A note on the two-color Rado numbers for $a(x-y) = bz$

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Abstract. Let $R(a(x-y) = bz)$ denote the least integer n such that for every 2-coloring of the set $\{1, 2, \ldots, n\}$ there exists a monochromatic solution to $a(x-y) = bz$. Recently, Gasarch, Moriarty and Tumma conjectured that $R(a(x-y) = bz) = b^2 + b + 1$, where $1 < a < b$. In this note, we confirm this conjecture.

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1 Introduction

Let $\mathbb{N}$ denote the set of natural numbers and let $[a, b] = \{n | n \in \mathbb{N}, a \leq n \leq b\}$. A function $\Delta : [1, n] \to [0, k-1]$ is referred to as a k-coloring of the set $[1, n]$. Given a k-coloring Δ and a linear equation $\mathcal{E}$ in m variables, a solution $(x_1, x_2, \ldots, x_m)$ to $\mathcal{E}$ is monochromatic if and only if

$$\Delta(x_1) = \Delta(x_2) = \cdots = \Delta(x_m). \tag{1.1}$$

In 1916, I. Schur [8] proved that for every integer $k \geq 2$, there exists a least integer $n = S(k)$ such that for every k-coloring of the set $[1, n]$, there exists a monochromatic solution to $x + y = z$. The integers $S(k)$ are called *Schur numbers*. In 1933, R. Rado [7] generalized the concept of Schur numbers to arbitrary systems of linear equations. Rado found necessary and sufficient conditions to determine if an arbitrary system of linear equations admits a monochromatic solution under every k-coloring of the natural numbers. For a linear equation $\mathcal{E}$, the least integer n, provided that it exists, such that for every k-coloring of the set $[1, n]$ there exists a monochromatic solution to $\mathcal{E}$ is called the k-color Rado number for $\mathcal{E}$. If such an integer n does not exist, then the k-color Rado number for $\mathcal{E}$ is infinite. Rado numbers are also referred to as generalized Schur numbers. In recent years the exact Rado numbers for several families of equations have been found [2–4, 6].

The reader may consult the book [5] by B.M. Landman and A. Robertson for a survey of results on Rado numbers.

H. Harborth and S. Maasberg [3, 4] completely characterized the 2-color Rado numbers for equations of the form $a(x+y) = bz$. Motivated by the results of H. Harborth and S. Maasberg, W. Gasarch, R. Moriarty and N. Tumma [1] characterized the 2-color Rado numbers of equations of the form $a(x-y) = bz$. Let $R(a(x-y) = bz)$ represent the 2-color Rado numbers for the equation $a(x-y) = bz$. W. Gasarch, R. Moriarty and N. Tumma [1] proved the following two theorems.

Theorem 1.1. *For* $1 \leq b < a$, *we have*

$$R(a(x-y) = bz) = a^2. \tag{1.2}$$

Theorem 1.2. *For* $1 < a < b$, *we have*

$$R(a(x-y) = bz) \geq b^2 + b + 1. \tag{1.3}$$

Gasarch, Moriarty and Tumma [1] conjectured further that the inequality in (1.3) is actually equality. In this note, we confirm this conjecture; namely, we establish the following theorem.

Theorem 1.3. *For* $1 < a < b$, *we have*

$$R(a(x-y) = bz) = b^2 + b + 1. \tag{1.4}$$

2 Proof of Theorem 1.3

It may be assumed that a and b are relatively prime since any common factors could be reduced, creating the same equation. It follows from the facts $1 < a < b$ that there exists only one integer r such that $a|(b+r)$ and $1 \leq r \leq a-1$. Also, by elementary number theory, the fact $1 < a < b$ implies that there exist integers k_1 and k_2 such that $1 \leq k_1 < b$, $1 \leq k_2 < a$ and

$$k_1 a - k_2 b = 1. \tag{2.1}$$

By (1.3), in order to prove Theorem 1.3, it suffices to prove that

$$R(a(x-y) = bz) \leq b^2 + b + 1. \tag{2.2}$$

Assume by way of a contradiction that there exists a coloring $\Delta : [1, b^2+b+1] \rightarrow [0, 1]$ that does not admit a monochromatic solution to $a(x-y) = bz$.

Without loss of generality we may assume that $\Delta(a) = 0$. We will now consider two cases on the possible values of $\Delta(b+r)$.

Case 1: $\Delta(b+r) = 0$. We first prove the following claim:
Claim 1: if $1 \leq k \leq b-1$ and $\Delta(ka) = 0$, then $\Delta(ka+a) = 0$. The facts $\Delta(a) = 0$ and $\Delta(b+r) = 0$ imply that $\Delta(r) = \Delta(2b+r) = 1$. It follows from the facts $\Delta(ka) = 0$ and $\Delta(b+r) = 0$ that $\Delta(kb+b+r) = 1$ or else $(kb+b+r, b+r, ka)$ is a monochromatic solution to $a(x-y) = bz$. Note that the inequality $1 \leq k \leq b-1$ implies that $kb+b+r < b^2+b+1$. If $\Delta(ka+a) = 1$, then $(kb+b+r, r, ka+a)$ is a monochromatic solution to $a(x-y) = bz$, so we may assume that $\Delta(ka+a) = 0$. This proves Claim 1.

By Claim 1, we have $\Delta(a) = \Delta(2a) = \cdots = \Delta(ab) = 0$. Particularly, we have $\Delta(k_1 a) = \Delta(k_2 b+1) = \Delta(k_2 a) = \Delta(a^2) = \Delta((b+1-k_2)a) = 0$. By (2.1), we know that $(k_1 a, 1, k_2 a)$ is a solution to $a(x-y) = bz$ which yields that $\Delta(1) = 1$. The facts $\Delta(a) = \Delta(a^2) = 0$ imply that $\Delta(ab+a) = 1$ or else $(ab+a, a, a^2)$ is a monochromatic solution to $a(x-y) = bz$. It follows from the facts $\Delta(1) = 1$ and $\Delta(ab+a) = 1$ that $\Delta(b^2+b+1) = 0$ otherwise $(b^2+b+1, 1, ab+a)$ is a monochromatic solution to $a(x-y) = bz$. Now we have $\Delta(k_2 b+1) = \Delta((b+1-k_2)a) = \Delta(b^2+b+1) = 0$ and $(b^2+b+1, k_2 b+1, (b+1-k_2)a)$ is a monochromatic solution to $a(x-y) = bz$. This is a contradiction.

Case 2: $\Delta(b+r) = 1$. Since $(\frac{b(b+r)}{a} + b + r, b+r, b+r)$ is a solution to $a(x-y) = bz$, we have $\Delta(\frac{b(b+r)}{a} + b + r) = 0$. If $\Delta(\frac{b(b+r)}{a} + r) = 0$, then $(\frac{b(b+r)}{a} + b + r, \frac{b(b+r)}{a} + r, a)$ is a monochromatic solution to $a(x-y) = bz$, so we may assume that $\Delta(\frac{b(b+r)}{a} + r) = 1$. Combining $\Delta(\frac{b(b+r)}{a} + r) = 1$ and $\Delta(b+r) = 1$, we have $\Delta(\frac{2b(b+r)}{a} + r) = 0$ or else $(\frac{2b(b+r)}{a} + r, \frac{b(b+r)}{a} + r, b+r)$ is a monochromatic solution to $a(x-y) = bz$. Note that $\frac{2b(b+r)}{a} + r \leq b^2+b+1$. It follows from the facts $\Delta(a) = 0$ and $\Delta(\frac{2b(b+r)}{a} + r) = 0$ that $\Delta(\frac{2b(b+r)}{a} - b + r) = 1$ otherwise $(\frac{2b(b+r)}{a} + r, \frac{2b(b+r)}{a} - b + r, a)$ is a monochromatic solution to $a(x-y) = bz$. Since $(\frac{2b(b+r)}{a} - b + r, b+r, 2b+2r-2a)$ is a solution to $a(x-y) = bz$, we have $\Delta(2b+2r-2a) = 0$. Now, we are ready to prove the following claim:
Claim 2: if $2 \leq k \leq b$ and $\Delta(ka) = 0$, then $\Delta((k-1)a) = 0$. It follows from the facts $\Delta(a) = 0$ and $\Delta(ka) = 0$ that $\Delta(kb+a) = 1$ or else $(kb+a, a, ka)$ is a monochromatic solution to $a(x-y) = bz$. Note that the inequality $k \leq b$ implies that $ka+b < b^2+b+1$. If $\Delta(b+a) = 0$, then $(b+a, a, a)$ is a monochromatic solution to $a(x-y) = bz$, so we may assume that $\Delta(b+a) = 1$. Since $(kb+a, b+a, (k-1)a)$ solves the equation $a(x-y) = bz$, we see that $\Delta((k-1)a) = 0$. This proves Claim 2.

By Claim 2 and the fact $\Delta(2b+2r-2a)=0$, we obtain

$$\Delta(2b+2r-2a)=\Delta(2b+2r-3a)=\cdots=\Delta(2a)=\Delta(a)=0. \quad (2.3)$$

Note that $a|(b+r)$ and $b+r \leq 2b+2r-2a$. By (2.3), we have $\Delta(b+r)=0$. However, in Case 2, we assume that $\Delta(b+r)=1$. This is a contradiction and this completes the proof of Theorem 1.3.

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References

[1] W. Gasarch, R. Moriarty and N. Tumma, New upper and lower bounds on the Rado numbers, arXiv:1206.4885.

[2] S. Guo and Z.W. Sun, Determination of the two-color Rado number for $a_1x_1+\cdots+a_mx_m=x_0$, J. Combin. Theory Ser. A 115 (2008) 345–353.

[3] H. Harborth and S. Maasberg, Rado numbers for $a(x+y)=bz$, J. Combin. Theory Ser. A 80 (2) (1997) 356–363.

[4] H. Harborth and S. Maasberg, All two-color Rado numbers for $a(x+y)=bz$, Discrete Math. 197/198 (1999) 397–407.

[5] B.M. Landman and A. Robertson, Ramsey Theory on the Integers, American Mathematical Society, Providence, RI, 2004.

[6] K. Myers and A. Robertson, Some two color, four variable Rado numbers, Adv. Appl. Math. 41 (2008) 214–226.

[7] R. Rado, Studien zur Kombinatorik, Math. Z. 36 (1933) 424–480.

[8] I. Schur, Über die kongruenz $x^m+y^m \equiv z^m \pmod{p}$, Jahresber. Deutsch. Math. Verein. 25 (1916) 114–117.