# A note on the LEL-equienergetic graphs*

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**Abstract:** Let $G$ be a graph with $n$ vertices and $\mu_1, \mu_2, \ldots, \mu_n$ be the Laplacian eigenvalues of $G$. The Laplacian-energy-like graph invariant $LEL(G) = \sum_{i=1}^{n} \sqrt{\mu_i}$, has been defined and investigated in [1]. Two non-isomorphic graphs $G_1$ and $G_2$ of the same order are said to be LEL-equienergetic if $LEL(G_1) = LEL(G_2)$. In [2], three pairs of LEL-equienergetic non-cospectral connected graphs are given. It is also claimed[2] that the LEL-equienergetic non-cospectral connected graphs are relatively rare. It is natural to consider the question: Whether the number of the LEL-equienergetic non-cospectral connected graphs is finite? The answer is negative, because we shall construct a pair of LEL-equienergetic non-cospectral connected graphs of order $n$, for all $n \geq 12$ in this paper.

# 1 Introduction

Let $G = (V, E)$ be a simple connected graph with $n$ vertices and $m$ edges. In general, if $m = n + c - 1$, then $G$ is called a *c-cyclic* graph. Specially, a 1-cyclic graph, i.e., $m = n$, is known as a *unicyclic graph.*

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Let the adjacency matrix, degree matrix of $G$ be $A(G) = [a_{ij}]$, $D(G) = diag\{d(v_1), d(v_2), \cdots, d(v_n)\}$, respectively. The *Laplacian matrix* of $G$ is $L(G) = D(G) - A(G)$. Let $\lambda_1, \lambda_2, ..., \lambda_n$ be the adjacency spectrum of $G$, and $\mu_1, \mu_2, ..., \mu_n$ be the Laplacian spectrum of $G$. The Lapalacian characteristic polynomial of $G$ is denoted by $\Phi(G, \lambda)$, i.e., $\Phi(G, \lambda)=\det(\lambda I - L(G))$.

The *energy* $E(G)$ of a graph $G$ is defined[3] as $E(G) = \sum_{i=1}^{n} |\lambda_i|$. This quantity has a long known application in molecular-orbital theory of organic molecules (see [3-5]) and has been much investigated (see [6-9]). Two non-isomorphic graphs $G_1$ and $G_2$ of the same order are said to be *equienergetic* [10] if $E(G_1) = E(G_2)$. Clearly, cospectral graphs are equienergetic, but such case is of no interest. In [11], a pair of equienergetic non-cospectral connected graphs of order $n$ for $n \geq 8$ is given. For other results on equienergetic graphs see [12-14] and the references therein.

The *Laplacian energy* $LE(G)$ of a graph $G$ has been defined [15] as $LE(G) = \sum_{i=1}^{n} |\mu_i - \frac{2m}{n}|$. Similarly as the graph energy, two non-isomorphic graphs $G_1$ and $G_2$ of the same order are said to be *LE-equienergetic* if $LE(G_1) = LE(G_2)$ (see [2]). The *Laplacian-energy-like invariant* of a graph $G$, denoted as $LEL(G) = \sum_{i=1}^{n} \sqrt{\mu_i}$, has been defined and investigated in [1]. In [2], two non-isomorphic graphs $G_1$ and $G_2$ of the same order are said to be *LEL-equienergetic* if $LEL(G_1) = LEL(G_2)$. The quantities $E(G)$, $LE(G)$ and $LEL(G)$ were found to have a number of analogous properties[1,15], for the (chemical) application background of the LEL-equienergetic graphs see [1-2].

In [2], a pair of LE-equienergetic non-cospectral connected graphs of order $n$ for $n \geq 4$ and three pairs of LEL-equienergetic non-cospectral connected graphs are given. It is also claimed[2] that the LEL-equienergetic non-cospectral connected graphs are relatively rare. It is natural to consider the question: Whether the number of the LEL-equienergetic non-cospectral connected graphs is finite? The answer is negative, because we shall construct a pair of LEL-equienergetic non-cospectral connected graphs of order $n$, for all $n \geq 12$ in this paper. Moreover, we identify a pair of LE-equienergetic non-cospectral connected unicyclic graphs of order $n$, for all $n \geq 7$.

# 2 Main results

**Theorem 2.1** *There exists a pair of LE-equienergetic non-cospectral, connected unicyclic graphs of order $n$, for all $n \geq 7$.*

**Proof.** Let $G_1$ and $G_2$ be the connected unicyclic graphs as shown in Fig.

1. By an elementary calculation, we have

(1a) $\Phi(G_1,\lambda)=\lambda(\lambda-1)^{n-7}(\lambda-3)(\lambda^2-3\lambda+1)(\lambda^3-(n+1)\lambda^2+(3n-5)\lambda-n)$.

(2a) $\Phi(G_2,\lambda)=\lambda(\lambda-1)^{n-6}(\lambda^2-5\lambda+5)(\lambda^3-(n+1)\lambda^2+(3n-5)\lambda-n)$.

By equalities (1a) and (2a), it is easy to see that $LE(G_1)=LE(G_2)$ for all $n\geq 7$.

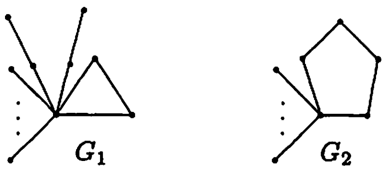


Fig. 1

Let $G_1\cup G_2$ denote the graph consisting of two (disconnected) components $G_1$ and $G_2$, and $kG$ denote the graph consisting of $k$ ($k>0$ be an integer) copies of the graph $G$. The join $G_1\vee G_2$ of graphs $G_1$ and $G_2$ is the graph having vertex set $V(G_1\vee G_2)=V(G_1\cup G_2)$ and edge set $E(G_1\vee G_2)=E(G_1)\cup E(G_2)\cup\{(u,v):u\in V(G_1),v\in V(G_2)\}$. Let $K_n$, $K_{1,n-1}$ denote the complete graph, and the star of order $n$, respectively. Specially, $K_1$ denotes an isolated vertex.

**Lemma 2.1** [16] *If an isolated vertex is connected by edges to all the vertices of a graph $G$ of order $n$, then the Laplacian eigenvalues of the resultant graph are as follows: one of the eigenvalues is $n+1$, the other eigenvalues can be obtained by incrementing the eigenvalues of the old graph $G$ by 1 except the lowest one, and 0 as another eigenvalue.*

**Example 2.1** *The Laplacian spectrum of $K_2\cup 2K_1$ is $(2,0,0,0)$, then the Laplacian spectrum of $(K_2\cup 2K_1)\vee K_1$ is $(5,3,1,1,0)$ by* Lemma 2.1.

**Theorem 2.2** (1) *Let $H_1=(K_3\cup K_{1,6}\cup(n-11)K_1)\vee K_1$ and $H_2=(K_{1,7}\cup K_{1,2}\cup(n-12)K_1)\vee K_1$, then $H_1$ and $H_2$ is a pair of LEL-equienergetic non-cospectral, connected graphs of order $n$, for all $n\geq 12$.* (2) *Let $H_3=(7K_3\cup(n-22)K_1)\vee K_1$ and $H_4=(K_8\cup(n-9)K_1)\vee K_1$, then $H_3$ and $H_4$ is a pair of LEL-equienergetic non-cospectral, connected graphs of order $n$, for all $n\geq 22$.* (3) *Let $H_5=(4K_7\cup 4K_3\cup(n-41)K_1)\vee K_1$ and $H_6=(K_{17}\cup(n-18)K_1)\vee K_1$, then $H_5$ and $H_6$ is a pair of LEL-equienergetic non-cospectral, connected graphs of order $n$, for all $n\geq 41$.* (4) *Let $H_7=(20K_2\cup 5K_3\cup(n-56)K_1)\vee K_1$ and $H_8=(K_{11}\cup(n-12)K_1)\vee K_1$, then $H_7$ and $H_8$ is a pair of LEL-equienergetic non-cospectral, connected graphs of order $n$, for all $n\geq 56$.*

**Proof.** In the proof of this Theorem, we use $S(G)$ to denote the Laplacian spectrum of $G$. Recall that $S(K_n) = (\underbrace{n, n, \cdots, n}_{n-1}, 0)$, and $S(K_{1,n-1}) = (n, \underbrace{1, 1, \cdots, 1}_{n-2}, 0)$, where $n \geq 2$.

(1) By Lemma 2.1, we have $S(H_1) = (n, 8, 4, 4, 2, 2, 2, 2, 2, \underbrace{1, 1, \cdots, 1}_{n-10}, 0)$, $S(H_2) = (n, 9, 4, 2, 2, 2, 2, 2, 2, 2, \underbrace{1, 1, \cdots, 1}_{n-11}, 0)$. Thus, $LEL(H_1) = LEL(H_2)$.

(2) By Lemma 2.1, it follows that $S(H_3) = (n, \underbrace{4, 4, \cdots, 4}_{14}, \underbrace{1, 1, \cdots, 1}_{n-16}, 0)$, and $S(H_4) = (n, \underbrace{9, 9, \cdots, 9}_{7}, \underbrace{1, 1, \cdots, 1}_{n-9}, 0)$. Thus, $LEL(H_3) = LEL(H_4)$.

(3) Lemma 2.1 implies that $S(H_5) = (n, \underbrace{8, 8, \cdots, 8}_{24}, \underbrace{4, 4, \cdots, 4}_{8} \underbrace{1, 1, \cdots, 1}_{n-34}, 0)$, and $S(H_6) = (n, \underbrace{18, 18, \cdots, 18}_{16}, \underbrace{1, 1, \cdots, 1}_{n-18}, 0)$. Thus, $LEL(H_5) = LEL(H_6)$.

(4) Lemma 2.1 implies that $S(H_7) = (n, \underbrace{4, 4, \cdots, 4}_{10}, \underbrace{3, 3, \cdots, 3}_{20} \underbrace{1, 1, \cdots, 1}_{n-32}, 0)$, and $S(H_8) = (n, \underbrace{12, 12, \cdots, 12}_{10}, \underbrace{1, 1, \cdots, 1}_{n-12}, 0)$. Thus, $LEL(H_7) = LEL(H_8)$.

It is well-known that $\sum_{i=1}^{n} \mu_i = 2m$ (for example, see [1]). This implies that $H_1$ and $H_2$ are two 9-cyclic graphs. Thus, we have

**Corollary 2.1** *There exists a pair of LEL-equienergetic non-cospectral, connected* 9*-cyclic graphs of order* $n$*, for all* $n \geq 12$.


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