Bounds on the k-independence and k-chromatic numbers of graphs.

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Abstract

For an integer $k \geq 1$ and a graph $G = (V, E)$, a subset S of the vertex set V is k-independent in G if the maximum degree of the subgraph induced by the vertices of S is less or equal $k - 1$. The k-independence number $\beta_k(G)$ of G is the maximum cardinality of a k-independent set of G.

A set S of V is k-Co-independent in G if S is k-independent in the complement of G. The k-Co-independence number $\omega_k(G)$ of G is the maximum size of a k-Co-independent set in G. The sequences (β_k) and (ω_k) are weakly increasing.

We define the k-chromatic number or k-independence partition number $\chi_k(G)$ of G as the smallest integer m such that G admits a partition of its vertices into m k-independent sets and the k-Co-independence partition number $\theta_k(G)$ of G as the smallest integer m such that G admits a partition of its vertices into m k-Co-independent sets. The sequences (χ_k) and (θ_k) are weakly decreasing.

In this paper we mainly present bounds on these four parameters. Some of them are extensions of well-known classical results.

Keywords: k-independence, k-Co-independence, k-chromatic number, k-Co-independence partition number.

1 Terminology and introduction

We consider finite, undirected and simple graphs $G = (V, E)$ of order $|V| = n(G)$ and size $|E| = m(G)$. The open neighborhood of a vertex $v \in V$

is $N_G(v) = \{u \in V \mid uv \in E\}$, i.e, the set of all vertices adjacent with v. If $S \subseteq V(G)$, then $N_G(S) = \cup_{v \in S} N_G(v)$ is the open neighborhood of S. The closed neighbourhoods of v and S are $N_G[v] = N_G(v) \cup \{v\}$ and $N_G[S] = N_G(S) \cup S$. The degree of a vertex v of G is $d_G(v) = |N_G(v)|$. By $\Delta(G)$ and $\delta(G)$ we denote the maximum degree and the minimum degree of G. If $S \subseteq V$, then $G[S]$ denotes the subgraph induced by the vertex set S. If $S \subset V$ and $x \in V \setminus S$, then we denote by $d_S(x)$ the number of edges from x to S.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two disjoint graphs. Their union $G = G_1 \cup G_2$ has the vertex set $V = V_1 \cup V_2$ and the edge set $E = E_1 \cup E_2$. Their join $G_1 + G_2$ consists of $G_1 \cup G_2$ together with the edge set $\{uv \mid u \in V_1,\ v \in V_2\}$. The composition $G = G_1[G_2]$ has $V = V_1 \times V_2$ as its vertex set and $u = (u_1, u_2)$ is adjacent with $v = (v_1, v_2)$ whenever (u_1 is adjacent with v_1) or ($u_1 = v_1$ and u_2 is adjacent with v_2). The cycle of order n is denoted by C_n.

For any parameter $\mu(G)$ associated to a graph property $\mathcal{P}$, we refer to a set of vertices with property $\mathcal{P}$ and cardinality $\mu(G)$ as a $\mu(G)$-set. An independent set S is a set of vertices whose induced subgraph has no edge. In [11, 12] Fink and Jacobson defined a generalization of the concepts of domination and independence. For an integer $k \geq 1$ and a graph $G = (V, E)$, a subset D of V is k-dominating if every vertex in $V \backslash D$ has at least k neighbors in D. The k-domination number $\gamma_k(G)$ of G is the minimum cardinality of a k-dominating set of G. A subset S of V is k-independent in G if $\Delta(G[S]) < k$. The k-independence number $\beta_k(G)$ of G is the maximum cardinality of a k-independent set of G. Since every k-independent set is $(k+1)$-independent, the sequence (β_k) is weakly increasing and thus

$$\beta(G) = \beta_1(G) \leq \beta_2(G) \leq \ldots \leq \beta_\Delta(G) < \beta_{\Delta+1}(G) = n.$$

More details and results on k-independence and k-domination can be found in [2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 14, 18].

A set $S \subseteq V(G)$ is k-Co-independent in G if S is k-independent in the complement $\overline{G}$ of G; that is $\Delta(\overline{G[S]}) < k$. The k-Co-independence number $\omega_k(G)$ of G is the maximum size of a k-Co-independent set in G. Also the sequence (ω_k) is weakly increasing and so

$$\omega(G) = \omega_1(G) \leq \omega_2(G) \leq \ldots \leq \omega_{n-\delta-1}(G) < \omega_{n-\delta}(G) = n.$$

We define the k-chromatic number or k-independence partition number $\chi_k(G)$ of G as the smallest integer m such that G admits a partition of its vertices into m k-independent sets and the k-Co-independence partition number $\theta_k(G)$ of G as the smallest integer m such that G admits a partition of its vertices into m k-Co-independent sets. The sequences (χ_k) and (θ_k) are weakly decreasing and therefore

$$\chi(G) = \chi_1(G) \geq \chi_2(G) \geq \ldots \geq \chi_\Delta(G) > \chi_{\Delta+1}(G) = 1.$$

as well as

$$\theta(G) = \theta_1(G) \geq \theta_2(G) \geq \ldots \geq \theta_{n-\delta-1}(G) > \theta_{n-\delta}(G) = 1.$$

For $k = 1$, the k-chromatic number of G is the chromatic number $\chi(G)$ of G, and the k-Co-independence partition number of G is the clique partition number $\theta(G)$ of G.

Since a k-Co-independent set S of G is a k-independent set of $\overline{G}$, we deduce that $\Delta(\overline{G[S]}) < k$ and $\Delta(\overline{G[S]}) + \delta(G[S]) = |S| - 1$. Thus $\delta(G[S]) > |S| - k - 1$. Equivalently, a set S is a k-Co-independent set if $\delta(G[S]) > |S| - k - 1$.

Observation 1 *Every graph G satisfies $\omega_k(G) = \beta_k(\overline{G})$ and $\theta_k(G) = \chi_k(\overline{G})$.*

When no confusion can arrise, we often write: V, E, n, $d(v)$, $N(v)$, Δ, δ, ... for $V(G)$, $E(G)$, $n(G)$, $d_G(v)$, $N_G(v), \Delta(G), \delta(G), \ldots$

In this paper we present lower and upper bounds on $\beta_k(G)$, $\omega_k(G)$, $\chi_k(G)$ and $\theta_k(G)$. The special case $k = 1$ mostly leads to well known classical results.

2 Relations between $\beta_k, \omega_k, \chi_k, \theta_k$

It is well known that $\omega(G) \leq \chi(G)$ and $\beta(G) \leq \theta(G)$ for every graph G. In the following we extend these inequalities.

Theorem 1 *If G is a graph such that $k \leq \min(\Delta(G), \Delta(\overline{G}))$, then*

$$\omega_k(G) \leq (2k-1)\chi_k(G).$$

Proof. Let $S_1, S_2, \ldots, S_{\chi_k(G)}$ be a partition of the vertex set V into $\chi_k(G)$ k-independent sets. If B is a $\omega_k(G)$-set of G, then $\Delta(\overline{G[B]}) \leq k-1$. If we define $A_i = B \cap S_i$ for all $i = 1, 2, \ldots, \chi_k(G)$, then A_i is a k-independent set in G as well as in $\overline{G}$ or $A_i = \emptyset$ for $i = 1, 2, \ldots, \chi_k(G)$. Thus $\Delta(G[A_i]) \leq k-1$ and $\Delta(\overline{G[A_i]}) \leq k-1$. Since $2m(G[A_i]) = \sum_{v \in A_i} d_{G[A_i]}(v) \leq |A_i|(k-1)$ and $2m(\overline{G[A_i]}) = \sum_{v \in A_i} d_{\overline{G[A_i]}}(v) \leq |A_i|(k-1)$, we obtain

$$\frac{|A_i|(|A_i|-1)}{2} = m(G[A_i]) + m(\overline{G[A_i]}) \leq |A_i|(k-1).$$

This implies $|A_i| \leq 2k-1$, and we deduce that

$$\omega_k(G) = |B| = \sum_{i=1}^{\chi_k(G)} |B \cap S_i| = \sum_{i=1}^{\chi_k(G)} |A_i| \leq (2k-1)\chi_k(G).$$

This completes the proof of Theorem 1. ∎

The complement of the composition $G = C_{2p}[M_{2k-1}]$, where M_{2k-1} is a $(k-1)$-regular graph of order $2k-1$ with k an odd intger and C_{2p} the cycle of order $2p$, is extremal for Theorem 1, because $\omega_k(\overline{G}) = p(2k-1)$ and $\chi_k(\overline{G}) = p$.

Observation 1 and Theorem 1 imply the next corollary.

Corollary 1 *If G is a graph such that $k \leq \min(\Delta(G), \Delta(\overline{G}))$, then*

$$\beta_k(G) \leq (2k-1)\theta_k(G).$$

Let k be an odd integer, and let $G = G_1[G_2]$ be the composition, where G_1 is a graph such that $\beta(G_1) = \theta(G_1)$ and G_2 is a $(k-1)$-regular graph of order $2k-1$. Then we can see that G satisfies $\beta_k(G) = (2k-1)\theta_k(G)$.

In the book of C. Berge [1] we can find the inequalities $\chi(G)\beta(G) \geq n$ and $\chi(G) + \beta(G) \leq n+1$ for every graph. In the following we generalize these results.

Observation 2 *If G is a graph with $k \leq \Delta(G)$, then*

$$\chi_k(G)\beta_k(G) \geq n.$$

Proof. Let $S_1, S_2, \ldots, S_{\chi_k(G)}$ be a partition of the vertex set V into $\chi_k(G)$ k-independent sets. Since every S_i is a k-independent set, we conclude that $n = |S_1| + |S_2| + \ldots + |S_{\chi_k}| \leq \chi_k(G)\beta_k(G)$. ∎

Let $n \geq 2$ be an even integer, and let H_n be an $(n-2)$-regular graph of order n (that means that H_n is a complement of the union of $\frac{n}{2}$ copies of K_2). Since $\beta_k(H_{pk}) = k$ and $\chi_k(H_{pk}) = p$ when k is even, the graph H_{pk} is extremal for Observation 2.

Observations 1 and 2 imply the next corollary.

Corollary 2 *Let G be a graph. If $k \leq \Delta(\overline{G})$, then*

$$\theta_k(G)\omega_k(G) \geq n.$$

Corollary 3 *Let G be a graph. If $k \leq \Delta(G)$, then*

$$\chi_k(G) + \beta_k(G) \geq 2\sqrt{n}.$$

Proof. Observation 2 leads to $\chi_k(G) + \beta_k(G) \geq \chi_k(G) + \frac{n}{\chi_k(G)}$. A simple calculation shows that the minimum of the function $f(x) = x + \frac{n}{x}$ is $2\sqrt{n}$ when $0 < x \leq n$. Hence we arrive at $\chi_k(G) + \beta_k(G) \geq \chi_k(G) + \frac{n}{\chi_k(G)} \geq 2\sqrt{n}$, and the desired bound is proved. ■

Let $k \geq 2$ be an even integer, and let H_{k^2} be a $(k^2 - 2)$-regular graph of order k^2. Since $\beta_k(H_{k^2}) = k$ and $\chi_k(H_{k^2}) = k$, the graph H_{k^2} is extremal for Corollary 3.

Corollary 4 *Let G be a graph. If $k \leq \Delta(\overline{G})$ then*

$$\theta_k(G) + \omega_k(G) \geq 2\sqrt{n}.$$

Theorem 2 *Let $G = (V, E)$ be a graph of order n. If $k \leq \Delta(G)$, then*

$$k\chi_k(G) + \beta_k(G) \leq n + 2k - 1.$$

Proof. Let S be a $\beta_k(G)$-set of G. Then $V \backslash S$ can be partioned into $\left\lceil \frac{|V\backslash S|}{k} \right\rceil$ sets of size at most k. Hence those sets are k-independent sets. Since $\left\lceil \frac{|V\backslash S|}{k} \right\rceil = \left\lfloor \frac{|V\backslash S| - 1}{k} \right\rfloor + 1$, we deduce that

$$\begin{aligned} \chi_k(G) &\leq \left\lceil \frac{|V\backslash S|}{k} \right\rceil + 1 \leq \left\lfloor \frac{|V\backslash S| - 1}{k} \right\rfloor + 2 \\ &\leq \frac{|V\backslash S| - 1}{k} + 2 = \frac{n - \beta_k(G) - 1}{k} + 2. \end{aligned}$$

This inequality chain yields to $k\chi_k(G) + \beta_k(G) \leq n + 2k - 1$, and the desired bound is proved. ■

Let k be an odd intger, and let M_{2k-1} be a $(k-1)$-regular graph of order $2k-1$. If we define $G = K_{kp+1} + M_{2k-1}$, then we observe that $\beta_k(G) = 2k-1$ and $\chi_k(G) = p + 2$. It follows that $k\chi_k(G) + \beta_k(G) = n(G) + 2k - 1$, and thus the bound of Theorem 2 is sharp.

Observation 1 and Theorem 2 imply the next result.

Corollary 5 *Let G be a graph. If $k \leq \Delta(\overline{G})$, then*

$$k\theta_k(G) + \omega_k(G) \leq n + 2k - 1.$$

Corollary 6 *Let G be a graph. If $k \le \Delta(G)$, then*

$$\chi_k(G)\beta_k(G) \le \frac{(n+2k-1)^2}{4k}.$$

Proof. Theorem 2 leads to $\beta_k(G) \le n+2k-1-k\chi_k(G)$. Thus we obtain $\chi_k(G)\beta_k(G) \le \chi_k(G)(n+2k-1-k\chi_k(G))$. A simple calculation shows that the maximum of the function $f(x) = x(n+2k-1-kx)$ is $\frac{(n+2k-1)^2}{4k}$ when $1 \le x \le n$. Thus $\chi_k(G)\beta_k(G) \le \chi_k(G)(n+2k-1-k\chi_k(G)) \le \frac{(n+2k-1)^2}{4k}$. ∎

The star $K_{1,2k}$ (that is a connected graph of order $n = 2k+1$ with $2k$ vertices of degree one) is extremal for Theorem 2 and Corollary 6, because $\beta_k(K_{1,2k}) = 2k$, $\chi_k(K_{1,2k}) = 2$, $k\chi_k(G) + \beta_k(G) = 4k = n + 2k - 1$ and $\chi_k(G)\beta_k(G) = 4k = \frac{(2k+1+2k-1)^2}{4k}$.

Corollary 7 *Let G be a graph. If $k \le \Delta(\overline{G})$, then*

$$\theta_k(G)\omega_k(G) \le \frac{(n+2k-1)^2}{4k}.$$

3 Bounds for $\beta_k, \omega_k, \chi_k, \theta_k$

The next result by Favaron [9] is the main tool for the proofs of our next two theorems.

Theorem 3 *(Favaron [9] 1985) If G is a graph, then every k-independent set D of G such that $k|D| - |E(G[D])|$ is maximum is a k-dominating set of G.*

Theorem 4 *Let G be a graph. If $k \le \Delta(G)$, then*

$$\chi_k(G) \le \frac{\Delta(G)+k}{k}.$$

Proof. Let $S_1, S_2, \ldots, S_p$ be a partition of the vertex V such that S_1 is a k-independent set and a k-dominating set of G. In addition, let S_i be a k-independent set and a k-dominating set in $G[V - \bigcup_{j=1}^{i-1} S_j]$. In view of Theorem 3, such a partition exists. Then $d_{S_i}(x) \ge k$ for every vertex $x \in S_p$ and each $i \in \{1, 2, \ldots, p-1\}$. This implies that $d_G(x) \ge k(p-1)$ for each $x \in S_p$, and consequently $\Delta \ge k(p-1) \ge k(\chi_k(G) - 1)$. This leads to the desired upper bound for $\chi_k(G)$. ∎

Let $G = v + H_k^1 + H_k^2 + \ldots + H_k^p$, where H_k^i is a copy of a $(k-2)$-regular graph H_k of even order k for every $i = 1, 2, \ldots, p$. Then G is extremal for Theorem 4, because $\chi_k(G) = p + 1$ and $\frac{\Delta(G) + k}{k} = \frac{pk + k}{k} = p + 1$.

Corollary 8 *Let G be a graph. If $k \leq \Delta(\overline{G})$, then*

$$\theta_k(G) \leq \frac{\Delta(\overline{G}) + k}{k}.$$

Theorem 4 and Observation 2 immediately imply the following well known bound given by Hopkins and Staton [15].

Corollary 9 *(Hopkins and Staton [15]* 1986*) If G is a graph, and $1 \leq k \leq \Delta(G)$ then*

$$\beta_k(G) \geq \frac{n}{\left(1 + \left\lfloor \frac{\Delta(G)}{k} \right\rfloor\right)}.$$

Theorem 5 *If G is a graph such that $\Delta(G) \geq k$, then*

$$\chi_k(G) \leq \sqrt{\frac{2m(G)}{k^2} + \left(\frac{k-2}{2k}\right)^2} + \frac{3k-2}{2k}.$$

Proof. Let $S_1, S_2, \ldots, S_p$ be a partition of the vertex set V such that S_1 is a k-independent set and a k-dominating set of G. In addition, let S_i be a k-independent set and a k-dominating set of $G[V - \bigcup_{j=1}^{i-1} S_j]$ for $i = 2, 3, \ldots, p$. By Theorem 3, such a partition exists. Since S_i is a k-dominating set of $G[V - \bigcup_{j=1}^{i-1} S_j]$ for $i = 1, 2, \ldots, p-1$, it follows that $d_{S_i}(x) \geq k$ for each $x \in V - \bigcup_{j=1}^{i} S_j$ and each $i = 1, 2, \ldots, p-1$. Furthermore, we observe that $|S_i| \geq k$ for each $i = 1, 2, \ldots, p-1$, and therefore we obtain

$$\begin{aligned} m(G) &\geq k|S_2| + 2k|S_3| + \ldots + (p-2)k|S_{p-1}| + (p-1)k|S_p| \\ &\geq k^2(1 + 2 + \ldots + (p-2)) + k(p-1) \\ &= \frac{k^2(p-1)(p-2) + 2k(p-1)}{2}. \end{aligned}$$

and thus

$$\begin{aligned} 2m(G) &\geq k(p-1)(k(p-2) + 2) \\ &\geq k^2 \left((p-1) - \frac{(k-2)}{2k} + \frac{k-2}{2k}\right)\left((p-1) - \frac{k-2}{2k} - \frac{k-2}{2k}\right) \end{aligned}$$

$$\frac{2m(G)}{k^2} \geq \left(p - 1 - \frac{k-2}{2k}\right)^2 - \left(\frac{k-2}{2k}\right)^2$$

Which implies that

$$\sqrt{\frac{2m(G)}{k^2}+\left(\frac{k-2}{2k}\right)^2} \geq p-1-\frac{k-2}{2k}.$$

The last inequality and a simple calculation lead to

$$\chi_k(G) \leq p \leq \sqrt{\frac{2m(G)}{k^2}+\left(\frac{k-2}{2k}\right)^2}+\frac{3k-2}{2k},$$

and hence the theorem is proved. ■

Let t, k be integers such that $k \geq 1$ and $t \geq 2$, and let G be a complete t-partite graph with the partite sets $V_1, V_2, \dots, V_t$ such that $|V_1| = |V_2| = \dots |V_{t-1}| = k$ and $|V_t| = 1$. Then $n(G) = k(t-1)+1$, $\chi_k(G) = t$ and $2m(G) = k(t-1)(k(t-2)+2)$. This leads to

$$\sqrt{\frac{2m(G)}{k^2}+\left(\frac{k-2}{2k}\right)^2} = t-\frac{3k-2}{2k},$$

and thus

$$\chi_k(G) = t = \sqrt{\frac{2m(G)}{k^2}+\left(\frac{k-2}{2k}\right)^2}+\frac{3k-2}{2k}.$$

This example shows that Theorem 5 is best possible.

Since $\Delta(G) = k(t-1)$, this example also shows that Theorem 4 is best possible.

Corollary 10 *If G is a graph such that $\Delta(\overline{G}) \geq k$, then*

$$\theta_k(G) \leq \sqrt{\frac{2m(\overline{G})}{k^2}+\left(\frac{k-2}{2k}\right)^2}+\frac{3k-2}{2k}.$$

Theorem 5 immediately implies the following well known bound given by P. Hansen [13].

Corollary 11 *(Hansen [13] 1979) If G is a graph, then*

$$\chi(G) \leq \sqrt{2m(G)+\frac{1}{4}}+\frac{1}{2}.$$

Lemma 1 *Let $n, p \geq 1$ and $r, t \geq 0$ be integers such that $n = tp + r$ and $r < p$. If $x_1, x_2, \ldots, x_p \geq 1$ are integers with $\sum_{i-1}^{p} x_i = n$, then*

$$\sum_{i=1}^{p} x_i^2 \geq \frac{n^2 - r^2}{p} + r. \tag{1}$$

Proof. Assume, without loss of generality, that $x_1 \geq x_2 \geq \ldots \geq x_p$. First we will show that the sum in (1) is minimum when $x_1 \leq x_p + 1$. Suppose that $x_1 \geq x_p + 2$ and define $x_1' = x_1 - 1$, $x_p' = x_p + 1$ and $x_i' = x_i$ for $2 \leq i \leq p - 1$. Obviously, $\sum_{i=1}^{p} x_i' = n$ but

$$\begin{aligned}\sum_{i=1}^{p} x_i^2 - \sum_{i=1}^{p} x_i'^2 &= x_1^2 - (x_1 - 1)^2 + x_p^2 - (x_p + 1)^2 \\ &= 2(x_1 - x_p - 1) \geq 2.\end{aligned}$$

Consequently, the sum in (1) is minimum when $x_1 \leq x_p + 1$. Then $x_i = t$ for $r + 1 \leq i \leq p$ and $x_i = t + 1$ for $1 \leq i \leq r$. Using $t = \frac{(n-r)}{p}$, we obtain

$$\begin{aligned}\sum_{i=1}^{p} x_i^2 &= \sum_{i=1}^{r} (t+1)^2 + \sum_{i=r+1}^{p} t^2 \\ &= r(t+1)^2 + (p-r)t^2 \\ &= t(pt + 2r) + r \\ &= \frac{n-r}{p}(n - r + 2r) + r \\ &= \frac{n^2 - r^2}{p} + r,\end{aligned}$$

and the proof of Lemma 1 is complete. ■

Theorem 6 *Let G be a graph of order n, and assume that $n = t\chi_k(G) + r$ with integers $t \geq 0$ and $0 \leq r < \chi_k(G)$. If $k \leq \Delta(G)$, then*

$$\chi_k(G) \geq \max\left\{\frac{n^2 - r^2}{n^2 - 2m(G) + (k-1)n - r}, \frac{n^2}{n^2 - 2m(G) + (k-1)n}\right\}. \tag{2}$$

Proof. Let $S_1, S_2, \ldots, S_p$ be a partition of V into $p = \chi_k(G)$ k-independent sets. Applying Lemma 11, we obtain

$$\begin{aligned} n(n-1) &= 2m(G) + 2m(\overline{G}) \geq 2m(G) + \sum_{i=1}^{p} 2m(\overline{G[S_i]}) \\ &\geq 2m(G) + \sum_{i=1}^{p} |S_i|(|S_i| - k) = 2m(G) + \sum_{i=1}^{p} |S_i|^2 - k \sum_{i=1}^{p} |S_i| \\ &\geq 2m(G) - kn + \frac{n^2 - r^2}{p} + r. \end{aligned} \tag{3}$$

Using $p = \chi_k(G)$, this easily leads to

$$\chi_k(G) \geq \frac{n^2 - r^2}{n^2 - 2m(G) + (k-1)n - r}. \tag{4}$$

In addition, we deduce from (3) that

$$n(n-1) \geq 2m(G) - kn + \frac{n^2 - r^2}{p} + r \geq 2m(G) - kn + \frac{n^2}{p},$$

and this yields

$$\chi_k(G) \geq \frac{n^2}{n^2 - 2m(G) + (k-1)n}. \tag{5}$$

Combining (4) and (5), we obtain the desired bound (2). ∎

Let p, k, r be integers such that $k \geq 1$, $p \geq 2$ and $r < p$ and let $G = H_{k+1}^1 + H_{k+1}^2 + \ldots + H_{k+1}^r + K_k^{r+1} + K_k^{r+2} + \ldots + K_k^p$ be a complete p-partite graph with the partite sets $H_{k+1}^1, H_{k+1}^2, \ldots, H_{k+1}^r, K_k^{r+1}, K_k^{r+2}, \ldots, K_k^p$; where H_{k+1}^i is a copy of a $(k-1)$-regular graph H_{k+1} of even order $k+1$ and K_k^i is a copy of a clique K_k of order k. We can see that $n = kp + r$, $\chi_k(G) = p$ and $2m(G) = n\,(n-1) - r\,(k+1)$. A simple calculation shows that $\frac{n^2 - r^2}{n^2 - 2m(G) + (k-1)n - r} = p$. Then G is extremal for Theorem 6.

Let $H_{p(k+1)}$ be a $(p(k+1) - 2)$-regular graph of order $p(k+1)$ with k odd. Then $H_{p(k+1)}$ is also an extremal graph for Theorem 6.

Because of $\theta_k(\overline{G}) = \chi_k(G)$ and $2m(G) + 2m(\overline{G}) = n^2 - n$, inequality (2) implies the next corollary.

Corollary 12 *Let G be a graph of order n, and assume that $n = t\theta_k(G) + r$ with integers $t \geq 0$ and $0 \leq r < \theta_k(G)$. If $k \leq \Delta(G)$, then*

$$\theta_k(G) \geq \max\left\{ \frac{n^2 - r^2}{2m(G) + kn - r}, \frac{n^2}{2m(G) + kn} \right\}.$$

The following well known bound given by Meyers and Liu [16] is a special case of Theorem 6.

Corollary 13 *(Meyers and Liu [16]* 1972*) If G is a graph of order n, then*

$$\chi(G) \geq \frac{n^2}{n^2 - 2m(G)}.$$

4 Nordhaus-Gaddum type results

In their now classical 1956 paper [17], Nordhaus and Gaddum established the inequality $\chi(G) + \chi(\overline{G}) \leq n + 1$. Improvements and generalizations of this inequality can be found in Section 9.1 of the monograph [14] by Haynes, Hedetniemi and Slater.

Theorem 7 *(Chartrand, Schuster [4]* 1974*) For any graph G of order n, we have:*

$$\beta(G) + \beta(\overline{G}) \leq n + 1 \text{ and}$$
$$\beta(G)\beta(\overline{G}) \leq \left\lceil \frac{n^2 + 2n}{4} \right\rceil.$$

Next we present generalizations of these inequalities.

Theorem 8 *If G is a graph of order n such that $k \leq \min(\Delta(G), \Delta(\overline{G}))$, then*

$$\beta_k(G) + \beta_k(\overline{G}) \leq n + 2k - 1 \text{ and}$$
$$\beta_k(G)\beta_k(\overline{G}) \leq \frac{(n + 2k - 1)^2}{4}.$$

Proof. Let S be a $\beta_k(G)$-set of G, and let B be a $\beta_k(\overline{G})$-set of $\overline{G}$. If $A = B \cap S$, then $n \geq |S| + |B| - |A|$. Since $|A| \leq 2k - 1$ (see the proof of Theorem 1), it follows that $\beta_k(G) + \beta_k(\overline{G}) \leq n + 2k - 1$, and the first inequality is proved. This implies that

$$\begin{aligned}(n + 2k - 1)^2 &\geq \left(\beta_k(G) + \beta_k(\overline{G})\right)^2 \\ &= \left(\beta_k(G) - \beta_k(\overline{G})\right)^2 + 4\beta_k(G)\beta_k(\overline{G}) \\ &\geq 4\beta_k(G)\beta_k(\overline{G}),\end{aligned}$$

and this leads to the second inequality. ■

Let k be an odd integer, and let M_{2k-1} be a $(k-1)$-regular graph of order $n = 2k-1$. We can see that $\beta_k(M_{2k-1}) + \beta_k(\overline{M_{2k-1}}) = n + 2k - 1$ and $\beta_k(M_{2k-1})\beta_k(\overline{M_{2k-1}}) = \frac{(n+2k-1)^2}{4}$. Therefore M_{2k-1} is extremal for the first and the second inequality of Theorem 8.

Theorem 9 *If G is a graph of order n such that $k \leq \min(\Delta(G), \Delta(\overline{G}))$, then*

$$\chi_k(G)\chi_k(\overline{G}) \geq \frac{n}{2k-1} \text{ and}$$
$$\chi_k(G) + \chi_k(\overline{G}) \geq 2\sqrt{\frac{n}{2k-1}}.$$

Proof. Let $S_1, S_2, \ldots, S_{\chi_k(G)}$ be a partition of the vertex set V into $\chi_k(G)$ k-independents sets. Then each S_i is a k-Co-independent set of $\overline{G}$ and Theorem 1 implies that $|S_i| \leq \omega_k(\overline{G}) \leq (2k-1)\,\chi_k(\overline{G})$. Therefore we obtain $n = \sum_{i=1}^{\chi_k(G)} |S_i| \leq (2k-1)\,\chi_k(\overline{G})\chi_k(G)$,and the first inequality is proved. Now it follows that

$$\begin{aligned} 4\frac{n}{2k-1} &\leq 4\chi_k(G)\chi_k(\overline{G}) \\ &\leq \left(\chi_k(G) - \chi_k(\overline{G})\right)^2 + 4\chi_k(G)\chi_k(\overline{G}) \\ &= \left(\chi_k(G) + \chi_k(\overline{G})\right)^2, \end{aligned}$$

and this leads to the second inequality. ■

Let k be an odd integer, and let $G = C_4[M_{2k-1}]$ be the composition, where C_4 is the cycle of order 4 and M_{2k-1} is a $(k-1)$-regular graph of order $2k-1$. Then this composition is extremal for the first and the second inequality of Theorem 9.

Since $\Delta(\overline{G}) = n - \delta(G) - 1$, Theorem 4 yields the following Nordhaus-Gaddum bound.

Corollary 14 *Let G be a graph of order n. If $k \leq \min(\Delta(G), \Delta(\overline{G}))$, then*

$$\chi_k(G) + \chi_k(\overline{G}) \leq \frac{\Delta(G) - \delta(G) + n + 2k - 1}{k}.$$

If $\Delta(G) + \Delta(\overline{G}) \leq n$, then $0 \leq \Delta(G) - \delta(G) \leq 1$.
So, if $\Delta(G) - \delta(G) = 0$, then $\chi_k(G) + \chi_k(\overline{G}) \leq \frac{n+2k-1}{k}$,
And if $\Delta(G) - \delta(G) = 1$, then $\chi_k(G) + \chi_k(\overline{G}) \leq \frac{n+2k}{k}$.

Conjecture 1 *If G is a graph of order n, then*

$$\chi_k(G) + \chi_k(\overline{G}) \leq \left\lceil \frac{n+2k-1}{k} \right\rceil .$$

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