On the Cordiality of D(1, 3) Graphs*

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Abstract Let $d_G(x)$ be the degree of a vertex x in a graph G. A graph G is called $D(i_1, ..., i_k)$ graph, if $\{d_G(x)|x \in V(G)\} = \{i_1, ..., i_k\}$.In this paper, a necessary and sufficient condition for a connected D(1, 3) graph to be cordial is given.

Keywords Cordial graph D(1, 3) graph
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1 Introduction

In the following discourse, we consider finite, undirected simple graphs and 0-1 labelings only. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, we define a 0-1 label f on $V(G)$ by giving each $v \in V(G)$ a label $f(v) = 0$ or 1, and denote $V_i = V_i(G) = V_i(f, G) = \{v|v \in V(G), f(v) = i\}, i = 0, 1$. From a labeling f on V(G), we derive a 0-1 label on E(G) by giving each $uv \in E(G)$ a label $f(uv) = |f(u) - f(v)|$. Let $E_{ii} = E_{ii}(G) = E_{ii}(f, G) = \{uv|uv \in E(G), f(u) = f(v) = i\}, i = 0, 1, E_1 = E_1(G) = E_1(f, G) = \{uv|uv \in E(G), |f(u) - f(v)| = 1\}, E_0 = E_{00} \cup E_{11}$. Let $v_i(G) = v_i(f, G)$, $e_i(G) = e_i(f, G)$, e_{ii} be the cardinal numbers of V_i, E_i, E_{ii} respectively. For a set of vertices V_i, we use $G[V_i]$ to denote the induced subgraph of G whose vertex set is V_i and whose edge set is the subset of E(G) consisting of those edges with both ends in V_i. Note that $E_{ii} = E(G[V_i])$.

Definition 1[1] For a graph G, if there exists a 0-1 labeling f on G such that $|v_0(f, G) - v_1(f, G)| \leq 1$ and $|e_0(f, G) - e_1(f, G)| \leq 1$, then G is said to be cordial, and f is a cordial labeling.

Definition 1.2 Let $d_G(x)$ be the degree of the vertex x in a graph G. A graph G is called $D(i_1, ..., i_k)$ graph, if $\{d_G(x)|x \in V(G)\} = \{i_1, ..., i_k\}$.

The cordiality of 3-regular graph has been discussed in [2], in this paper, we give a necessary and sufficient condition for a D(1, 3) graph to be cordial.

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2 Preliminaries

For a labeling f of a graph G and a vertex $x \in V(G), e_i(x) = e_i(f,\ x)$ is the number of edges labeled i incident to x, $i = 1,\ 0$, and $\phi(x) = e_0(x) - e_1(x)$. It is easy to see that $\phi(x) \in \{\pm 1,\ \pm 3\}$ for each vertex x of a D(1, 3) graph, and $\sum_{x \in V(G)} \phi(x) = 2[e_0(G) - e_1(G)]$.

Lemma 1 Let G be a graph with $|G| \geq 2$, and f be a labeling of the graph G. If $\{x, y\} \subset V(G)$ and $f(x) \neq f(y)$, then
(1) $e_0(G)$ decreases $\phi(x)$ when the labeling of x is changed;
(2) (a). $e_0(G)$ decreases $\phi(x) + \phi(y)$ when the labels of x and y are exchanged and $xy \notin E(G)$; (b). $e_0(G)$ decreases $\phi(x) + \phi(y) + 2$ when the labels of x and y are exchanged and $xy \in E(G)$.

Proof. Consider another labeling g, let $g(v) = f(v)$ for $v \neq x$ and $g(x) = 1 - f(x)$, $\{v, x\} \subset V(G)$. We see that $g(e) \neq f(e)$ for the edge e which is incident to x and $g(e) = f(e)$ for the edge e which is not incident to x. It implies that $e_0(f,\ G) - e_0(g,\ G) = e_0(f,\ x) - e_0(g,\ x) = e_0(f,\ x) - e_1(f,\ x) = \phi(x)$.
(2) (a).By applying (1) twice we can obtain the statement.
(b).When we change the label of the vertex x, $e_0(G)$ decreases $\phi(x)$, $e_0(y)$ increases 1 and $e_1(y)$ decreases 1 . So $e_0(G)$ decreases $\phi(y) + 2$ again after changing the label of the vertex y. Thus we have (b).

Lemma 2 Let G be a $D(1,\ 3)$ graph of order $2n (n \geq 4)$, there exists a labeling f, such that, $v_0(G) = v_1(G)$ and $e_0(G) \geq e_1(G)$.

Proof. Consider G^c(the complementary graph of G), since the maximum degree of G, $\Delta(G) \leq 3$, we have the minimum degree of G^c, $\delta(G^c) \geq 2n - 1 - 3 \geq n$. By Dirac Theorem[3][4], G^c is a Hamiltonian graph. Let $C_{x_1 y_1 x_2 y_2\ \ldots\ x_n y_n}$ be a Hamiltonian cycle of G^c, then $x_i y_i \notin E(G)$, $i = 1,\ \ldots,\ n$. Let f_1 be a labeling with $f_1(x_i) = 0$, $f_1(y_i) = 1$,if $e_0(f_1, G) \geq e_1(f_1, G)$, then $f = f_1$ is just the labeling we need. Otherwise, $\sum_{i=1}^{n} [\phi(x_i) + \phi(y_i)] = 2[e_0(f_1, G) - e_1(f_1, G)] < 0$, we can find a number $i \in \{1,\ 2,\ \ldots,\ n\}$ such that, $\phi(x_i) + \phi(y_i) < 0$. By interchanging the labels of x_i and y_i only, we get a new labeling f_2. According to Lemma 1(2)(a), $e_0(f_2,\ G) > e_0(f_1,\ G)$ .If $e_0(f_2,\ G) \geq e_1(f_2,\ G)$, then f_2 is desired, else if $e_0(f_2,\ G) < e_1(f_2,\ G)$, go on with the above process we can get a desired labeling f, such that $v_0(f, G) = v_1(f, G)$ and $e_0(f, G) \geq e_1(f, G)$.

Lemma 3 Let G be a $D(1,\ 3)$ graph and f be a labeling of the graph $G(|G| = 2n, n \geq 4)$, and $v_0(G) = v_1(G) = n$, $e_0(G) > e_1(G)$, then there are two vertices $x \in V_0(G)$ and $y \in V_1(G)$ such that $\phi(x) > 0$ and $\phi(y) > 0$.

Proof(proof by contradiction). Suppose that $\phi(x) = e_0(x) - e_1(x) < 0$ for every vertex $x \in V_0(G)$, it is easy to see that each vertex x with degree 1 in $V_0(G)$ has an incident edge with label 1, and each vertex x with degree 3 in $V_0(G)$ has two or three incident edges with label 1. Let k be the number of vertices with degree 1 in $V_0(G)$, then $n - k$ is the number of vertices with degree 3 in $V_0(G)$. Since $\phi(x) < 0$ for every vertex $x \in V_0(G)$, then $e_1(G) \geq k + 2(n-k) = 2n - k$ and as $e(G) = \frac{1}{2} \sum_{v \in V(G)} d_G(v) \leq \frac{1}{2}[k \times 1 + (2n-k) \times 3] = 3n - k$, we see that $e_0(G) = e(G) - e_1(G) \leq 3n - k - (2n-k) = n \leq n + (n-k) \leq e_1(G)$. It contradicts $e_0(G) > e_1(G)$, then there is a vertex $x \in V_0(G)$, such that $\phi(x) > 0$; similarly, there is a vertex $y \in V_1(G)$, such that $\phi(y) > 0$.

Lemma 4 Let f be a labeling of a connected $D(1, 3)$ graph $G(|G| = 2n, n \geq 4)$, and $v_0(G) = v_1(G) = n$, $e_0(G) > e_1(G)$, then there exist vertices $x \in V_0(G)$, $y \in V_1(G)$, after exchanging the labels of x and y only, $e_0(G)$ decreases 2.

Proof. We distinguish the following four cases.

Case 1. There exist $x^* \in V_0(G)$ and $y^* \in V_1(G)$, s.t. $\phi(x^*) = \phi(y^*) = 3$. Since G is connected, G contains a path $Px^*u_1u_2...u_sy^*$ with $f(u_1) = 0$ and $f(u_s) = 1$. Obviously, there are two vertices u_i and $u_j (i < j)$, such that $f(x^*) = f(u_1) = f(u_2) = ... = f(u_i) = 0$, $f(u_{i+1}) = 1$ and $f(y^*) = f(u_s) = f(u_{s-1}) = ... = f(u_j) = 1$, $f(u_{j-1}) = 0$. It is easy to see that $\phi(u_i) \in \{\pm 1\}$ and $\phi(u_j) \in \{\pm 1\}$. Since $f(u_i) = 0$, $u_iy^* \notin E(G)$. If $\phi(u_i) = -1$,by Lemma 1(2)(a), u_i and y^* are desired.Hence we may assume that $\phi(u_i) = 1$, similarly, we can assume that $\phi(u_j) = 1$. Since G is a $D(1, 3)$ graph, there is a vertex z with degree 1, without loss of generality, we assume that $z \in V_0(G)$. If $\phi(z) = -1$, z and y^* are desired, else if $\phi(z) = 1$, as $e_0(u_j) - e_1(u_j) = \phi(u_j) = 1$, u_j has only one neighbor in $V_0(G)$, obviously, $u_{j-1} \neq z$. By Lemma 1(2)(a), z and u_j are desired.

Case 2. $\phi(u) < 3$ for every $u \in V(G)$. Since $\sum_{u \in V(G)} \phi(u) = 2[e_0(G) - e_1(G)] > 0$, $|V(G)| \geq 8$ and $\phi(u) \in \{-3, -1, 1\}$ for each $u \in V(G)$. It yields at least five vertices u_i with $\phi(u_i) = 1$, $i = 1, ..., 5$. Obviously, there are at least three vertices in $V_0(G)$ or in $V_1(G)$. Without loss of generality, we assume that $\{u_1, u_2, u_3\} \subset V_0(G)$. By Lemma 3, there exists a vertex $u_4 \in V_1(G)$, for $\phi(u_4) = 1$, there is at least one vertex $u_i (i = 1, 2, 3)$ such that $u_iu_4 \notin E(G)$. By Lemma 1(2)(a), u_i and u_4 are desired.

Case 3. There exists $x^* \in V_0(G)$, s.t. $\phi(x^*) = 3$ and $\phi(y) < 3$ for each $y \in V_1(G)$. By Lemma 3, we know that $V_1' = \{y | y \in V_1(G), \phi(y) = 1\} \neq \Phi$. If $y \in V_1'$ and $d_G(y) = 1$, then $e_0(y) = 1$, $e_1(y) = 0$ and $d_{G[V_1']}(y) = 1$; If $y \in V_1'$ and $d_G(y) = 3$, then $e_0(y) = 2$, $e_1(y) = 1$ and $d_{G[V_1']}(y) = 2$. So each component of $G[V_1']$ is a cycle or a path. Note that such a path can not be K_2, otherwise, this path has only two vertices z, w with $\phi(z) = \phi(w) = 1$, it implies that $d_G(z) = d_G(w) = 1$. It contradicts the connectivity of the graph G. Hence $G[V_1']$ must have a vertex y_1 with $d_{G[V_1']}(y_1) = 2$. Let the neighbor set of y_1 in $G[V_1']$ be $\{y_2, y_3\}$ and x be the neighbor of y_1 in $G[V_0]$(see Figure 1(a)). Obviously, $\phi(x) \neq 3$. If $\phi(x) = 1$, then x has just one neighbor in $V_1(G)$. It implies that $xy_2 \notin E(G)$. By Lemma 1(2)(a), x and y_2 are desired. If $\phi(x) = -1$, by Lemma 1(2)(b), x and y_1 are what we need. We assume that $\phi(x) = -3$, when the neighbor set of x, $N_G(x) = \{y_1, y_2, y_3\}$(see Figure 1(b)), we prove Lemma 4 as follows. For G can not contain a 3-regular subgraph $G[\{x, y_1, y_2, y_3\}]$, we know

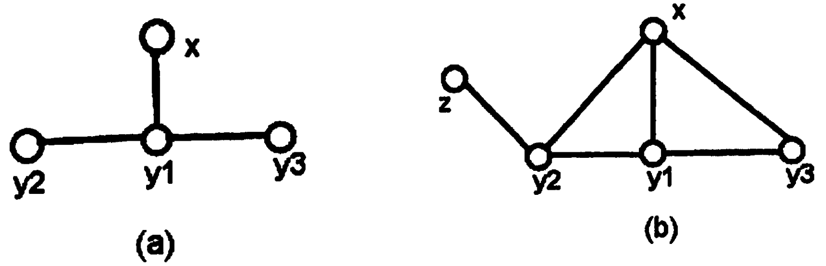

Figure 1: Two steps in the proof of Case 3

that $y_2y_3 \notin E(G)$. Let z be the third neighbor of y_2, since $\phi(y_2) = 1$, $f(z) = 1$, by interchanging the labels of x and y_2, $e_0(G)$ states the same, while $\phi(z)$ has a new value -1, by Lemma 1(2)(a), x^* and z are desired. When $N_G(x) \neq \{y_1, y_2, y_3\}$, we can assume that $xy_2 \notin E(G)$. By interchanging the labels of x and y_1, $\phi(y_2)$ has a new value -1. Thus x^* and y_2 are desired.

Case 4. There exists $y^* \in V_1(G)$, s.t. $\phi(y^*) = 3$ and $\phi(x) < 3$ for each $x \in V_0(G)$. It is easy to see that Case 4 can be proved in the same way as Case 3. The proof is completed.

3 Main Result

Theorem Let G be a connected $D(1, 3)$ graph , then G is cordial if and only if $e(G) + v(G) \not\equiv 2$ $(mod\ 4)$.

Proof.(1) $|G| \geq 8$. necessity. Seoud and Abdel Maqusoud[5][6] proved that if every vertex of G has odd degree, then G is not cordial when $v(G) + e(G) \equiv 2$ $(mod\ 4)$. From this, the necessity is obvious.

sufficiency. Since every vertex of G has an odd degree, we can suppose that $|G| = 2k (k \geq 4)$. By Lemma 2, there is a labeling f, s.t. $|V_0(G)| = |V_1(G)| = k$, and $e_0(G) \geq e_1(G)$. Suppose that there are r vertices in $V_0(G)$ with degree 3, and $k - r$ vertices in $V_0(G)$ with degree 1; there are s vertices in $V_1(G)$ with degree 3, and $k - s$ vertices in $V_0(G)$ with degree 1. Let $e_1(G) = t$, then $e_{00}(G) = \frac{3r+k-r-t}{2} = r + \frac{k-t}{2}$; in a similar manner, $e_{11}(G) = s + \frac{k-t}{2}$, so $2|(k - t)$, and $v(G) + e(G) = 2k + [(r + s + k - t) + t] = 3k + r + s$, $e_0(G) - e_1(G) = (r + s + k - t) - t = r + s + k - 2t$, hence $e(G) + v(G) - [e_0(G) - e_1(G)] = 2(k - t) \equiv 0$ $(mod\ 4)$. If $e(G) + v(G) \not\equiv 2$ $(mod\ 4)$, then $e_0(G) - e_1(G) \not\equiv 2$ $(mod\ 4)$. Furthermore, since $e_0(G) + e_1(G) = e(G)$ is a constant, $e_0(G) - e_1(G)$ decreases 4 when $e_0(G)$ decreases 2, using Lemma 4, we can obtain a labeling g of the graph G such that $e_0(G) - e_1(G) \in \{-1, 0, 1, 2\}$, for $e_0(G) - e_1(G) \not\equiv 2$ ($mod\ 4$), we have $|e_0(G) - e_1(G)| \leq 1$. Thus, we can get a cordial labeling of G, so G is cordial.

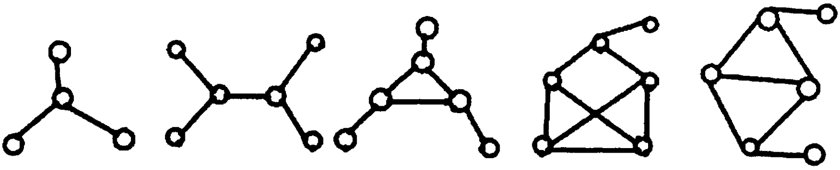

Figure 2: Five cases of D(1, 3) graphs when $|G| < 8$

(2) $|G| < 8$. There are only five kinds of D(1, 3) graphs, see Figure 2,it is easy to check that the statement of the Theorem is valid to these cases.

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