A degree sum condition for the existence of a path-factor

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Abstract

Let G be a connected graph of order n, and suppose that $n = \sum_{i=1}^{k} n_i$, where $n_1, n_2, \ldots, n_k$ are integers with at least two. A spanning subgraph is called a path-factor if each component of it is a path of order at least two. In [Y. Chen, F. Tian, B. Wei, Degree sums and path-factors in graphs, Graphs and Combin. **17** (2001), 61–71.], Chen et al. gave a degree sum condition for the existence of a path-factor consisting of paths of order $n_1, n_2, \ldots, n_k$. In this paper, for 2-connected graphs, we generalize this result.

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1 Introduction

In this paper, all graphs are finite undirected graphs without loops or multiple edges. For standard graph theoretic terminology not explained in this paper, we refer the reader to [1]. Let G be a graph. For convenience, we abbreviate $|V(G)|$ by $|G|$. We denote by $N_G(x)$ the neighborhood of a vertex x in G, and let $d_G(x) := |N_G(x)|$. Let H be a subgraph of G. For $x \in V(G)$, let $N_H(x) := N_G(x) \cap V(H)$, and $d_H(x) := |N_H(x)|$. For a positive integer k, if there exists an independent set of order k, then we let $\sigma_k(G)$ denote the minimum degree sum of an independent set of k vertices of G; otherwise we let $\sigma_k(G) := +\infty$. Let $p(G)$ and $c(G)$ be the order of a longest path and a longest cycle of G, respectively. We define $\mathrm{diff}(G) := p(G) - c(G)$. A subgraph H of G is said to be *k-dominating* if $d_H(x) \geq k$ holds for any vertex $x \in V(G-H)$. We denote by $\overrightarrow{C}$ a cycle C with a given orientation, and by $\overleftarrow{C}$ a cycle C with a reverse orientation. Let C be a cycle with a given orientation. For $u, v \in V(C)$, we denote by $u\overrightarrow{C}v$ a path from u to v along $\overrightarrow{C}$. The reverse sequence of $u\overrightarrow{C}v$ is denoted by $v\overleftarrow{C}u$. For $u \in V(C)$, we denote the h-th successor and the h-th predecessor of u on $\overrightarrow{C}$ by u^{+h} and u^{-h}, respectively. We abbreviate u^{+1} and u^{-1} by u^+ and u^-, respectively. For $X \subseteq V(C)$, we define $X^+ := \{x^+ : x \in X\}$ and $X^- := \{x^- : x \in X\}$, respectively. Let $n, n_1, n_2, \ldots, n_k$ be integers. If $n = \sum_{i=1}^{k} n_i$ and $n_i \geq 2$ $(1 \leq i \leq k)$, then we call $(n_1, n_2, \ldots, n_k)$ a *k-partition* of n.

In [3], M. El-Zahar proposed the following conjecture.

Conjecture 1.1 (El-Zahar [3]) *Let G be a graph of order n, and $(n_1, n_2, \ldots, n_k)$ be a k-partition of n such that $n_i \geq 3$ for any $1 \leq i \leq k$. If $\delta(G) \geq \lceil \frac{1}{2}n_1 \rceil + \cdots + \lceil \frac{1}{2}n_k \rceil$, then G has a spanning subgraph consisting of vertex-disjoint cycles of lengths $n_1, n_2, \ldots, n_k$.*

Conjecture 1.1 has not been settled yet, but, concerning this conjecture, Johansson [5] considered an analogous condition to have vertex-disjoint paths instead of cycles. A spanning subgraph is called a *path-factor* if each component of it is a path of order at least two.

Theorem 1.2 (Johansson [5]) *Let G be a connected graph of order n, and $(n_1, n_2, \ldots, n_k)$ be a k-partition of n. If $\delta(G) \geq \lfloor \frac{1}{2}n_1 \rfloor + \cdots + \lfloor \frac{1}{2}n_k \rfloor$, then G contains a path-factor consisting of paths of order $n_1, n_2, \ldots, n_k$.*

For integers $n_1, n_2, \ldots, n_k$, we let $\lambda(n_1, n_2, \ldots, n_k) := |\{n_i : n_i \text{ is even}, 1 \leq i \leq k\}|$. For convenience, we abbreviate $\lambda(n_1, n_2, \ldots, n_k)$ by λ. It is easy to see that Theorem 1.2 is equivalent to the following theorem.

Theorem 1.3 (Johansson [5]) *Let G be a connected graph of order n, and $(n_1, n_2, \ldots, n_k)$ be a k-partition of n. If $\delta(G) \geq (n - k + \lambda)/2$, then G contains a path-factor consisting of paths of order $n_1, n_2, \ldots, n_k$.*

In [2], Chen, Tian and Wei gave a σ_3 condition for the existence of a path-factor with given length.

Theorem 1.4 (Chen et al. [2]) *Let G be a connected graph of order n, and $(n_1, n_2, \ldots, n_k)$ be a k-partition of n. If $n \geq 3(k - \lambda) + 4$ and $\sigma_3(G) \geq 3(n - k + \lambda)/2 - 2$, then G contains a path-factor consisting of paths of order $n_1, n_2, \ldots, n_k$.*

In this paper, we give a σ_4 condition for the existence of a path-factor with given length.

Theorem 1.5 *Let G be a 2-connected graph of order n, and $(n_1, n_2, \ldots, n_k)$ be a k-partition of n. If $n \geq 3(k - \lambda) + 4$ and $\sigma_4(G) \geq 2(n - k + \lambda) - 3$, then one of the following holds;*

(i) *G contains a path-factor consisting of paths of order $n_1, n_2, \ldots, n_k$.*

(ii) *$G \in \{(3K_1 \cup K_2) + 2K_1, (3K_1 \cup K_2) + K_2\}$ and $k = 1$.*

In fact, we prove the following theorem which is more general than Theorem 1.5.

Theorem 1.6 *Let G be a 2-connected graph of order $n \geq \sum_{i=1}^{k} n_i$, where $n_1, n_2, \ldots, n_k$ are integers with at least two. If $n \geq 3(k - \lambda) + 4$ and $\sigma_4(G) \geq 2(n - k + \lambda) - 3$, then one of the following holds;*

(i) *G contains k vertex-disjoint paths of order $n_1, n_2, \ldots, n_k$.*

(ii) *$G \in \{(3K_1 \cup K_2) + 2K_1, (3K_1 \cup K_2) + K_2\}$, $k = 1$ and $n = n_1$.*

Note that $(3K_1 \cup K_2) + 2K_1$ and $(3K_1 \cup K_2) + K_2$ do not have a path of order 7.

In Theorem 1.5, the lower bound of $\sigma_4(G)$ is best possible in the following sense. Let $(n_1, n_2, \ldots, n_k)$ be a k-partition of n. Let $m := \lfloor \frac{1}{2} n_1 \rfloor + \cdots + \lfloor \frac{1}{2} n_k \rfloor - 1 = (n - k + \lambda)/2 - 1$, $m \geq 2$ and $G_1 := K_{m,n-m}$. Then $n - m \geq 4$ and $m \leq n - m$ since $m \geq 2$ and $n \geq 3(k - \lambda) + 4$. Hence G_1 is 2-connected and $\sigma_4(G_1) = 2(n - k + \lambda) - 4$. Since each path contributes at least $\lfloor \frac{1}{2} n_i \rfloor$ vertices to the partite set of order m, G_1 does not have a path-factor consisting of paths of order $n_1, n_2, \ldots, n_k$.

Moreover, 2-connectedness is necessary. Let G_2 be a graph obtained from $2K_1$ and K_{n-2} by joining one vertex of K_{n-2} and each vertex of $2K_1$. Then $\sigma_4(G_2) = +\infty$. But G_2 does not have a path-factor consisting of paths of order $n_1, n_2, \ldots, n_k$, where $(n_1, n_2, \ldots, n_k)$ is a k-partition of n with $n_i \neq 3$ for any $1 \leq i \leq k$.

2 Proof of Theorem 1.6

To prove Theorem 1.6, we will use the following theorem and two lemmas.

Theorem 2.1 (Kawarabayashi et al. [6]) *Let G be a 2-connected graph of order n. If $\sigma_4(G) \geq (4n-2)/3$, then $\mathrm{diff}(G) \leq 1$, or G has a hamiltonian path.*

In fact, the following lemma in [5] deals with only the case where $|G| = \sum_{i=1}^{k} n_i$. However, the same argument works also when $|G| > \sum_{i=1}^{k} n_i$.

Lemma 2.2 (Johansson [5]) *Let G be a graph of order $n \geq n'$, and $(n_1, n_2, \ldots, n_k)$ be a k-partition of n'. Suppose that G contains an $(n' - k + \lambda)/2$-dominating path. Then G contains k vertex-disjoint paths of order $n_1, n_2, \ldots, n_k$.*

Lemma 2.3 (Erdős and Gallai [4]) *Let G be a graph, and let C be a cycle of G, and $u, v \in V(C)$ with $u \neq v$. If $d_C(u^+) + d_C(v^+) \geq |C| + 1$, then G has a path P from u to v such that $V(P) = V(C)$.*

Proof of Theorem 1.6.

Let G be a graph satisfying the assumption of Theorem 1.6. Suppose that G does not satisfy the statements (i) and (ii). Let $n' := \sum_{i=1}^{k} n_i$. Note that $n' \equiv k - \lambda \pmod 2$. Then by Lemma 2.2, G does not have an $(n' - k + \lambda)/2$-dominating path. Since $n \geq 3(k - \lambda) + 4$,

$$\sigma_4(G) \geq 2(n - k + \lambda) - 3 \geq (4n - 1)/3. \tag{2.1}$$

Then the following fact holds by Theorem 2.1 and since G contains no k vertex-disjoint paths of order $n_1, n_2, \ldots, n_k$.

Fact 1 $\mathrm{diff}(G) \leq 1$.

Let C be a longest cycle of G, and let $Y := V(G - C)$. Choose such a cycle C so that $\sum_{y \in Y} d_G(y)$ is as large as possible.

The following claim holds by Fact 1 and the maximality of $|C|$. The proofs of the statements (i)–(iv) are very easy, and hence we omit the proof of them.

Claim 2 (i) *Y is an independent set.*

(ii) *$N_C(y) \cap N_C(y')^+ = \emptyset$ and $N_C(y) \cap N_C(y')^- = \emptyset$ for any $y, y' \in Y$.*

(iii) *$u^+v^+ \notin E(G)$ and $u^-v^- \notin E(G)$ for any $u, v \in N_C(Y)$.*

(iv) *$N_{C'}(u^+) \cap N_{C'}(v^+)^+ = \emptyset$ for any $y \in Y$ and $u, v \in N_C(y)$ with $u \neq v$, where $C' := u^+\overrightarrow{C}v$.*

(v) *$N_{C'}(y') \cap N_{C'}(v^+)^+ = \emptyset$ for any $y, y' \in Y$ with $y \neq y'$ and $u, v \in N_C(y)$ with $u \neq v$, where $C' := u^+\overrightarrow{C}v$.*

Proof. If there exists $w \in N_{C'}(y') \cap N_{C'}(v^+)^+$, then $P := y'w\overrightarrow{C}vyu\overleftarrow{C}v^+w^-\overleftarrow{C}u^+$ is a path such that $|P| = |C| + 2$, which contradicts Fact 1. Hence the statement (v) holds. □

Note that $N_C(y) = N_G(y)$ for any $y \in Y$ by Claim 2 (i). Let $Y_1 := \{y \in Y : d_G(y) \leq (n' - k + \lambda)/2 - 1\}$. Suppose that $|Y_1| \leq 1$. If $|Y_1| = 0$, then let $P := w\overrightarrow{C}w^-$ for some $w \in V(C)$. Else if $|Y_1| = 1$, say $Y_1 = \{y\}$, then let $P := yw\overrightarrow{C}w^-$ for some $w \in N_C(y)$. Then P is an $(n' - k + \lambda)/2$-dominating path, a contradiction. Thus $|Y_1| \geq 2$. Since $\sigma_4(G) \geq 2(n - k + \lambda) - 3 \geq 2(n' - k + \lambda) - 3$, $|Y_1| \leq 3$. Hence $2 \leq |Y_1| \leq 3$. Let $Y_1 := \{y_1, \ldots, y_l\}$ $(d_G(y_1) \leq \cdots \leq d_G(y_l) =: t)$. Let $N_G(y_l) := \{u_1, u_2, \ldots, u_t\}$. We may assume that $u_1, u_2, \ldots, u_t$ occur in this order along $\overrightarrow{C}$. For each $1 \leq i \leq t$, let $I_i := u_i^+\overrightarrow{C}u_{i+1}$, where $u_{t+1} := u_1$. By Claim 2 (ii), $|I_i| \geq 2$ for any $1 \leq i \leq t$.

Claim 3 *For any $1 \leq i \leq t$ such that $|I_i| = 2$, $d_G(u_i^+) \leq t$.*

Proof. For some $1 \leq i \leq k$, let $|I_i| = 2$ and $D := u_i y_l u_{i+1}\overrightarrow{C}u_i$. Then D is a longest cycle of G such that $V(D) = V(C) \cup \{y_l\} - \{u_i^+\}$. Hence by the choice of C, $d_G(u_i^+) \leq d_G(y_l) = t$. □

Claim 4 *$|I_i| \geq 3$ holds for any $1 \leq i \leq t$ except at most one.*

Proof. Suppose that there exist $1 \leq i < j \leq t$ such that $|I_i| \leq 2$ and $|I_j| \leq 2$. By Claim 2 (i)–(iii), $\{y_1, y_l, u_i^+, u_j^+\}$ is an independent set. By Claim 3, $d_G(u_i^+) \leq t$ and $d_G(u_j^+) \leq t$. Hence $d_G(y_1) + d_G(y_l) + d_G(u_i^+) + d_G(u_j^+) \leq 4t \leq 2(n' - k + \lambda) - 4 \leq 2(n - k + \lambda) - 4$, a contradiction. □

By Claim 4,

$$n - |Y| = |C| \geq 3(t - 1) + 2 = 3t - 1. \tag{2.2}$$

Hence

$$t \leq (n - |Y| + 1)/3. \tag{2.3}$$

Claim 5 *For any $1 \le i \le t$, there exists $y \in V(G) - \{y_1, y_l, u_i^+, u_{i+1}^+\}$ such that $\{y, y_1, y_l, u_i^+, u_{i+1}^+\}$ is an independent set and $d_G(y) \le t$.*

Proof. Suppose not for some i. Without loss of generality, we may assume that $i = 1$. Then by the definition of Y_1 and Claim 2 (i)–(iii), $l = 2$. Let $C_j := u_j^+ \overrightarrow{C} u_{3-j}$ for $j = 1, 2$. Suppose that $N_{C_j}(u_j^+)^- \cap N_{C_j}(u_{3-j}^+)^+ \ne \emptyset$ for some $j = 1, 2$, and let $y \in N_{C_j}(u_j^+)^- \cap N_{C_j}(u_{3-j}^+)^+$. Then $\{y, y_1, y_2, u_1^+, u_2^+\}$ is an independent set by Claim 2 (i)–(iii), and hence $d_G(y) > t$. Let $D := u_j^+ y^+ \overrightarrow{C} u_{3-j} y_2 u_j \overleftarrow{C} u_{3-j}^+ y^- \overleftarrow{C} u_j^+$. Then D is a longest cycle of G such that $V(D) = V(C) \cup \{y_2\} - \{y\}$. Hence by the choice of C, $d_G(y) \le d_G(y_2) = t$, a contradiction. Therefore $N_{C_j}(u_j^+)^- \cap N_{C_j}(u_{3-j}^+)^+ = \emptyset$ for any $j = 1, 2$. By Claim 2 (iii) and (v), $N_{C_j}(y_1) \cap (N_{C_j}(u_j^+)^- \cup N_{C_j}(u_{3-j}^+)^+) = \emptyset$ for any $j = 1, 2$. Moreover, $N_{C_j}(y_1) \cup N_{C_j}(u_j^+)^- \cup N_{C_j}(u_{3-j}^+)^+ \subseteq V(C_j) \cup \{u_{3-j}^+\}$ for any $j = 1, 2$. These imply that $d_G(y_1) + d_G(u_1^+) + d_G(u_2^+) = d_C(y_1) + d_C(u_1^+) + d_C(u_2^+) \le |C| + 2$ by Claim 2 (i) and (ii).

Suppose that $N_{C_2}(u_2^+)^- \cap N_{C_2}(y_2)^+ \ne \emptyset$, and let $y \in N_{C_2}(u_2^+)^- \cap N_{C_2}(y_2)^+$. Then $\{y, y_1, y_2, u_1^+, u_2^+\}$ is an independent set by Claim 2 (i)–(iii), and hence $d_G(y) > t$. Let $D := u_2^+ \overrightarrow{C} y^- y_2 u_2 \overleftarrow{C} y^+ u_2^+$. Then D is a longest cycle of G such that $V(D) = V(C) \cup \{y_2\} - \{y\}$. Hence by the choice of C, $d_G(y) \le d_G(y_2) = t$, a contradiction. Therefore $N_{C_2}(u_2^+)^- \cap N_{C_2}(y_2)^+ = \emptyset$. Since $N_{C_1}(y_2) = \{u_2\}$, we have $N_{C_1}(u_1^+)^- \cap N_{C_1}(y_2)^+ = \emptyset$. By Claim 2 (iii) and (iv), $N_{C_j}(u_{3-j}^+) \cap (N_{C_j}(u_j^+)^- \cup N_{C_j}(y_2)^+) = \emptyset$ for any $j = 1, 2$. Moreover, $N_{C_j}(u_{3-j}^+) \cup N_{C_j}(u_j^+)^- \cup N_{C_j}(y_2)^+ \subseteq V(C_j) \cup \{u_{3-j}^+\}$ for any $j = 1, 2$. These imply that $d_G(y_2) + d_G(u_1^+) + d_G(u_2^+) \le |C| + 2$ by Claim 2 (i) and (ii).

Hence by the definition of Y_1 and since $|C| \le n - |Y_1| = n - 2$,

$$\begin{aligned}
2(2(n-k+\lambda)-3) &\le 2\big(d_G(y_1) + d_G(y_2) + d_G(u_1^+) + d_G(u_2^+)\big) \\
&= d_G(y_1) + d_G(y_2) \\
&\quad + \big(d_G(y_1) + d_G(y_2) + 2(d_G(u_1^+) + d_G(u_2^+))\big) \\
&\le (n' - k + \lambda - 2) + 2|C| + 4 \\
&\le n - k + \lambda - 2 + 2(n-2) + 4 \\
&= 3n - k + \lambda - 2. \qquad (2.4)
\end{aligned}$$

This implies that $n \leq 3(k-\lambda)+4$, and hence $n = 3(k-\lambda)+4$ by the assumption of Theorem 1.6. Thus equalities hold in (2.4). The equality $d_G(y_1)+d_G(y_2)+d_G(u_1^+)+d_G(u_2^+) = 2(n-k+\lambda)-3$ implies that $\sigma_4(G) = 2(n-k+\lambda)-3$, and the equality $|C| = n-2$ implies that $|Y| = 2$, and the equality $n'-k+\lambda = n-k+\lambda$ implies that $n = n'$. Moreover, by the definition of Y_1, the equality $d_G(y_1)+d_G(y_2) = n'-k+\lambda-2$ implies that $d_G(y_j) = t = (n'-k+\lambda)/2-1 = (n-k+\lambda)/2-1 = (k-\lambda)+1$ for any $j = 1,2$. Then $n-|Y| = 3(k-\lambda)+4-2 = 3(k-\lambda)+3-1 = 3t-1$. Hence the equality holds in (2.2). This implies that there exists $1 \leq j_0 \leq t$ such that $|I_{j_0}| = 2$ and $|I_h| = 3$ holds for any $1 \leq h \leq t$ with $h \neq j_0$. By Claim 2 (ii), $N_G(y_1) = N_G(y_2)$.

Let $P_1 := u_{j_0}^{+3}u_{j_0}^{+4}$, $Q_1 := y_1u_{j_0}^{+5}\overrightarrow{C}u_{j_0}^{+2}y_2$, and $P_2 := y_1u_{j_0}\overrightarrow{C}u_{j_0}^{+}$, $Q_2 := y_2u_{j_0}^{+2}\overrightarrow{C}u_{j_0}^{-}$, and $P_3 := y_1u_{j_0}^{+2}\overrightarrow{C}u_{j_0}^{+4}$, $Q_3 := y_2u_{j_0}^{+5}\overrightarrow{C}u_{j_0}^{+}$, and $P_4 := y_1u_{j_0}\overrightarrow{C}u_{j_0}^{+2}y_2$, $Q_4 := u_{j_0}^{+3}\overrightarrow{C}u_{j_0}^{-}$, and $P_5 := y_1u_{j_0}^{+2}\overrightarrow{C}u_{j_0}^{+5}y_2$, $Q_5 := u_{j_0}^{+6}\overrightarrow{C}u_{j_0}^{+}$. Then P_j and Q_j are vertex-disjoint paths of G such that $V(G) = V(P_j) \cup V(Q_j)$ and $|P_j| = 1+j$ for any $1 \leq j \leq 5$. Since G contains no k vertex-disjoint paths of order $n_1, n_2, \ldots, n_k$, these imply that $n_j \geq 7$ for any $1 \leq j \leq k$. Hence $3(k-\lambda)+4 = n = \sum_{j=1}^{k} n_j \geq 7k$, that is, $k = 1$ and $\lambda = 0$. Then $n = n_1 = 7$, $t = 2$ and $\sigma_4(G) = 2(n-k+\lambda)-3 = 9$. By Claim 2 (iii), $u_{j_0}^{+}u_{j_0}^{+3}, u_{j_0}^{+}u_{j_0}^{-} \notin E(G)$. These imply that $G \in \{(3K_1 \cup K_2)+2K_1, (3K_1 \cup K_2)+K_2\}$. Hence the statement (ii) of Theorem 1.6 holds, a contradiction. $\square$

Claim 6 *$|I_i| \geq 4$ holds for any $1 \leq i \leq t$ except at most one.*

Proof. Suppose that there exist $1 \leq i < j \leq t$ such that $|I_i| \leq 3$ and $|I_j| \leq 3$. Then by Claim 2 (i)–(iii), $\{y_1, y_l, u_{i+1}^-, u_{j+1}^-\}$ is an independent set. Then $d_G(u_{i+1}^-)+d_G(u_{j+1}^-) \geq 2(n-k+\lambda)-3-(d_G(y_1)+d_G(y_l)) \geq 2(n'-k+\lambda)-3-2((n'-k+\lambda)/2-1) = (n'-k+\lambda)-1$. Hence we may assume that $d_G(u_{i+1}^-) \geq (n'-k+\lambda)/2$. This implies that $|I_i| = 3$ by Claim 3. By symmetry, we may assume that $i = 1$. Note that $u_2^- = u_1^{+2}$.

Subclaim 6.1 *$u_i^- \notin N_C(u_1^+)$ holds for any $1 \leq i \leq t$ with $i \neq 2$.*

Proof. Suppose not, and take $u_i^- \in N_C(u_1^+)$ for some $1 \leq i \leq t$ with $i \neq 2$. Then $D := u_i \overrightarrow{C} u_1^+ u_i^- \overleftarrow{C} u_2 y_l u_i$ is a longest cycle such that $V(D) = V(C) \cup \{y_l\} - \{u_1^{+2}\}$. Since $d_G(y_l) < (n' - k + \lambda)/2 \leq d_G(u_1^{+2})$, this contradicts the choice of C. $\square$

By Claim 5, there exists $y \in V(G) - \{y_1, y_l, u_1^+, u_2^+\}$ such that $d_G(y) \leq t$ and $\{y, y_1, y_l, u_1^+\}$ is an independent set. Hence

$$d_G(u_1^+) \geq \sigma_4(G) - d_G(y) - d_G(y_1) - d_G(y_l) \geq \frac{4n-1}{3} - 3t. \qquad (2.5)$$

By Claim 2 (iii), $N_C(y_l)^+ \cap N_C(u_1^+) = \emptyset$. By Claim 4, $|N_C(y_l)^+ \cap N_C(y_l)^-| \leq 1$. By Subclaim 6.1, $|N_C(y_l)^- \cap N_C(u_1^+)| \leq 1$. Therefore, by (2.3) and (2.5),

$$\begin{aligned} n - |Y| = |C| &\geq |N_C(y_l)^+| + |N_C(y_l)^-| + |N_C(u_1^+)| - 2 \\ &\geq 2t + \frac{4n-1}{3} - 3t - 2 \geq \frac{4n-1}{3} - \left(\frac{n - |Y| + 1}{3}\right) - 2. \qquad (2.6) \end{aligned}$$

This implies that $|Y| \leq 2$, and hence $|Y| = 2$. Thus equalities hold in (2.5) and (2.6). By (2.1), the equality $\sigma_4(G) = (4n-1)/3$ implies that $n = 3(k-\lambda)+4$. Hence the equalities $d_G(y_1) = d_G(y_2) = t$ and the equality $t = (n - |Y| + 1)/3$ imply that $d_G(y_i) = t = (n-1)/3 = (n-4)/3 + 1 = (k - \lambda) + 1 = (n - k + \lambda)/2 - 1$ for $i = 1, 2$. Then by the definition of Y_1, $(n' - k + \lambda)/2 - 1 \geq d_G(y_2) = (n - k + \lambda)/2 - 1 \geq (n' - k + \lambda)/2 - 1$, and hence $n = n'$. By arguing as in the proof of Claim 5, we can see that the statement (ii) of Theorem 1.6 holds, a contradiction. $\square$

By Claim 6, $n - |Y| = |C| \geq 4(t-1)+2$. Hence we have $t \leq n/4 - (|Y| - 2)/4$. For each $i = 1, 2$, by Claim 5, there exists y such that $\{y, y_1, y_l, u_i^+\}$ is an independent set and $d_G(y) \leq t$. Therefore by Claim 2 (ii), for any $i = 1, 2$,

$$d_C(u_i^+) = d_G(u_i^+) \geq \sigma_4(G) - 3t \geq \frac{7}{12}n - \frac{1}{3} + \frac{3}{4}(|Y| - 2).$$

Hence we have

$$d_C(u_1^+) + d_C(u_2^+) \geq 2\left(\frac{7}{12}n - \frac{1}{3} + \frac{3}{4}(|Y| - 2)\right) \geq 2\left(\frac{7}{12}(|C| + 2) - \frac{1}{3}\right) \geq |C| + 1.$$

By Lemma 2.3, G has a path P from u_1 to u_2 such that $V(P) = V(C)$. Hence G has a cycle of length more than $|C|$ since $u_i \in N_G(y_l)$ for each $i = 1, 2$, a contradiction. This completes the proof of Theorem 1.6. $\square$

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