphic graphs with the same status sequence can be found in [1, 3, 6]. A graph is *status injective* [1, 2, 4, 5] if its status sequence consists of distinct numbers. Pachter [4] showed that for any given finite simple connected graph and any positive integer N, there exist N non-isomorphic status injective graphs, each with the same status sequence and containing the given graph as an induced subgraph. As mentioned in [5], a path is uniquely determined by its status sequence. It was also conjectured in [5] that a tree and a non-tree graph can not have the same status sequence. We

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are interested in the following problem. If a given connected graph is not uniquely determined by its status sequence, how can one construct another graph such that the two graphs are non-isomorphic but have the same status sequence? In particular, how can one construct a non-isomorphic tree with the same status sequence as a given tree? Towards this goal, this paper shows that if a given graph satisfies certain conditions, then a simple modification of this graph yield a non-isomorphic graph with the same status sequence.

2 Main result

This section begins with the following theorem.

Theorem 2.1 *Let C be a connected graph and x, y a pair of distinct vertices in C with $s_C(x) = s_C(y)$. Let A and B be two connected graphs of the same order and $a \in V(A), b \in V(B)$. Suppose G is the graph constructed from A, B and C by identifying x with a and y with b, and G' is the graph constructed from A, B and C by identifying x with b and y with a. Then G and G' have the same status sequence. Furthermore, if $deg_A a \neq deg_B b$ and $deg_C x \neq deg_C y$, then G and G' have different degree sequences, and hence are non-isomorphic.*

Proof. Denote the two vertices in G obtained by identifying x in C with a in A and y in C with b in B by x and y, respectively. And denote the two vertices in G' obtained by identifying x in C with b in B and y in C with a in A by x and y, respectively. Note that in G and G' both x and y are cut vertices. Now show that $s_G(z) = s_{G'}(z)$ for all z. We distinguish the two cases.

Case 1. $z \in V(A) \bigcup V(B) - \{a, b\}$.

Without loss of generality let $z \in V(A) - \{a\}$. It is easily seen that

$$\begin{aligned} s_G(z) &= s_A(z) + s_C(x) + (|V(C)| - 1) \cdot d_A(a, z) \\ &\quad + s_B(b) + (|V(B)| - 1) \cdot (d_C(y, x) + d_A(a, z)), \end{aligned}$$

and

$$\begin{aligned} s_{G'}(z) &= s_A(z) + s_C(y) + (|V(C)| - 1) \cdot d_A(a, z) \\ &\quad + s_B(b) + (|V(B)| - 1) \cdot (d_C(x, y) + d_A(a, z)). \end{aligned}$$

By the assumption $s_C(x) = s_C(y)$, we obtain $s_G(z) = s_{G'}(z)$.

Case 2. $z \in V(C)$.

It is easily seen that

$$\begin{aligned} s_G(z) &= s_C(z) + s_A(a) + (|V(A)| - 1) \cdot d_C(x, z) \\ &\quad + s_B(b) + (|V(B)| - 1) \cdot d_C(y, z), \end{aligned}$$

and

$$\begin{aligned} s_{G'}(z) &= s_C(z) + s_A(a) + (|V(A)| - 1) \cdot d_C(y, z) \\ &\quad + s_B(b) + (|V(B)| - 1) \cdot d_C(x, z). \end{aligned}$$

By the assumption $|V(A)| = |V(B)|$, we obtain $s_G(z) = s_{G'}(z)$.

From above cases we see that the two graphs G and G' have the same status sequence. Now show that if $deg_A a \neq deg_B b$ and $deg_C x \neq deg_C y$, then G and G' have different degree sequences. It is clear that if $z \notin \{x, y\}$ then $deg_G z = deg_{G'} z$. So it suffices to show that $\{deg_G x, deg_G y\} \neq \{deg_{G'} x, deg_{G'} y\}$. Suppose, on the contrary, that $\{deg_G x, deg_G y\} = \{deg_{G'} x, deg_{G'} y\}$. Then either $deg_G x = deg_{G'} x$ or $deg_G x = deg_{G'} y$. If $deg_G x = deg_{G'} x$, that is, $deg_A a + deg_C x = deg_B b + deg_C x$, then $deg_A a = deg_B b$. This contradicts the assumption that $deg_A a \neq deg_B b$. If $deg_G x = deg_{G'} y$, that is, $deg_A a + deg_C x = deg_A a + deg_C y$, then $deg_C x = deg_C y$. This contradicts the assumption that $deg_C x \neq deg_C y$. Thus $\{deg_G x, deg_G y\} \neq \{deg_{G'} x, deg_{G'} y\}$. □

For a graph G with certain conditions, the following corollary gives a non-isomorphic graph G' with the same status sequence as G.

Corollary 2.2 *Let G be a connected graph and x, y a pair of distinct cut vertices of G with $s_G(x) = s_G(y)$. Suppose that $A_1, A_2, \cdots, A_p (p \geq 1)$ are some components of $G - x$ such that $y \notin \bigcup_{i=1}^{p} V(A_i)$, and $B_1, B_2, \cdots, B_q (q \geq 1)$ are some components of $G - y$ such that $x \notin \bigcup_{j=1}^{q} V(B_j)$. Let A and B be the subgraphs of G induced by the vertex sets $\{x\} \bigcup (\bigcup_{i=1}^{p} V(A_i))$ and $\{y\} \bigcup (\bigcup_{j=1}^{q} V(B_j))$, respectively. Let C be the subgraph of G induced by the vertex set $(V(G) - (V(A) \bigcup V(B))) \bigcup \{x, y\}$. Suppose G' is the graph constructed from A, B and C by identifying the vertex x in A with the vertex y in C, and the vertex y in B with the vertex x in C. If $|V(A)| = |V(B)|$, then G' and G have the same status sequence. Furthermore, if $deg_A x \neq deg_B y$ and $deg_C x \neq deg_C y$, then G' and G have different degree sequences, and hence are non-isomorphic.*

Proof. By Theorem 2.1 it suffices to show that $s_C(x) = s_C(y)$.
Since

$$s_G(x) = s_A(x) + s_C(x) + s_B(y) + (|V(B)| - 1) \cdot d_G(y, x),$$

$$s_G(y) = s_B(y) + s_C(y) + s_A(x) + (|V(A)| - 1) \cdot d_G(x, y),$$

and by the assumption $s_G(x) = s_G(y)$ and $|V(A)| = |V(B)|$, we have $s_C(x) = s_C(y)$. □

Note that in Corollary 2.2 G' is a tree if G is a tree. A *caterpillar* is a tree which contains a path such that each vertex not on the path is adjacent to a vertex on the path. A *spider* is a tree of which one and only one vertex has degree exceeding 2. Applying Corollary 2.2, we give two examples. In Fig. 1 there are two non-isomorphic non-tree graphs with the same status sequence $\{23, 23, 24, 24, 24, 32, 32, 33, 33, 34, 34, 43, 43\}$. The number beside each vertex in Fig. 1 is the status of the vertex. The induced subgraphs A and B of G have vertex sets $\{a, b, x\}$ and $\{i, j, y\}$, respectively. In Fig. 2 there are two non-isomorphic caterpillars with the same status sequence $\{26, 26, 28, 32, 32, 36, 36, 40, 42, 42, 42, 50\}$. The induced subgraphs A and B of F have vertex sets $\{a, b, x\}$ and $\{d, e, y\}$, respectively. Fig. 2 shows that a caterpillar is not uniquely determined by its status sequence. However, [5] shows that whenever a tree T and a spider S have the same status sequence then $T \cong S$. If the conjecture [5] that a tree and a non-tree can not have the same status sequence is true, then a spider is uniquely determined by its status sequence.

G

G'

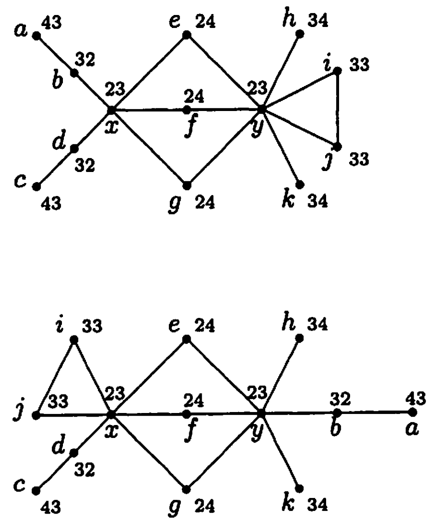

Fig. 1

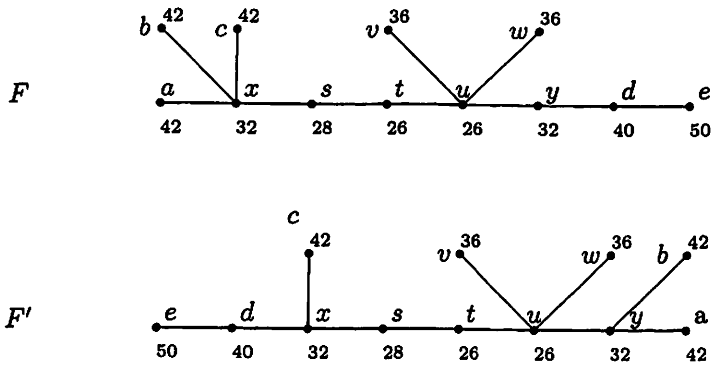

Fig. 2

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