ON GENERALIZED TRIPLE DERIVATIONS ON LATTICES

ZAFAR ULLAH, IMRAN JAVAID AND MUHAMMAD ANWAR CHAUDHARY*

ABSTRACT. In this paper we introduce the notion of a generalized triple derivation f, with an associated triple derivation d, on a lattice and investigate some related results. Among some other results we prove that "Let $(L, \wedge, \vee)$ be a distributive lattice and f be a generalized triple derivation, with associated triple derivation d, on L. Then the following conditions are equivalent for all $x, y, z \in L$: (a) f is an isotone generalized triple derivation on L, (b) $f_{x\wedge y\wedge z} = f_x \wedge f_y \wedge f_z$, (c) $f_x \vee f_y \vee f_z = f_{x\vee y\vee z}$."

1. INTRODUCTION

In 1940, G. Birkhoff [6] gave the notion of a lattice. The distributive lattices were introduced by *Grätzer* in 1971. In 1979, R. E. Hoffmann gave the concept of a partially ordered set(poset). Lattices play an important role in many fields such as information theory and cryptanalysis (see [5, 8, 12, 18] and references there in).

Posner[17] studied the notion of a derivation on rings. Moreover,in the past few decades several researchers have studied this notion in rings and near rings.Braser[7] and Hvala[16] introduced the concept of a generalized derivation in rings. This notion has been further studied by Gölbaşi and E. Koc, N. Argaç and E. Albas (see [2, 10, 13, 14] and references there in).

Many researchers have studied analytic and algebraic properties of lattices (see [3, 4, 6, 11, 12, 15, 18, 19, 9, 20] and references there in). Xin et al. [19] studied the notion of a derivation, previously studied for rings, near rings and C^*- algebras, for lattices and discussed some related properties. Alshehri[1] studied generalized derivations in the context of lattices.

In this paper the concept of a generalized triple derivation, with an associated triple derivation, on lattices is introduced and some related identities

---

Key words and phrases. Lattice, modular lattice,distributive lattice,derivation, triple derivation, generalized triple derivation.

2010 *Mathematics Subject Classification.* 06B35, 06B99.

Email: zafarbhatti73@gmail.com, imranjavaid45@gmail.com

*Corresponding author: chaudhry@bzu.edu.pk.

are investigated.This concept is more general than the concept of a generalized derivation.

The motivation behind this paper is to initiate a study of the properties of generalized triple derivations, with associated triple derivations, on lattices and prove certain results.

2. Preliminaries

Definition 2.1. *Let L be a non empty set endowed with operations $\wedge$ and $\vee$. Then $(L, \wedge, \vee)$ is called a lattice if it satisfies the following conditions for all $x, y, z \in L$:*
(i) $x \wedge x = x$, $x \vee x = x$;
(ii) $x \wedge y = y \wedge x$, $x \vee y = y \vee x$;
(iii) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$, $(x \vee y) \vee z = x \vee (y \vee z)$;
(iv) $(x \wedge y) \vee x = x$, $(x \vee y) \wedge x = x$.

Definition 2.2. *A lattice $(L, \wedge, \vee)$ is called a distributive lattice if it satisfies the following conditions for all $x, y, z \in L$:*
(v) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$;
(vi) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.
It is known that in a lattice the above two conditions (v) and (vi) are equivalent [6].

Definition 2.3. *Let $(L, \wedge, \vee)$ be a lattice. A binary relation $\leq$ on L is defined by: $x \leq y$ if and only if $x \wedge y = x$, $x \vee y = y$.*

Definition 2.4. *A lattice $(L, \wedge, \vee)$ is called a modular lattice if it satisfies the following conditions for all $x, y, z \in L$:*
(vii) If $x \leq z$, then $x \vee (y \wedge z) = (x \vee y) \wedge z$.

The following lemma is already known.

Lemma 2.5. *Let $(L, \wedge, \vee)$ be a lattice. Let the binary relation $\leq$ be as in definition 2.3. Then $(L, \leq)$ is a partially ordered set(poset) and for any $x, y \in L$, $x \wedge y$ is the g.l.b of $\{x, y\}$ and $x \vee y$ is the l.u.b of $\{x, y\}$.*

Definition 2.6. *An ideal I of the lattice $(L, \wedge, \vee)$ is a non-empty subset I of L satisfying the properties:*
(viii) $x \leq y, y \in I \Rightarrow x \in I$;
(ix) $x, y \in I \Rightarrow x \vee y \in I$.

Let F be a derivation of any type on a lattice L. In the sequel we shall write F_x for $F(x)$, $x \in L$, and F_I for $F(I)$, $I \subseteq L$.

Definition 2.7. *Let $(L, \wedge, \vee)$ be a lattice. A function $d : L \to L$ is called a derivation on L if $d_{x \wedge y} = (d_x \wedge y) \vee (x \wedge d_y)$, for all $x, y \in L$.*

Definition 2.8. *Let $(L, \wedge, \vee)$ be a lattice. A function $f : L \to L$ is called a generalized derivation on L if there exists a derivation $d : L \to L$ such that for all $x, y \in L$:*

$f_{x\wedge y} = (f_x \wedge y) \vee (x \wedge d_y)$.

Definition 2.9. *Let $(L, \wedge, \vee)$ and $(M, \wedge, \vee)$ be lattices. A function α : $L \to M$ is called an homomorphism if it satisfies the following conditions for all $x, y \in L$:*

(x) $\alpha(x \wedge y) = \alpha(x) \wedge \alpha(y)$,

(xi) $\alpha(x \vee y) = \alpha(x) \vee \alpha(y)$.

If $M = L$ then α is called an endomorphism. If α satisfies x(xi) then it is called $\wedge$-homomorphism ($\vee$-homomorphism).

3. Generalized triple derivations

In this section we describe the concept of a generalized triple derivation, with associated triple derivation d, on a lattice L and prove our results regarding this notion.

Definition 3.1. *Let $(L, \wedge, \vee)$ be a lattice. A function $d : L \to L$ is called a triple derivation on L if :*

$d_{x\wedge y\wedge z} = (d_x \wedge y \wedge z) \vee (x \wedge d_y \wedge z) \vee (x \wedge y \wedge d_z)$ *for all* $x, y, z \in L$.

Definition 3.2. *Let $(L, \wedge, \vee)$ be a lattice. A function $f : L \to L$ is called a generalized triple derivation, with associated triple derivation d, on L if* $f_{x\wedge y\wedge z} = (f_x \wedge y \wedge z) \vee (x \wedge d_y \wedge z) \vee (x \wedge y \wedge d_z)$ *for all* $x, y, z \in L$.

Obviously every derivation on a lattice $(L, \wedge, \vee)$ is a triple derivation and every generalized derivation, with associated derivation d, is a generalized triple derivation with associated triple derivation d. The following examples shows that the converse of above mentioned results are not true in general.

Example 3.3. *Every triple derivation is not a derivation.*

Let L be a lattice of Figure 1. Let $d : L \to L$ be defined by

$$d_x = \begin{cases} 0, & x = 0, 1, b \\ b, & x = a, c. \end{cases}$$

Let $(x, y, z) = (a, b, c)$. *Then* $d_{x\wedge y\wedge z} = d_{a\wedge b\wedge c} = d_c = b$ *and*
$(d_x \wedge y \wedge z) \vee (x \wedge d_y \wedge z) \vee (x \wedge y \wedge d_z) = (d_a \wedge b \wedge c) \vee (a \wedge d_b \wedge c) \vee (a \wedge b \wedge d_c) = (b \wedge b \wedge c) \vee (a \wedge 0 \wedge c) \vee (a \wedge b \wedge b) = c \vee 0 \vee b = b$.
Now $d_{x\wedge y} = d_{a\wedge b} = d_b = 0$ *and*
$(d_x \wedge y) \vee (x \wedge d_y) = (d_a \wedge b) \vee (a \wedge d_b) = (a \wedge 0) \vee (b \wedge b) = 0 \vee b = b$.
So D is not a derivation.

Example 3.4. *Every generalized triple derivation is not a generalized derivation.*

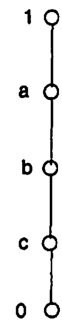

FIGURE 1. Lattice

Let L be a lattice of Figure 1. We define mappings $d : L \to L$ and $f : L \to L$ by

$$d_x = \begin{cases} 0, & x = 0, 1, b \\ b, & x = a, c \end{cases}$$

and

$$f_x = \begin{cases} x, & x = a, 0 \\ b, & x = c \\ c, & x = 1, b. \end{cases}$$

Let $(x, y, z) = (a, b, c)$. Then $f_{x\wedge y\wedge z} = f_{a\wedge b\wedge c} = f_c = b$ and
$(f_x \wedge y \wedge z) \vee (x \wedge d_y \wedge z) \vee (x \wedge y \wedge d_z) = (f_a \wedge b \wedge c) \vee (a \wedge d_b \wedge c) \vee (a \wedge b \wedge d_c) =$
$(a \wedge b \wedge c) \vee (a \wedge 0 \wedge c) \vee (a \wedge b \wedge b) = c \vee 0 \vee b = b.$
Now $f_{x\wedge y} = f_{a\wedge b} = f_b = c$ and
$(f_x \wedge y) \vee (x \wedge d_y) = (f_a \wedge b) \vee (a \wedge d_b) = (a \wedge b) \vee (a \wedge 0) = b \vee 0 = b.$
So f is not a generalized derivation.

Definition 3.5. *Let $(L, \wedge, \vee)$ be a lattice and f a generalized triple derivation,with associated triple derivation d, on L. Then*
(a) f is called an isotone generalized triple derivation if $x \le y$ implies $D_x \le D_y$.
(b) If f is one-to-one, then f is called a monomorphic generalized triple derivation.
(c) If f is onto, then f is called an epic generalized triple derivation.

Proposition 3.6. *Let $(L, \wedge, \vee)$ be a lattice and f a generalized triple derivation, with associated triple derivation d, on L. Then the following hold for all $x, y, z \in L$:*
(a) $d_x \le f_x \le x$,
(b) $f_x \wedge f_y \wedge f_z \le f_{x\wedge y\wedge z} \le f_x \vee f_y \vee f_z$,
(c) If I is an ideal of L with $I \subseteq L$, then $f_I \subseteq I$,

(d) If L has a least element 0, then $f_0 = 0$,
(e) If L has a greatest element 1 then $f_x = (f_1 \wedge x) \vee d_x$.

Proof. (a) Let $x \in L$. Then
$d_x = d_{x\wedge x\wedge x} = (d_x \wedge x \wedge x) \vee (x \wedge d_x \wedge x) \vee (x \wedge x \wedge d_x) = d_x \wedge x$, which implies

$$(1) \qquad d_x \leq x.$$

Further, $f_x \wedge d_x = f_{x\wedge x\wedge x} \wedge d_x = ((f_x \wedge x \wedge x) \vee (x \wedge d_x \wedge x) \vee (x \wedge x \wedge d_x)) \wedge d_x$. The last relation along with (1) and definition 2.1(iv) implies $f_x \wedge d_x = ((f_x \wedge x) \vee d_x) \wedge d_x = d_x$, which implies

$$(2) \qquad d_x \leq f_x.$$

Also $f_x \vee x = f_{x\wedge x\wedge x} \vee x = ((f_x \wedge x \wedge x) \vee (x \wedge d_x \wedge x) \vee (x \wedge x \wedge d_x)) \vee x$. Using (1) and definition 2.1(iii and iv), from the last relation we get
$f_x \vee x = ((f_x \wedge x) \vee d_x) \vee x = (f_x \wedge x) \vee (d_x \vee x) = (f_x \wedge x) \vee x = x$, which implies

$$(3) \qquad f_x \leq x.$$

Using (1), (2) and (3), we get $d_x \leq f_x \leq x$.
(b) Let $x, y, z \in L$. Then
$f_{x\wedge y\wedge z} = (f_x \wedge y \wedge z) \vee (x \wedge d_y \wedge z) \vee (x \wedge y \wedge d_z)$, which implies
$f_x \wedge y \wedge z \leq f_{x\wedge y\wedge z}$. Since $f_y \leq y$ for all $y \in L$, therefore $f_x \wedge f_y \wedge f_z \leq f_x \wedge y \wedge z$. Thus

$$(4) \qquad f_x \wedge f_y \wedge f_z \leq f_{x\wedge y\wedge z}.$$

Since $f_x \wedge y \wedge z \leq f_x$, $x \wedge d_y \wedge z \leq d_y \leq f_y$ and $x \wedge y \wedge d_z \leq d_z \leq f_z$, therefore

$$(5) \qquad f_{x\wedge y\wedge z} \leq f_x \vee f_y \vee f_z.$$

Using (4) and (5), we get
$f_x \wedge f_y \wedge f_z \leq f_{x\wedge y\wedge z} \leq f_x \vee f_y \vee f_z$.
(c) Let $x \in I$. Since $f_x \leq x$, therefore $f_x \in I$ for all $x \in I$. Thus $f_I \subseteq I$.
(d) Since 0 is the least element, then (a) gives $0 \leq d_x \leq f_x \leq x$, which implies $f_0 = 0$.
(e) For each $x \in L$ we have $d_x \leq x \leq 1$. Thus
$f_x = f_{1\wedge x\wedge x} = (f_1 \wedge x \wedge x) \vee (1 \wedge d_x \wedge x) \vee (1 \wedge x \wedge d_x) = (f_1 \wedge x) \vee d_x$. □

Proposition 3.7. *Let $(L, \wedge, \vee)$ be a lattice with greatest element 1. Let f be a generalized triple derivation, with associated triple derivation d, on L. Then for all $x \in L$:*
(a) If $f_1 \leq x$, then $f_1 \leq f_x$,
(b) If $f_1 \geq x$, then $f_x = x$.

Proof. (a) Let $x \in L$ and $f_1 \leq x$. Then proposition 3.6(e) along with definition 2.1(iv) implies $f_x \wedge f_1 = ((f_1 \wedge x) \vee d_x) \wedge f_1 = (f_1 \vee d_x) \wedge f_1 = f_1$, which gives $f_1 \leq f_x$.
(b) Let $x \in L$ and $f_1 \geq x$. Using proposition 3.6(a and e), we have
$f_x = (f_1 \wedge x) \vee d_x = x \vee d_x = x$. □

Proposition 3.8. *Let $(L, \wedge, \vee)$ be a lattice and f a generalized triple derivation, with associated triple derivation d, on L. Then the following hold for all $x, y, z \in L$:*
(a) $f_x = (f_{x\vee y\vee z} \wedge x) \vee d_x$.
(b) If $y \leq x \wedge z$ and $f_x = x$ then $f_y = y$,
(c) If L has a greatest element 1, then $f_1 = 1$ if and only if $f_x = x$.

Proof. (a) Let $x, y \in L$. Then
$f_x = f_{(x\vee y\vee z)\wedge x\wedge x} = (f_{x\vee y\vee z} \wedge x \wedge x) \vee ((x \vee y \vee z) \wedge d_x \wedge x)) \vee ((x \vee y \vee z) \wedge x \wedge d_x))$. The last relation along with proposition 3.6(a) implies $f_x = (f_{x\vee y\vee z} \wedge x) \vee d_x$.
(b) Let $y \leq (x \wedge z)$ and $f_x = x$. Then
$f_y = f_{x\wedge y\wedge z} = (f_x \wedge y \wedge z) \vee (x \wedge d_y \wedge z) \vee (x \wedge y \wedge d_z) = y \vee d_y \wedge (y \wedge d_z) = y \wedge (y \wedge d_z) = y$.
(c) Let $f_x = x$. Then obviously $f_1 = 1$.
Conversely let $f_1 = 1$. Since $x \leq 1$ and $f_1 = 1$. Then (b) implies $f_x = x$. □

Remark 3.9. *Let the lattice $(L, \wedge, \vee)$ have a least element 0. Then f is a monomorphic as well as epic generalized triple derivation, with associated triple derivation $0 : L \to L$ defined by $0_x = 0$ for all $x \in L$.*

Theorem 3.10. *Let $(L, \wedge, \vee)$ be a lattice and f a generalized triple derivation, with associated triple derivation d, on L. Then the following hold for all $x, y, z \in L$:*
(a) $f_x^2 = f_x$.
(b) $f_x = x$ if only if $f_{x\vee y\vee z} = (f_x \vee y \vee z) \wedge (x \vee f_y \vee z) \wedge (x \vee y \vee f_z)$.

Proof. (a) Consider,
$f_x^2 = f(f_x) = f(x \wedge x \wedge f_x) = (f_x \wedge x \wedge f_x) \vee (x \wedge d_x \wedge f_x) \vee (x \wedge x \wedge d_{f_x}) = f_x \vee d_x \vee d_{f_x}$. Since $d_x \leq f_x \leq x$ and $d_{f_x} \leq f_x$, therefore $f_x^2 = f_x$.
(b) Let $f_x = x$. Then
$f_{x\vee y\vee z} = x \vee y \vee z = (x \vee y \vee z) \wedge (x \vee y \vee z) \wedge (x \vee y \vee z)$
$= (f_x \vee y \vee z) \wedge (x \vee f_y \vee z) \wedge (x \vee y \vee f_z)$.
Conversely, let $f_{x\vee y\vee z} = (f_x \vee y \vee z) \wedge (x \vee f_y \vee z) \wedge (x \vee y \vee f_z)$.
Replacing y and z by x in the last equation, we get $f_x = x$. □

Theorem 3.11. *Let the lattice $(L, \wedge, \vee)$ have a greatest element 1. Let f be a generalized triple derivation, with associated triple derivation d, on L. Then the following statements are equivalent for all $x, y, z \in L$:*
(a) f is an isotone generalized triple derivation on L,

(b) $f_x = x \wedge f_1$,
(c) $f_{x\wedge y\wedge z} = f_x \wedge f_y \wedge f_z$,
(d) $f_{x\vee y\vee z} \leq f_x \vee f_y \vee f_z$.

Proof. **(a)** $\Rightarrow$ **(b)** Assume that f is an isotone, then $f_x \leq f_1$. Since $f_x \leq x$, therefore $f_x \leq x \wedge f_1$. Further, Proposition 3.6(e) gives $f_x = (f_1 \wedge x) \vee d_x$, which implies $f_1 \wedge x \leq f_x$. Thus $f_x = x \wedge f_1$.
(b) $\Rightarrow$ **(c)** Let $f_x = x \wedge f_1$. Then
$f_x \wedge f_y \wedge f_z = (x \wedge f_1) \wedge (y \wedge f_1) \wedge (z \wedge f_1) = (x \wedge y \wedge z) \wedge f_1 = f_{x\wedge y\wedge z}$.
(c) $\Rightarrow$ **(a)** Let $f_{x\wedge y\wedge z} = f_x \wedge f_y \wedge f_z$ and $x \leq (y \wedge z)$. Then
$f_x = f_{x\wedge y\wedge z} = f_x \wedge f_y \wedge f_z$, which implies $f_x \leq f_y \wedge f_z = f_{y\wedge z}$. Hence f is an isotone.
(a) $\Rightarrow$ **(d)** Since f is an isotone therefore $f_x \leq f_{x\vee y\vee z}$, $f_y \leq f_{x\vee y\vee z}$ and $f_z \leq f_{x\vee y\vee z}$. Hence $f_x \vee f_y \vee f_z \leq f_{x\vee y\vee z}$.
(d) $\Rightarrow$ **(a)** Let $f_x \vee f_y \vee f_z \leq f_{x\vee y\vee z}$ and $x \leq (y \vee z)$. Then
$f_x \vee f_y \vee f_z \leq f_{x\vee y\vee z} = f_{y\vee z}$, which implies $f_x \leq f_{y\vee z}$. Hence f is an isotone. □

Theorem 3.12. *Let $(L, \wedge, \vee)$ be a modular lattice and f a generalized triple derivation, with associated triple derivation d, on L. Then the following conditions are equivalent for all $x, y, z \in L$:*
(a) f is an isotone generalized triple derivation on L,
(b) $f_{x\wedge y\wedge z} = f_x \wedge f_y \wedge f_z$.

Proof. **(a)** $\Rightarrow$ **(b)** Assume that (a) holds. Then for $x, y, z \in L$
$f_{x\wedge y\wedge z} \leq f_x$, $f_{x\wedge y\wedge z} \leq f_y$ and $f_{x\wedge y\wedge z} \leq f_z$. Thus $f_{x\wedge y\wedge z} \leq f_x \wedge f_y \wedge f_z$.
Since L is modular and $f_x \wedge y \wedge z \leq x \wedge y \wedge z \leq x$, therefore
$f_{x\wedge y\wedge z} = (f_x \wedge y \wedge z) \vee (x \wedge d_y \wedge z) \vee (x \wedge y \wedge d_z) = ((f_x \wedge y \wedge z) \vee (d_y \wedge z) \wedge x) \geq ((f_x \wedge y \wedge z) \wedge x) \vee (x \wedge y \wedge d_z)$. Since $f_x \wedge y \wedge z \leq x$ and L is modular, therefore $f_{x\wedge y\wedge z} \geq (f_x \wedge y \wedge z) \vee (x \wedge y \wedge d_z) = ((f_x \wedge y \wedge z) \vee (y \wedge d_z)) \wedge x \geq (f_x \wedge y \wedge z) \wedge x = f_x \wedge y \wedge z \geq f_x \wedge f_y \wedge f_z$. Thus, $f_{x\wedge y\wedge z} \geq f_x \wedge f_y \wedge f_z$.
(b) $\Rightarrow$ **(a)** Since f is a $\wedge$-homomorphism, so it is an isotone. □

Theorem 3.13. *Let $(L, \wedge, \vee)$ be a distributive lattice and f a generalized triple derivation, with associated triple derivation d, on L. Then the following conditions are equivalent for all $x, y, z \in L$:*
(a) f is an isotone generalized triple derivation on L,
(b) $f_{x\wedge y\wedge z} = f_x \wedge f_y \wedge f_z$,
(c) $f_x \vee f_y \vee f_z = f_{x\vee y\vee z}$.

Proof. Since a distributive lattice is a modular lattice, therefore Theorem 3.12 implies that conditions (a) and (b) are equivalent.
(a) $\Rightarrow$ **(c)** Assume that f is an isotone generalized triple derivation. Since $d_x \leq f_x \leq f_{x\vee y\vee z}$ and $d_y \leq f_y \leq f_{x\vee y\vee z}$, therefore Proposition 3.8(a) implies

$f_x = (f_{x\vee y\vee z} \wedge x) \vee d_x = (f_{x\vee y\vee z} \vee d_x) \wedge (x \vee d_x) = f_{x\vee y\vee z} \wedge x$. Thus $f_x \vee f_y \vee f_z = (f_{x\vee y\vee z} \wedge x) \vee (f_{x\vee y\vee z} \wedge y) \vee (f_{x\vee y\vee z} \wedge z) = f_{x\vee y\vee z} \wedge (x \vee y \vee z) = f_{x\vee y\vee z}$.

(c) $\Rightarrow$ **(a)** Since f is a $\wedge$-homomorphism, so it is an isotone. □

4. Conclusion

We have introduced the concept of a generalized triple derivation on lattices. It has been shown that for a distributive lattice $(L, \wedge, \vee)$ and a generalized triple derivation f, with associated triple derivation d on L, the following conditions are equivalent for all $x, y, z \in L$: (a) f is an isotone generalized triple derivation on L, (b) $f_{x\wedge y\wedge z} = f_x \wedge f_y \wedge f_z$, (c) $f_x \vee f_y \vee f_z = f_{x\vee y\vee z}$.

Acknowledgment. The authors are thankful to the Bahauddin Zakariya University, Multan, Pakistan for the support provided during this work. The authors are also thankful to the referee(s) for suggestions which led to the improvement of the paper.

References

1. N.O. Alshehri, Generalized Derivations of Lattices, *Int. J. Contemp. Math. Sciences* , **5**(2010), 629-640.
2. N. Argaç and E. Albas, Generalized derivations of prime rings, *Algebra Coll.*, **11**(2004), 399-410
3. R. Balbes and P. Dwinger, Distributive Lattices, *University of Missouri Press, Columbia, United State*, 1974.
4. A.J. Bell, The co-information lattice, *in: 4th International Symposium on Independent Component Analysis and Blind Signal Sepration (ICA2003), Nara, Japan*,(2003), 921-926.
5. H.E. Bell and L.C. Kappe, Rings in which derivations satisfy certain algebraic conditions, *Acta Math. Hungar*, **53**(1989), 339-346.
6. G. Birkhoof, Lattice Theory, *American Mathematical Society, New York*, (1940).
7. M. Bresar, On the distance of the composition of the two derivations to the generalized derivations, *Glasgow Math. J.*, **33**(1991), 89-93.
8. C. Carpineto and G. Romano, Information retrieval through hybird navigation of lattice representations, *Int. J. Human Computers Studies*, **45**(1996), 553-578.
9. Y. Ceven and M.A. Öztürk, On f-derivations of lattices, *Bull. Korean Math. Soc.*, **45**(2008), 701-707.
10. M. A. Chaudhry and Z. Ullah, On generalized (α, β)-derivations on lattices, *Quaest. Math.*, **34**(2011), 417-424.
11. C. Degang, Z. Wenxiu, D. Yeung and E.C.C. Tsang, Rough approximations on a complete distributive lattice with application to generalized rough sets, *Informat. Sci.*, **176**(2006), 1829-1848.
12. G. Durfee, Cryptanalysis of RSA using algebraic and lattice methods, *A dissertation submitted to the Department of Computer Sciences and the committe on graduate studies of Stanford University*,(2002), 1-114.
13. Ö. Gölbaşi and K. Kaya, On Lie ideal with generalized derivations, *Siberian. Math. J.*, **47**(2006), 862-866.

14. Ö. Gölbaşi and E. Koç, Generalized derivations on Lie ideal in prime rings, *Turk. J. Math.*, **33**(2009), 1-6.
15. A. Honda and M. Grabisch, Entropy of capacities on lattices and set systems, *Informat. Sci.*, **176**(2006), 3472-3489.
16. B. Hvala, Generalized derivations in rings, *Common.Alg.*,**26**(1998), 1147-1166.
17. E. Posner, Derivations in prime rings, *Proc. Amer. Math. Soc.*, **8**(1957), 1093-1100.
18. R.S. Sandhu, Role hierarchies and constraints for lattice-based access controls *in: Proceeding of the 4th European Symposium on Research in computer Security, Rome, Italy* , (1996), 65-79.
19. X.L. Xin, T.Y. Liu, and J.H. Lu, On derivations of lattices, *Informat. Sci.*, **178**(2008), 307-316.
20. J. Zhan and T.Y. Liu, On f-derivations of BCI-algebras, *Int. J. Math. Sci.*, **11**(2005), 1675-1684.

CENTRE FOR ADVANCED STUDIES IN PURE AND APPLIED MATHEMATICS, BAHAUDDIN ZAKARIYA UNIVERSITY MULTAN, PAKISTAN.