Metamorphosis problems for graph designs

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Abstract

A λ-fold G-design of order n is a pair (X, B), where X is a set of n vertices and B is a collection of edge disjoint copies of the simple graph G, called blocks, which partitions the edge set of λK_n (the undirected complete graph with n vertices) with vertex set X.

Let (X, B) be a G-design and H be a subgraph of G. For each block $b \in B$, partition b into copies of H and $G \setminus H$ and place the copy of H in $B(H)$ and the edges belonging to the copy of $G \setminus H$ in $D(G \setminus H)$. Now if the edges belonging to $D(G \setminus H)$ can be arranged into a collection $D(H)$ of copies of H, then $(X, B(H) \cup D(H))$ is a λ-fold H-design of order n and is called a *metamorphosis* of the λ-fold G-design (X, B) into a λ-fold H-design and denoted by $(G > H) - M_\lambda(n)$.

In this paper, the existence of a $(G > H) - M_\lambda(n)$ for graph designs will be presented, variations of this problem will be explained and recent developments will be surveyed.

AMS classification: 05B05, 05C51.

Keywords: G-design; Metamorphosis; Full metamorphosis; Simultaneous metamorphosis.

1 Introduction

A λ-fold G-design of order n is a pair (X, B), where X is a set of n vertices and B is a collection of edge disjoint copies of the simple graph G, called blocks, which partitions the edge set of λK_n (the undirected complete graph with n vertices) with vertex set X. A λ-fold G-design is also called a λ-fold G-system.

Let (X, B) be a G-design and H be a subgraph of G. For each block $b \in B$, partition b into copies of H and $G \setminus H$ and place the copy of H in $B(H)$ and the edges belonging to the copy of $G \setminus H$ in $D(G \setminus H)$. Now if the edges belonging to $D(G \setminus H)$ can be arranged into a collection $D(H)$ of copies of H, then $(X, B(H) \cup D(H))$ is a λ-fold H-design of order n and is called a *metamorphosis* of the λ-fold G-design (X, B) into a λ-fold H-design.

The first papers, in which the term "metamorphosis" was introduced for graph designs, were published in the beginning of 21st century for the case $(G, H) = (K_4, C_4)$ by Lindner and Street [25] (where C_4 is a 4-cycle) and for the case $(G, H) = (K_4, K_3)$ by Lindner and Rosa [26]. But the metamorphosis problem was first considered from a different perspective by Bryant earlier in 1996 in [12] while solving a problem on nested Steiner triple systems. Then, various researchers; Billington, Dancer, Küçükçifçi, Lindner and Rosa contributed to the problem and gave constructions for $G = K_4$ and all possible subgraphs H of G in a series of papers. Afterwards, variations of the problem such as full metamorphosis, simultaneous metamorphosis and metamorphosis of maximum packings have been studied.

In this paper, these problems will be presented and recent developments will be explained. The problems will be presented in 4 sections, followed by a conclusion.

Before presenting the results on metamorphosis problems we need to introduce few more definitions and notations.

The metamorphosis of a λ-fold G-design into a λ-fold H-design will be denoted by $(G > H)-M_\lambda(n)$, where n is the order of the G-design. The set of all integers n such that there exists a $(G > H) - M_\lambda(n)$ will be denoted by Meta$(G > H, \lambda)$.

For a graph G, $V(G)$ will denote the set of vertices of G and $E(G)$; the set of edges of G.

An m-cycle ($m \geq 3$) on the vertices $v_1, v_2, ..., v_m$ with the edge set $\{\{v_i, v_{i+1}\}| \ 1 \leq i \leq m-1\} \cup \{v_1, v_m\}$ will be denoted by $(v_1, v_2, ..., v_m)$ or $(v_1, v_m, v_{m-1}, ..., v_2)$ or any cyclic shift of these.

A $K_{1,t}$ star is the complete bipartite graph with the vertex set $\{c\} \cup \{v_1, v_2, ..., v_t\}$, the edge set $\{\{c, v_1\}, \{c, v_2\}, ..., \{c, v_t\}\}$ and will be denoted by $(c : v_1, v_2, ..., v_t)$.

A k-path, P_k is a simple graph with k vertices, $v_1, v_2, ..., v_k$ and edges $\{v_i, v_{i+1}\}$, $1 \leq i \leq k-1$ with $v_i \neq v_j$, if $i \neq j$.

A kite T_e is a simple graph on 4 vertices consisting of a triangle with a pendant edge.

An i-wheel is the simple graph on $i+1$ vetices with $2i$ edges which is the union of the star $(c : v_1, v_2, ..., v_i)$ and the i-cycle $(v_1, v_2, ...v_i)$.

An i-windmill is a simple graph on $2i+1$ vertices consisting of i triangles all sharing one common vertex. A 2-windmill is also known as a bowtie.

2 Metamorphosis problems for block designs

In this section, we will summarize the necessary and sufficient conditions on λ and n for the existence of a metamorphosis of a λ-fold K_4-design into a λ-fold H-design, that is; the existence of a $(K_4 > H) - M_\lambda(n)$ for all possible subgraphs H of K_4, namely we will listMeta$(K_4 > H, \lambda)$.

First, consider the following two examples to illustrate the definition of a $(K_4 > H) - M_\lambda(n)$.

Example 2.1. $(K_4 > K_3) - M_\lambda(13)$ (*Metamorphosis of a K_4-design of order* 13 *into a triple system.*)

Let $(\mathbb{Z}_{13}, B)$ be the K_4-design of order 13 with the base block $\{0, 1, 5, 11\}$ (mod 13). Put the star isomorphic to $K_{1,3}$ centered at $1 + i$, for $i \in \mathbb{Z}_{13}$ from each cyclic shift of $\{0, 1, 5, 11\}$ in $D(G \setminus H)$ and rearrange the edges in $D(G \setminus H)$ to obtain $D(H) = \{(i, 1 + i, 4 + i) \mid 0 \le i \le 12\}$ (mod 13). Then $(\mathbb{Z}_{13}, B(H) \cup D(H))$ forms a K_3-design of order 13. □

Example 2.2. $(K_4 > C_4) - M_\lambda(49)$ (*Metamorphosis of a K_4-design of order* 49 *into a* 4-*cycle system.*)

Let $(\mathbb{Z}_{49}, B)$ be the K_4-design of order 49 with the base blocks $\{0, 1, 3, 8\}$, $\{0, 18, 4, 29\}$, $\{0, 21, 6, 33\}$, $\{0, 9, 19, 32\}$ (mod 49).

For each cyclic shift of the base block $\{a, b, c, d\}$, put the cycle (a, d, b, c) in $B(H)$, and the edges $\{a, b\}$, $\{c, d\}$ are left for $D(G \setminus H)$. The edges in $D(G \setminus H)$ can be rearranged to get $D(H)$ obtained by the base 4-cycles; $(0, 5, 4, 22)$ and $(0, 9, 34, 13)$.

Then $(X, B(H) \cup D(H))$ forms a 4-cycle system of order 49. □

In Table 1 we list all the connected subgraphs H of K_4, where the existence of a $(K_4 > H) - M_\lambda(n)$ has been proven. The cases where $H = P_4$ and $H = P_3$ are trivial since the edges of each block $\{a, b, c, d\}$ of a K_4-design can be partitioned into 2 P_4s as $[a, b, c, d]$ and $[b, d, a, c]$, and into 3 P_3s as $[a, b, c]$, $[b, d, a]$ and $[a, c, d]$.

Then in Tables 2, 3 and 4, we present the necessary and sufficient λ and n values for which a $(K_4 > H) - M_\lambda(n)$ exists.

Note that the necessary and sufficient conditions for the existence of a λ-fold K_4-design are

H	$\|E(H)\|$	References
$K_4 - e$	5	[19, 27]
T_e	4	[18]
C_4	4	[25]
K_3	3	[26]
$K_{1,3}$	3	[7]
P_4	3	trivial
P_3	2	trivial

Table 1: Meta($K_4 > H, \lambda$), where H are connected subgraphs of K_4

(1) $\lambda n(n-1) \equiv 0 \pmod{12}$ and

(2) $\lambda(n-1) \equiv 0 \pmod 3$.

Note also that the necessary and sufficient conditions for the existence of a λ-fold H-design are

(1) $\lambda n(n-1) \equiv 0 \pmod{2|E(H)|}$ and

(2) $\lambda(n-1) \equiv 0 \pmod d$, where $d = gcd(d(v_i))$ for $v_i \in V(H)$.

All these conditions must be satisfied for the existence of a $(K_4 > H) - M_\lambda(n)$.

These papers listed in Table 1 complete the problem for $G = K_4$ and all possible connected subgraphs H of G. The main ingredients in the constructions used to complete the problem are skew Room squares. The constructions include also pairwise balanced designs, commutative quasi-groups, self-orthogonal quasigroups and transversal designs.

Packings of metamorphoses of a λ-fold K_4-design into a λ-fold H-design for $H = C_4$ and $H = T_e$ have also been studied by Küçükçifçi, et al. in [20] and by Küçükçifçi in [21], respectively. Tables 5 and 6 present the necessary and sufficient λ and n values together with the possible leaves for which a metamorphosis of a λ-fold K_4-design into a maximum packing

λ (mod 30)	Meta($K_4 > K_4 \setminus e, \lambda$)
$1,7,11,13,17,19,23,29$	$1,16,25,40$ (mod 60)
$2,4,8,14,16,22,26,28$	$1,10$ (mod 15)
$3,9,21,27$	$0,1,5,16$ (mod 20)
$5,25$	$1,4$ (mod 12)
$6,12,18,24$	$0,1$ (mod 5)
$10,20$	1 (mod 3)
15	$0,1$ (mod 4)
0	$n \geq 4$

Table 2: Meta($K_4 > H, \lambda$) when $|E(H)| = 5$

of λK_n with graphs isomorphic to H exists, when $|E(H)| = 4$ that is; $Meta(K_4 > H, \lambda, L)$ for $H = C_4$ and $H = T_e$.

3 Full metamorphosis problems for block designs

In this section we will present very recent work on a variation of the metamorphoses problems for K_4-designs. Note that when we consider a $(K_4 > K_3) - M_\lambda(n)$, $K_4 \setminus K_3$ can be 4 different $K_{1,3}$. This lead us to the following definition:

Let (X, B) be a λ-fold K_4-design and label the elements of each block b with b_1, b_2, b_3, and b_4 (in any manner). For each $i = 1,2,3,4$ define a set of triangles H_i and a set of stars $G \setminus H_i$ as follows: for each block $b = [b_1, b_2, b_3, b_4]$ belonging to B, partition the edges of b into a triangle and a star centered at b_i and place the triangle in $B(H_i)$ and the star in $D(G \setminus H_i)$. Now if the stars in $D(G \setminus H_i)$ can be arranged into a collection of

$\lambda \pmod{12}$	Meta$(K_4 > C_4, \lambda)$	Meta$(K_4 > T_e, \lambda)$
$1, 5, 7$ or 11	$1 \pmod{24}$	1 or $16 \pmod{24}$
2 or 10	1 or $4 \pmod{12}$	1 or $4 \pmod{12}$
3 or 9	$1 \pmod 8$	0 or $1 \pmod 8$
4 or 8	$1 \pmod 3$	$1 \pmod 3$
6	0 or $1 \pmod 4$	0 or $1 \pmod 4$
0	$n \geq 4$	$n \geq 4$

Table 3: Meta$(K_4 > H, \lambda)$ when $|E(H)| = 4$

$\lambda \pmod 6$	Meta$(K_4 > K_3, \lambda)$	Meta$(K_4 > K_{1,3}, \lambda)$
$1, 5$	$1 \pmod{12}$	$1, 4 \pmod{12}$
$2, 4$	$1 \pmod 3$	$1 \pmod 3$
3	$1 \pmod 4$	$0, 1 \pmod 4$
0	$n \geq 4$	$n \geq 4$

Table 4: Meta$(K_4 > H, \lambda)$ when $|E(H)| = 3$

triples $D(H_i)$, then $(X, B(H_i) \cup D(H_i))$ is a $(K_4 > K_3) - M_\lambda(n)$, as defined in [26]. We will refer to $M_i = (X, B(H_i) \cup D(H_i))$ as the ith metamorphosis of (X, B). Observe that the center of the star corresponding to each block b is different in each metamorphosis M_i.

The *full metamorphosis* of a K_4-design (X, B) into a λ-fold triple system is a set of four metamorphoses $\{M_1, M_2, M_3, M_4\}$ and will be denoted by full$-(K_4 > K_3) - M_\lambda(n)$.

Example 3.1. *Full*$-(K_4 > K_3) - M_\lambda(25)$*:*

Let (X, B) be the K_4-design with vertex set $X = \mathbb{Z}_5 \times \mathbb{Z}_5$, and block set $B = \{[(0+i, 1+j), (0+i, 0+j), (2+i, 2+j), (1+i, 0+j)], [(0+i, 2+j),$

$\lambda \pmod{12}$	spectrum of K_4-designs		leave
1, 5, 7, 11	1	(mod 24)	∅
	4, 16		1-factor
	13		bowtie, 2 disjoint K_3, C_6
2, 10	1, 4	(mod 12)	∅
	7, 10		D_2,
3, 9	1	(mod 8)	∅
	0, 4		1-factor
	5		D_2
4, 8	1	(mod 3)	∅
0, 6	0, 1	(mod 4)	∅
	2, 3		D_2

Table 5: Meta($K_4 > C_4, \lambda, L$)

$(0+i, 0+j)$, $(4+i, 4+j)$, $(2+i, 0+j)] \mid 0 \le i \le 4, 0 \le j \le 4\}$, where addition is in modulo 5.

Then $D(H_i)$ is a partial STS, where the starter blocks for $D(H_1)$ are $\{(0,0),(0,1),(1,4)\}$, $\{(0,0),(0,2),(2,3)\}$; for $D(H_2)$ are $\{(0,0),(0,1),(1,1)\}$, $\{(0,0),(0,2),(2,2)\}$; for $D(H_3)$ are $\{(0,0),(2,1),(4,3)\}$, $\{(0,0),(1,1),(4,2)\}$; and for $D(H_4)$ are $\{(0,0),(1,4),(4,0)\}$, $\{(0,0),(1,2),(3,0)\}$. □

The problem of determining the existence of a full$-(K_4 > K_3) - M_\lambda(n)$ for all possible values of n and λ has been completely solved by Küçükçifçi et al. in [23].

Similarly, a full$-(K_4 > C_4) - M_\lambda(n)$ and a full$-(K_4 > T_e) - M_\lambda(n)$ have been defined and their existence for all possible values of n and λ have

λ (mod 12)	spectrum of K_4-designs	leave
1, 5, 7, 11	1, 16 (mod 24)	$\varnothing$
	4, 13 (mod 24)	P_3, E_2 (P_3 for $n = 4$)
2, 10	1, 4 (mod 12)	$\varnothing$
	7, 10 (mod 12)	$P_3, E_2, D_2,$
3, 9	0, 1 (mod 8)	$\varnothing$
	4, 5 (mod 8)	P_3, E_2, D_2
4, 8	1 (mod 3)	$\varnothing$
6	0, 1 (mod 4)	$\varnothing$
	2, 3 (mod 4)	P_3, E_2, D_2
0	all $n \geq 4$	$\varnothing$

Table 6: Meta($K_4 > T_e, \lambda, L$)

been completely solved by Küçükçifçi et al. in [22] and [24].

In the case $H = C_4$, each block of $(K_4 > C_4) - M_\lambda(n)$ can be partitioned into 3 different 2 disjoint edges. Therefore the full metamorphosis definition for this case will be as follows:

Label the elements of each block b with b_1, b_2, b_3, and b_4 (in any manner). For each block $b = [b_1, b_2, b_3, b_4]$ belonging to B, let $\{F_1, F_2, F_3\}$ be the 1-factorization of b defined by $F_1 = \{\{b_1, b_2\}, \{b_3, b_4\}\}$, $F_2 = \{\{b_1, b_3\}, \{b_2, b_4\}\}$, and $F_3 = \{\{b_1, b_4\}, \{b_2, b_3\}\}$. For each $i = 1, 2, 3$ define a set of 4-cycles H_i and a set of deleted edges $G \setminus H_i$ as follows: for each 1-factor F_i of b, place the 4-cycle $b \setminus F_i$ in $B(H_i)$ and F_i in $D(G \setminus H_i)$. Then

(X, H_i) is a partial λ-fold 4-cycle system. Now if the edges belonging to $D(G \setminus H_i)$ can be rearranged into a collection of 4-cycles $D(H_i)$, then $M_i = (X, B(H_i) \cup D(H_i))$ is a λ-fold 4-cycle system called the ith metamorphosis of (X, B). The full metamorphosis of (X, B) into a λ-fold 4-cycle system is the set of three metamorphoses $\{M_1, M_2, M_3\}$.

In the case $H = T_e$, each block of $(K_4 > T_e) - M_\lambda(n)$ can be partitioned into 3 disjoint P_3s. Therefore the full metamorphosis definition for this case will be as follows:

For each block b belonging to B let $\{P_1(b), P_2(b), P_3(b)\}$ be the partition of b into three paths of length two. For each $i = 1, 2, 3$ define a set of kites $B(H_i)$ and a set of deleted edges $D(G \setminus H_i)$ as follows: for each $P_i(b)$ of b, place the kite $b \setminus P_i(b)$ in $B(H_i)$ and the two edges belonging to the path $P_i(b)$ in $D(G \setminus H_i)$. Then $(X, B(H_i))$ is a partial λ-fold kite system. Now if the edges belonging to $D(G \setminus H_i)$ can be arranged into a collection of kites $D(H_i)$, then $M_i = (X, B(H_i) \cup D(H_i))$ is a λ-fold kite system called the ith metamorphosis of (X, B). The full metamorphosis of (X, B) into a λ-fold kite system is the set of three metamorphoses $\{M_1, M_2, M_3\}$.

The necessary and sufficient conditions for the existence of a full$-(K_4 > H) - M_\lambda(n)$ are the same as the necessary and sufficient conditions for the existence of a $(K_4 > H) - M_\lambda(n)$ given in Section 2.

The references for the existence of a full$-(K_4 > H) - M_\lambda(n)$ have been given in Table 7.

H	References
T_e	[24]
C_4	[22]
K_3	[23]

Table 7: Full-Meta$(K_4 > H, \lambda)$, where H are connected subgraphs of K_4

The constructions in [22, 23, 24] use group divisible designs and pairwise balanced designs.

4 Metamorphosis problems for G-designs

In the previous two sections, the results when the graph G is K_4 have been summarized. Now we will present the results obtained when G is a possible non-trivial subgraph of K_4, 4-wheel, 6-wheel, $K_4 + e$ and $K_{3,3}$. Table 8 summarizes the results obtained when G is a possible non-trivial subgraph of K_4. In the table "×"denotes H is not a subgraph of G and "*" denotes the construction of a $(G > H) - M_\lambda(n)$ is trivial.

(G,H)	$K_4 - e$	C_4	T_e	K_3	P_4	$K_{1,3}$	P_3	$2P_2$	P_2
$K_4 - e$	–	[28, 30, 36]	[13]	[38]	[39]	[39]	[39]	[39]	*
C_4	×	–	×	×	[38]	×	*	*	*
T_e	×	×	–	[29, 36]	[33]	[38]	*	[38]	*
K_3	×	×	×	–	×	×	[38]	×	*
P_4	×	×	×	×	–	×	[38]	[38]	*
$K_{1,3}$	×	×	×	×	×	–	[38]	×	*
P_3	×	×	×	×	×	×	–	×	*
$2P_2$	×	×	×	×	×	×	×	–	*

Table 8: Meta$(G > H, \lambda)$ when G is a proper subgraph of K_4

Note that a 3-wheel is the graph K_4. Therefore, in Section 2, the existence of a $(3\text{-}wheel > H) - M_\lambda(n)$ for all possible subgraphs H have been summarized. Table 9 gives the references for the existence of an $(i - wheel > H) - M_\lambda(n)$, when $i = 4$ and $i = 6$.

Finally, there are only two more cases considered; the case $(G, H) = (K_4 + e, K_4)$ is solved by Chang et al. in [14] and the case $(G, H) =$

(G,H)	$K_{1,i}$	C_i	i-windmill
4-wheel	[1] for $\lambda = 1$	[5]	[4]
6-wheel	[1] for $\lambda = 1$	[1] for $\lambda = 1$	[1] for $\lambda = 1$

Table 9: Meta(i-wheel $> H, \lambda$)

$(K_{3,3}, C_6)$ is solved by Billington and Quattrocchi in [6]. We believe up to date all the remaining cases are open.

5 Simultaneous metamorphosis problems for G-designs

In 2003 [1], Adams, Billington and Mahmoodian introduced a new perspective to the metamorphosis problems for graph designs. Let H_i, $1 \leq i \leq r$ be subgraphs of the graph G. If there are metamorphoses Meta($G > H_i, \lambda$)s with the same set of G-blocks, then we call them simultaneous metamorphoses of λ-fold G-designs into H_i-designs. In their paper the simultaneous metamorphosis of i-wheel systems into H_i-designs when $G = 3-wheel$ (i.e.; $G = K_3$), $H_1 = K_{1,3}$, $H_2 = K_3$, and $G = i - wheel$, $H_1 = K_{1,i}$, $H_2 = C_i$, $H_3 = i$-windmill, when $i = 4$ and 6 have been constructed. Very recently, G. Ragusa gave a complete answer to the existence of a λ-fold G-design having a complete simultaneous metamorphosis into H_i-designs when $G = T_e$, $H_1 = K_{1,3}$, $H_2 = K_3$, $H_3 = P_4$ in [34] and when $G = K_4$, $H_1 = C_4$, $H_2 = T_e$ in [35]. These are the three papers published up to date on the simultaneous metamorphosis problems for G-designs.

6 Conclusion and open problems

Although the existence problem for a $(G > H) - M_\lambda(n)$ when $G = K_4$ and G is one of all possible non-trivial subgraphs of K_4 is completely solved and there are two singular results for the two cases $((G, H) = (K_4 + e, K_4)$ and $(G, H) = (K_{3,3}, C_6))$ as we summarized in Section 4, for all other G and H, the problem remains open.

On the other hand, there are also isolated results on variations of the metamorphosis problems. In [32] Ling et al. worked on the generalization of the metamorphosis definition. They considered changing a G-design into an H-design by adding as few vertices as possible. In [8] Billington studied the extended metamorphosis of a complete bipartite design into a cycle system.

In [17] Gionfriddo and Lindner considered the problem of finding a metamorphosis from a 2-fold triple system to a 2-fold 4-cycle system for all admissible orders by pairing the triples of the two triple systems and removing the repeated edge from each pair and rearranging removed edges into a 2-fold 4-cycle system. The generalization of this problem is to find a metamorphosis from a 2-fold k-cycle system into a 2-fold $2k - 2$-cycle system. The case $k = 4$ is solved by Yazici in [37]. The case in which the starting two systems are a triple system and a 4-cycle system is solved by Billington in [9]. Recently Chang et al. [16] solved the problem of the metamorphosis of 2-fold triple systems into maximum 2-fold $(K_4 - e)$-packings. For all other k, we believe the problem is open.

Then Lindner et al. considered of finding a complete set of k metamorphoses from a fixed 2-fold k-cycle system paired in k different pairs into 2-fold $(2k - 2)$-cycle systems. They solved the case $k = 3$ in [31]. Then Billington et al. solved the case $k = 4$ in [11]. For all other k, we believe the problem is open. Recently, Billington et al. [10] also considered of pairing two stars to get a complete sets of metamorphoses into 4-cycles.

Furthermore interested reader is referred to [2, 3, 15] for problems that are inspired from metamorphosis problems.

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