On the second largest scrambling index of primitive matrices*

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Abstract

The scrambling index of an $n \times n$ primitive matrix A is the smallest positive integer k such that $A^k(A^T)^k > 0$, where A^T denotes the transpose of A. In 2009, M. Akelbek, and S. Kirkland gave an upper bound on the scrambling index of an $n \times n$ primitive matrix M in terms of its order n, and they also characterized the primitive matrices that achieve the upper bound. In this paper, we characterize primitive matrix which achieves the second largest scrambling index in terms of its order. Meanwhile we show that there exists gap in scrambling index set of primitive matrices.

AMS classification: 15A48, 05C20, 05C50, 05C75
Keywords: Scrambling index, Primitive matrix.

1 Introduction

A matrix A is called *nonnegative* if all its elements are nonnegative, and denoted by $A \geq 0$. A matrix A is called *positive* if all its elements are positive, and denoted by $A > 0$. We denote the $n \times n$ all ones matrix by J_n. The subscripts will be omitted whenever their values are clear from the context.

Let $D = (V, E)$ denote a *digraph* on n vertices. Loops are permitted, but multiple arcs are not. A $u \to v$ *walk* in D is a sequence of vertices $u, u_1, \ldots, u_p = v$ and a sequence of arcs $(u, u_1), (u_1, u_2), \ldots, (u_{p-1}, v)$, where the vertices and the arcs are not necessarily distinct. The *length* of

*Research supported by NNSF of China (No. 11071227), and Shanxi Scholarship Council of China (No. 2012-070).

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a walk W is the number of arcs in W. The length of a shortest cycle in D is called the *girth* of D. The notation $u \xrightarrow{k} v$ is used to indicate that there is a $u \to v$ walk of length k.

For an $n \times n$ nonnegative matrix $A = (a_{ij})$, the digraph $D(A)$ is the digraph with vertex set $V(D(A)) = \{1, 2, \ldots, n\}$, and (i, j) is an arc of $D(A)$ if and only if $a_{ij} \neq 0$. Then, for a positive integer $r \geq 1$, the (i, j)th entry of the matrix A^r is positive if and only if $i \xrightarrow{r} j$ in the digraph $D(A)$.

A digraph D is called *primitive* if for some positive integer k, there is a walk of length exactly k from each vertex u to each vertex v (possibly u again). Equivalently, a square nonnegative matrix A of order n is called primitive if there exists a positive integer r such that $A^r > 0$.

The *scrambling index* of a primitive digraph D, denoted by $k(D)$, is the smallest positive integer k such that for every pair of vertices u and v, there exists some vertex w such that $u \xrightarrow{k} w$ and $v \xrightarrow{k} w$ in D. An analogous definition for scrambling index can be given for nonnegative matrices. The scrambling index of a primitive matrix A, denoted by $k(A)$, is the smallest positive integer k such that any two rows of A^k have at least one positive element in a coincident position. The scrambling index of a primitive matrix A can also be equivalently defined as the smallest positive integer k such that $A^k(A^T)^k > 0$, where A^T denotes the transpose of A.

If A is the adjacency matrix of a primitive digraph D, then $k(D) = k(A)$. As a result, throughout the paper, where no confusion occurs, we use the digraph D and the adjacency matrix $A(D)$ interchangeably.

The other terminology and notation used here we follow [1].

Let D_1 and D_2 be primitive digraphs of order n in Fig. 1.1 and Fig. 1.2 respectively.

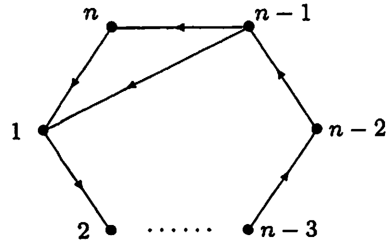

Fig. 1.1 The digraph D_1 (Wielandt digraph, $n \geq 3$)

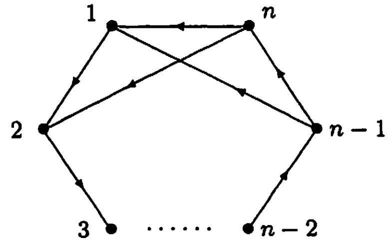

Fig. 1.2 The digraph D_2 ($n \geq 4$)

Denote

$$W_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad W_n = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ 1 & 0 & \cdots & \cdots & 0 & 1 \\ 1 & 0 & \cdots & \cdots & 0 & 0 \end{bmatrix},$$

where $n \geq 3$, and

$$H_n = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 1 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

where $n \geq 4$. Then $D(W_n) = D_1$ for $n \geq 3$, and $D(H_n) = D_2$ for $n \geq 4$.

In [2, 3], M. Akelbek, and S. Kirkland obtained an upper bound on the scrambling index of an $n \times n$ primitive matrix M in terms of its order n, and they also characterized all primitive matrices that achieve the upper bound.

Lemma 1.1 *([2]) Let A be a primitive matrix of order $n \geq 2$. Then*

$$k(A) \leq \lceil \frac{(n-1)^2+1}{2} \rceil.$$

Equality holds if and only if there is a permutation matrix P such that PAP^T is equal to W_2 or J_2 when $n = 2$ and W_n when $n \geq 3$.

In this paper, we characterize primitive matrices which achieve the second largest scrambling index in terms of its order $n \geq 5$. Meanwhile we show that there exists gap in scrambling index set of primitive matrices.

2 Main results

Let $X \subseteq V(D)$, and $R_t(X)$ denote the set of all those vertices which can be reached by a walk of length t in digraph D starting from some vertex in X.

Lemma 2.1 *Let D_2 be the primitive digraph of order $n \geq 4$ as given in Figure 1.2. Then $k(D_2) = \lceil \frac{(n-1)^2+1}{2} \rceil - 1$.*

Proof For simplicity, let $l = \lceil \frac{(n-1)^2+1}{2} \rceil - 1$.

Case 1. n is even.

In this case $l = (\frac{n}{2} - 1)n$. It is not difficult to verify that
$R_l(\{1\}) = \{1, 2, \ldots, \frac{n}{2}\}$,
$R_l(\{n\}) = \{n, 1, 2, \ldots, \frac{n}{2}\}$,
$R_l(\{i\}) = \{i, i+1, \ldots, i + \frac{n}{2} - 1\}$ for $2 \leq i \leq \frac{n}{2}$,
$R_l(\{\frac{n}{2}+1\}) = \{\frac{n}{2}+1, \frac{n}{2}+2, \ldots, n-1, n, 1\}$, and
$R_l(\{i\}) = \{i, i+1, \ldots, n-1, n, 1, 2, \ldots, i - \frac{n}{2}\}$ for $\frac{n}{2} + 2 \leq i \leq n-1$.
So $R_l(\{i\}) \cap R_l(\{j\}) \neq \phi$ for $i \neq j$, and it follows that $k(D_2) \leq l$.

On the other hand, $R_{l-1}(\{1\}) \cap R_{l-1}(\{\frac{n}{2}+1\}) = \phi$ since $R_{l-1}(\{1\}) = \{n, 1, 2, \ldots, \frac{n}{2} - 1\}$ and $R_{l-1}(\{\frac{n}{2}+1\}) = \{\frac{n}{2}, \frac{n}{2}+1, \ldots, n-1\}$. Consequently, $k(D_2) > l - 1$, and then $k(D_2) = l$.

Case 2. n is odd.

In this case $l = \frac{1}{2}(n-1)^2$. It is not difficult to verify that
$R_l(\{1\}) = \{1, n, n-1, \ldots, \frac{n+1}{2} + 1\}$,
$R_l(\{i\}) = \{i, i-1, \ldots, 1, n, n-1, \ldots, \frac{n+1}{2} + i\}$ for $2 \leq i \leq \frac{n-3}{2}$,
$R_l(\{\frac{n-1}{2}\}) = \{\frac{n-1}{2}, \frac{n-1}{2} - 1, \ldots, 2, 1, n\}$,
$R_l(\{\frac{n+1}{2}\}) = \{\frac{n+1}{2}, \frac{n+1}{2} - 1, \ldots, 1\}$,
$R_l(\{i\}) = \{i, i-1, \ldots, i - \frac{n-1}{2}\}$ for $\frac{n+3}{2} \leq i \leq n-1$, and
$R_l(\{n\}) = \{1, n, n-1, \ldots, n - \frac{n-1}{2}\}$.
So $R_l(\{i\}) \cap R_l(\{j\}) \neq \phi$ for $i \neq j$, and it follows that $k(D_2) \leq l$.

On the other hand, $R_{l-1}(\{1\}) \cap R_{l-1}(\{\frac{n+1}{2}\}) = \phi$ since $R_{l-1}(\{1\}) = \{n-1, n-2, \ldots, \frac{n+1}{2}\}$ and $R_{l-1}(\{\frac{n+1}{2}\}) = \{\frac{n+1}{2} - 1, \frac{n+1}{2} - 2, \ldots, 1, n\}$. Consequently, $k(D_2) > l - 1$, and then $k(D_2) = l$. □

Denote

$$K(n,s) = n - s + \begin{cases} \frac{s-1}{2}n, & \text{for } s \text{ is odd,} \\ \frac{n-1}{2}s, & \text{for } s \text{ is even.} \end{cases}$$

Note that $K(n, n-1) = \lceil \frac{(n-1)^2+1}{2} \rceil$, $K(n, n-2) < \lceil \frac{(n-1)^2+1}{2} \rceil - 1$ when $n \geq 7$, and the function $K(n,s)$ is increase with respect to s for $n \geq 5$.

Lemma 2.2 *([2, 3]) Let D be a primitive digraph with n vertices, and suppose that s is the girth of D with $s \geq 2$. If there is another cycle of length p, $s < p \leq n$, such that* $\gcd(s,p) = 1$*, then $k(D) \leq K(n,s)$. Furthermore, if $p < n$, then $k(D) < K(n,s)$.*

Lemma 2.3 *([3]) Let D be a primitive digraph with n vertices, and s be the girth of D with $s \geq 2$. Let $L(D) = \{s, a_1, a_2, \ldots, a_{r-1}, a_r\}$ be the set of distinct cycle lengths of D. If* $\gcd(s, a_i) \neq 1$ *for each $i = 1, 2, \ldots, r$, then $k(D) < K(n,s)$.*

Now we give the main result as follows.

Theorem 2.4 *Let D be a primitive digraph of order $n \geq 5$. Then $k(D) = \lceil \frac{(n-1)^2+1}{2} \rceil - 1$ if and only if D is isomorphic to D_2.*

Proof The sufficiency is following Lemma 2.1. Now we consider the necessity.

Let s be the girth of D. First we consider $n \geq 5$ and $n \neq 6$. For a primitive digraph D, we have $s \leq n-1$. When $s = 1$, it is easy to see that $k(D) \leq n-1 < K(n, n-2) < \lceil \frac{(n-1)^2+1}{2} \rceil - 1$. When $2 \leq s \leq n-2$, assume $L(D) = \{s, a_1, a_2, \ldots, a_{r-1}, a_r\}$. If there exists some a_i, without loss of generality, say a_1, such that $\gcd(s, a_1) = 1$, then by Lemma 2.2 and the fact that $K(n,s)$ is increase with respect to s, $k(D) \leq K(n,s) \leq K(n, n-2) < \lceil \frac{(n-1)^2+1}{2} \rceil - 1$. If $\gcd(s, a_i) \neq 1$ for each $i = 1, 2, \ldots, r$, then by Lemma 2.3, $k(D) < K(n,s) \leq K(n, n-2) < \lceil \frac{(n-1)^2+1}{2} \rceil - 1$. When $s = n-1$, apart from labeling of the vertices, there are only two primitive digraphs; they are D_1 and D_2 as given in Figures 1.1 and 1.2 respectively. By Lemmas 1.1 and 2.1, $k(D_1) = \lceil \frac{(n-1)^2+1}{2} \rceil$ and $k(D_2) = \lceil \frac{(n-1)^2+1}{2} \rceil - 1$. So the result is correct.

Now we consider $n = 6$. In this case, $K(n, n-1) = 13$, $K(n, n-1) - 1 = K(n, n-2) = 12$, and $K(n, n-3) = 9$. For a primitive digraph D, if $s \leq n-3$, then $K(D) \leq K(n, n-3) < 12$. If $s = n-2 = 4$, then by primitivity, digraph D must has a cycle of length 5. By Lemma 2.2, $k(D) < K(n, n-2) = 12$. If $s = n-1$, then we know that $k(D_1) = 13$ and $k(D_2) = 12$. So the result is correct. □

Remark 2.5 *Theorem 2.4 is not correct for $n = 4$. For example, let D be primitive digraph with vertex set $V(D) = \{1,2,3,4\}$, and arc set $E(D) = \{(1,2),(2,1),(1,3),(3,4),(4,1)\}$. Then $R_3(3) = \{2,3\}$, $R_3(2) = \{1,4\}$. So $K(D) \geq 4$. In fact, we can easily show $K(D) = 4$.*

In terms of matrix, Theorem 2.4 can be represent as follows.

Theorem 2.6 *Let A be a primitive matrix of order $n \geq 5$. Then*

$$k(A) = \lceil \frac{(n-1)^2+1}{2} \rceil - 1$$

if and only if there is a permutation matrix P such that PAP^T is equal to H_n.

From the proof of Theorem 2.4, the following result is clear.

Theorem 2.7 *There is no primitive digraph D on $n \geq 7$ vertices such that $K(n, n-2) < k(D) < \lceil \frac{(n-1)^2+1}{2} \rceil - 1$, that is, $\frac{n^2-3n+4}{2} < k(D) < \frac{(n-1)^2}{2}$ for n is odd, and $\frac{n^2-3n+6}{2} < k(D) < \frac{n^2-2n}{2}$ for n is even.*

References

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