The 2-color Rado Number of $x_1 + x_2 + \cdots + x_{m-1} = ax_m$

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Abstract

In 1982, Beutelspacher and Brestovansky proved that for every integer $m \geq 3$, the 2-color Rado number of the equation

$$x_1 + x_2 + \cdots + x_{m-1} = x_m$$

is $m^2 - m - 1$. In 2008, Schaal and Vestal proved that, for every $m \geq 6$, the 2-color Rado number of

$$x_1 + x_2 + \cdots + x_{m-1} = 2x_m$$

is $\lceil \frac{m-1}{2} \lceil \frac{m-1}{2} \rceil \rceil$. Here we prove that, for every integer $a \geq 3$ and every $m \geq 2a^2 - a + 2$, the 2-color Rado number of

$$x_1 + x_2 + \cdots + x_{m-1} = ax_m$$

is $\lceil \frac{m-1}{a} \lceil \frac{m-1}{a} \rceil \rceil$. For the case $a = 3$, we show that our formula gives the Rado number for all $m \geq 7$, and we determine the Rado number for all $m \geq 3$.

1. Introduction

A special case of the work of Richard Rado [5] is that for every integer $m \geq 3$ and all positive integers $a_1, \ldots, a_m$ there exists a smallest positive integer n with the following property: for every coloring of the elements of the set $[n] = \{1, \ldots, n\}$ with two colors, there exists a solution of the equation

$$a_1x_1 + a_2x_2 + \cdots + a_{m-1}x_{m-1} = a_mx_m$$

using elements of $[n]$ that are all colored the same. (Such a solution is called *monochromatic.*) The integer n is called the *2-color Rado number* of the equation.

In 1982, Beutelspacher and Brestovansky [1] proved that for every $m \geq 3$, the 2-color Rado number of

$$x_1 + x_2 + \cdots + x_{m-1} = x_m$$

is $m^2 - m - 1$. Since then, Rado numbers for a number of variations of this equation have also been determined. For example, in 2008 Guo and Sun [2] solved the problem for the equation

$$a_1x_1 + a_2x_2 + \cdots + a_{m-1}x_{m-1} = x_m,$$

for all positive integers $a_1, \ldots, a_{m-1}$. They proved (confirming a conjecture of Hopkins and Schaal [4]) that the 2-color Rado number is $aw^2 + w - a$, where $a = \min\{a_1, \ldots, a_{m-1}\}$ and $w = a_1 + \cdots + a_{m-1}$. In the same year, Schaal and Vestal [6] dealt with the equation

$$x_1 + x_2 + \cdots + x_{m-1} = 2x_m.$$

They proved, in particular, that for every $m \geq 6$, the 2-color Rado number is $\lceil \frac{m-1}{2} \lceil \frac{m-1}{2} \rceil \rceil$. Our main purpose in the present paper is to obtain an analogue of this result for all larger values of the coefficient on x_m. We prove the following result.

Theorem 1. For every integer $a \geq 3$ and every $m \geq 2a^2 - a + 2$, the 2-color Rado number of the equation

$$x_1 + x_2 + \cdots + x_{m-1} = ax_m$$

is $\lceil \frac{m-1}{a} \lceil \frac{m-1}{a} \rceil \rceil$.

Notation. We will denote $\lceil \frac{m-1}{a} \lceil \frac{m-1}{a} \rceil \rceil$ by $C(m, a)$, and we will denote the equation indicated in the statement of the theorem by $L(m, a)$.

To prove Theorem 1, we show first, in Section 2, that for all $a \geq 3$ and $m \geq 3$, $C(m, a)$ is a lower bound for the Rado number, i.e., either $C(m, a) = 1$ or there exists a 2-coloring of $[C(m, a) - 1]$ that admits no monochromatic solution of $L(m, a)$. Then, in Sections 3, 4 and 5, we show that for all $a \geq 3$ and $m \geq 2a^2 - a + 2$, $C(m, a)$ is an upper bound for the Rado number, i.e., every 2-coloring of $[C(m, a)]$ admits a monochromatic solution of $L(m, a)$.

In Section 6, we prove the following.

Theorem 2. The 2-color Rado number of $L(m, 3)$ is $C(m, 3)$ when $m \geq 7$. For $m = 6, 5, 4, 3$ the Rado number is, respectively, $5, 4, 1, 9$.

We confine ourselves to $m \geq 3$ because the Rado number of $L(2, a)$ fails to exist for $a \geq 3$, as we can see by using the argument used in [6] for $a = 2$.

Notation. In working with a fixed 2-coloring of a set, we will use the colors red and blue, and we will denote by R and B, respectively, the sets of elements colored red and blue.

2. Lower bounds, and some results for dealing with upper bounds

Proposition 1. For every $a \geq 3$ and $m \geq 3$, the 2-color Rado number of $L(m, a)$ is at least $C(m, a)$.

Proof. Since $m \geq 3$, the Rado number exists. If $m \leq a+1$ then $C(m, a) = 1$ and our claim is clear.

Now suppose that $m \geq a + 2$, so that $C(m, a) \geq 2$. We must show that there exists a 2-coloring of $[C(m, a) - 1]$ that yields no monochromatic solution of $L(m, a)$. We use the same coloring that Schaal and Vestal used in [5] to establish their lower bounds, but in our less specific situation it is easier to work directly from the meaning of $C(m, a)$ than from algebraic expressions for the ceiling function, as Schaal and Vestal did. We will use the fact that, for every real number r,

$$r > \lceil r \rceil - 1.$$

We 2-color $[C(m, a) - 1]$ by coloring all the elements of $[\lceil \frac{m-1}{a} \rceil - 1]$ red and all remaining elements blue. For any red elements $x_1, x_2, \ldots, x_m$ we have

$$\frac{x_1 + \cdots + x_{m-1}}{a} \geq \frac{m-1}{a} > \left\lceil \frac{m-1}{a} \right\rceil - 1 \geq x_m,$$

so there are no red solutions of $L(m, a)$. For any blue elements $x_1, x_2, \ldots, x_m$ we have

$$\frac{x_1 + \cdots + x_{m-1}}{a} \geq \frac{m-1}{a} \left\lceil \frac{m-1}{a} \right\rceil > C(m, a) - 1 \geq x_m,$$

so there are no blue solutions either. □

In considering upper bounds, we will often need to exhibit solutions of $L(m, a)$ in $[C(m, a)]$. To do this we will need to know that certain numbers are less than or equal to $C(m, a)$.

Lemma 1. Suppose $a \geq 3$ and $m \geq 2a^2 - a + 2$. Then the numbers $2m - 2$ and $a + 1$ are both less than or equal to $C(m, a)$.

Proof. By our assumption about the size of m, it is clear that $2m-2 \geq a+1$, so it will suffice to prove the inequality

$$2m - 2 \leq C(m, a).$$

We first consider the case $2a^2 - a + 2 \leq m \leq 2a^2 + 1$. In this case we can write $m = 2a^2 - a + b$, where $2 \leq b \leq a + 1$. We have $\frac{m-1}{a} = 2a - 1 + \frac{b-1}{a}$, and therefore $\lceil \frac{m-1}{a} \rceil = 2a$ and $C(m, a) = 4a^2 - 2a + 2b - 2 = 2m - 2$.

To prove the inequality when $m \geq 2a^2 + 2$, it will suffice to show that

$$\frac{(m-1)^2}{a^2} \geq 2m - 2.$$

This is equivalent to

$$m \geq \left(2 - \frac{2}{m}\right) a^2 + 2 - \frac{1}{m},$$

and is therefore clear, since $m \geq 2a^2 + 2$. □

In their treatment of upper bounds in [**6**], Schaal and Vestal proceeded by fixing the coloring of the element 1 and considering the two possibilities for the coloring of the element 2. In dealing with $a \geq 3$, we find it convenient to fix the coloring of the element $a - 2$ and consider the two options for the coloring of $a - 1$.

Convention. In dealing with 2-colorings of $[C(m, a)]$ in the following sections, we will assume without loss of generality that $a - 2 \in R$.

3. Monochromatic solutions when $a - 1 \in B$

Throughout this section we assume that $a \geq 3$ and $m \geq 2a^2 - a + 2$.

In this and the following sections, it will be convenient to have a compact notation for indicating solutions of $L(m, a)$.

Notation. If $n_1, \ldots, n_k$ are nonnegative integers whose sum is m, and $d_1, \ldots, d_k$ are elements of $[C(m, a)]$ such that we obtain a true equation from $L(m, a)$ by substituting d_1 for the variables $x_1, \ldots, x_{n_1}$, d_2 for the next n_2 variables, and so on, then we denote this true equation by

$$[n_1 \to d_1;\ n_2 \to d_2;\ \cdots;\ n_k \to d_k].$$

For example, the true instance

$$a + a + \cdots + a = a(m-1) \tag{1}$$

of $L(m,a)$ will be denoted by

$$[m-1 \to a;\ 1 \to m-1].$$

Suppose now that we have a 2-coloring of $[C(m,a)]$ that yields no monochromatic solution of $L(m,a)$. We seek a contradiction. We will proceed by noting a number of solutions of $L(m,a)$; all the numbers used in these solutions will be in $[C(m,a)]$ by Lemma 1.

Recall that we are assuming that $a-2 \in R$, and, in this section, that $a-1 \in B$.

Lemma 2. We have $m-2 \in R$ and $m-3 \in B$.

Proof. Since $a-1 \in B$ and there are no monochromatic solutions of $L(m,a)$ in $[C(m,a)]$, the solution

$$[m-2 \to a-1;\ 2 \to m-2] \tag{2}$$

tells us that $m-2 \in R$. Likewise, since $a-2 \in R$, the solution

$$[m-3 \to a-2;\ 3 \to m-3] \tag{3}$$

tells us that $m-3 \in B$. □

Lemma 3. We have $a \in R$ and $m-1 \in B$.

Proof. Since $a-1$ and $m-3$ are in B, the solution

$$[2a \to a-1;\ m-1-2a \to a;\ 1 \to m-3]$$

tells us that $a \in R$, and then solution (1) above tells us that $m-1 \in B$. □

Now if m is even we obtain the desired contradiction by observing that the solution

$$\left[\frac{m-2}{2} \to a-2;\ \frac{m-2}{2} \to a;\ 2 \to m-2\right] \tag{4}$$

is red. If m is odd we obtain a contradiction by considering the color of the element $a+1$. If $a+1 \in B$ then the blue solution

$$\left[\frac{m-1}{2} \to a-1;\ \frac{m-1}{2} \to a+1;\ 1 \to m-1\right] \tag{5}$$

yields a contradiction. If $a+1 \in R$ then the red solution

$$\left[\frac{m-1}{2} \to a-2;\ \frac{m-5}{2} \to a;\ 1 \to a+1;\ 2 \to m-2\right] \tag{6}$$

yields a contradiction.

We have proved the following proposition.

Proposition 2. If $a \geq 3, m \geq 2a^2 - a + 2$, and $a - 2 \in R$, then every 2-coloring of $[C(m, a)]$ with $a - 1 \in B$ yields a monochromatic solution of $L(m, a)$.

4. Consequences of assuming $a - 1 \in R$ and no monochromatic solutions

In this section we assume that $a \geq 3, m \geq 2a^2 - a + 2$, $a - 1 \in R$, and our 2-coloring of $[C(m, a)]$ yields no monochromatic solutions of $L(m, a)$. We derive some consequences that will be used in the next section.

Lemma 4. Each of $m - 1, m - 2, m - 3$ is in B, and $a \in R$.

Proof. Since $a - 1$ and $a - 2$ are in R, it follows from solutions (2) and (3) above that $m - 2$ and $m - 3$ are in B. To conclude the proof, it will suffice, by solution (1) above, to show that $a \in R$. But if $a \in B$, then by solution (1) we have $m - 1 \in R$, and then the solution

$$[m - 2a + 1 \to a - 2;\ 2a - 4 \to a - 1;\ 3 \to m - 1] \tag{7}$$

is red, a contradiction. □

Lemma 5. We have $1 \in R$.

Proof. The solution

$$[m - a \to 1;\ a - 2 \to m - 1;\ 2 \to m - 2] \tag{8}$$

would be blue if 1 were in B. □

Lemma 6. We have $2 \in R$.

Proof. We recall that, by Lemma 1, $2m - 2$ and all smaller numbers are available in $[C(m, a)]$ for use in producing solutions of $L(m, a)$.

From the solution

$$[m - a \to 1;\ a \to m - a]$$

we conclude that $m - a \in B$, and from the solution

$$[m - 2a + 5 \to a;\ a - 3 \to a - 1;\ a - 3 \to 1;\ 1 \to m - a + 2]$$

we conclude that $m-a+2 \in B$. If $2a \in B$, then from the solution

$$[a-2 \to m-a;\ m-a \to 2;\ 1 \to 2a;\ 1 \to m-a+2]$$

we infer that $2 \in R$.

For the remainder of the proof we assume that $2a \in R$ and $2 \in B$ and seek a contradiction. First, by doubling all the entries in solution (1) we conclude that $2(m-1) \in B$, and then by doubling all the entries in solution (8) we conclude that $2(m-2) \in R$. Using the solution

$$[m-2 \to 2(a-1);\ 2 \to 2(m-2)],$$

we see that $2(a-1) \in B$, and then by doubling all the entries in solution (7) we see that $2(a-2) \in R$.

If m is even, we get a contradiction by doubling all the entries in solution (4) to get a red solution. If m is odd, we double all the entries in solution (5) and conclude that $2(a+1) \in R$, and then we get a contradiction by doubling all the entries in solution (6) to get a red solution. □

Lemma 7. The numbers $m-3, m-2, m-1, \ldots, 2m-2$ are all in B.

Proof. By Lemma 4, we only need to prove this for $m, m+1, \ldots, 2m-2$.

Since $2 \in R$, if $2a \in B$ we can repeat all the steps in the last two paragraphs of the proof of Lemma 6, with all the colors reversed, to obtain a contradiction. Therefore $2a \in R$.

Now consider the number $m+k$, where $0 \le k \le m-2$. The solution

$$[m-k-2 \to a;\ k+1 \to 2a;\ 1 \to m+k]$$

shows that $m+k \in B$. □

Lemma 8. The numbers $1, \ldots, 2a-2$ are all in R.

Proof. We want to show that $2a-2j \in R$ for all integers j such that $2 \le 2j \le 2a-2$, and that $2a-(2j+1) \in R$ for all j such that $3 \le 2j+1 \le 2a-1$.

For $2a-2j$ we consider the solution

$$[m-(j+1) \to 2a-2j;\ j+1 \to 2m-2(j+1)]$$

and need to know that $2m-2(j+1) \in B$. This will be true by Lemma 7 if $2m-2(j+1) \ge m-3$, i.e., if $m \ge 2j-1$. But this inequality holds, since $2j \le 2a-2$.

For $2a-(2j+1)$ we consider the solution

$$[m-(j+2) \to 2a-(2j+1);\ j \to 2m-(2j+5);\ 1 \to m-2;\ 1 \to 2m-2(j+2)]$$

and need to know that $2m-(2j+5)$ and $2m-2(j+2)$ are in B. This will be true if $2m-(2j+5) \geq m-3$, i.e., if $m \geq 2j+2$. This inequality holds because $(2j+1) \leq 2a-1$. □

There is one more result that we will need in Section 5.

Lemma 9. If d is an integer such that $a|d$ and $m-1 \leq d \leq 2m-2$, then $\frac{d}{a} \in B$.

Proof. Write $d = m-1+k$, with $0 \leq k \leq m-1$. Then the solution

$$\left[m-1-k \to 1;\ k \to 2;\ 1 \to \frac{d}{a}\right]$$

shows that $\frac{d}{a} \in B$. □

5. Monochromatic solutions when $a-1 \in R$

In this section we suppose that $a \geq 3, m \geq 2a^2-a+2, a-1 \in R$ and there are no monochromatic solutions of $L(m,a)$ in $[C(m,a)]$. We again seek a contradiction.

We will use the results of Section 4, and we will also need algebraic expressions for $C(m,a)$.

Lemma 10. Let $m = ua^2+va+c$, with u as large as possible and $0 \leq v, c \leq a-1$.

(i) If $c=1$ then $C(m,a) = \frac{(m-1)^2}{a^2}$.

(ii) If $c=0$ then $C(m,a) = \frac{m^2-m+va}{a^2}$.

(iii) If $2 \leq c \leq a-1$ then $C(m,a) = \frac{m^2+(a-c-1)m+c-ac-vac+va+ta^2}{a^2}$, where $t = \left\lceil \frac{(c-1)(v+1)}{a} \right\rceil$.

Proof. If $c=1$ then $a|(m-1)$, so the claim is clear from the definition of $C(m,a)$. If $c=0$ then

$$C(m,a) = \left\lceil \frac{m-1)}{a} \cdot \frac{m}{a} \right\rceil = \left\lceil \frac{m^2-m}{a^2} \right\rceil = \frac{m^2-m+va}{a^2},$$

since $a^2|m^2$ and va is the smallest number we can add to m^2-m to produce a multiple of a^2.

If $2 \leq c \leq a-1$, then

$$C(m,a) = \left\lceil \left(ua + v + \frac{c-1}{a}\right)(ua+v+1) \right\rceil$$

so

$$C(m,a) = (ua+v)^2 + (ua+v) + (c-1)u + t.$$

Replacing $ua+v$ by $(\frac{m-c}{a})$ and u by $\frac{m-va-c}{a^2}$, and simplifying, we obtain the final claim of the lemma. □

The three descriptions of $C(m,a)$ in Lemma 10 lead us to consider three cases.

Case 1: $m \equiv 1$ (mod a)

In this case we have $\frac{m-1}{a} \in B$, by Lemma 9. Since $1 \in R$, we can use an idea from [6] and let s be an integer such that $s \in R$, $s+1 \in B$, and $s+1 \leq \frac{m-1}{a}$. Then $\frac{m-1}{a}(s+1) \leq C(m,a)$ by Lemma 10, and the solution

$$\left[m-1 \to s+1;\ 1 \to \frac{m-1}{a}(s+1)\right]$$

shows that $\frac{m-1}{a}(s+1) \in R$. We now obtain a contradiction by noting that the solution

$$\left[\frac{m-1}{a} - a \to a-1;\ a-1 \to a;\ (a-1)\frac{m-1}{a} \to s;\ 2 \to \frac{m-1}{a}(s+1)\right]$$

is red.

Case 2: $m \equiv 0$ (mod a)

In this case we have $\frac{m}{a} \in B$ by Lemma 9. We choose an s such that $s \in R$, $s+1 \in B$, and $s+1 \leq \frac{m}{a}$. Noting that

$$C(m,a) - \left(\frac{m-a}{a}\right)\frac{m}{a} = \frac{m^2 - m + va}{a^2} - \frac{m^2 - am}{a^2} = \frac{(a-1)m + va}{a^2},$$

we consider the element

$$\alpha = \frac{m-a}{a}(s+1) + \frac{(a-1)m+va}{a^2} \leq \left(\frac{m-a}{a}\right)\frac{m}{a} + \frac{(a-1)m+va}{a^2},$$

so $\alpha \leq C(m,a)$. Noting that $\frac{m}{a}+1 \in B$ by Lemma 9, we see that $\alpha \in R$ by considering the solution

$$\left[m-a \to s+1;\ v \to \frac{m}{a}+1;\ a-1-v \to \frac{m}{a};\ 1 \to \alpha\right].$$

We now obtain a red solution of $L(m, a)$ (and therefore a contradiction) by assigning the value α to x_{m-1} and x_m and the value s to $(a-1)(\frac{m-a}{a})$ other variables, and showing that we can assign values in R to the remaining $\frac{m}{a} + a - 3$ variables to complete the solution. In fact we will show that we can accomplish this by using only values in the set $[2a-2]$. These values are all in R by Lemma 8.

The values assigned to the remaining variables must add up to

$$\frac{a-1}{a}(m-a) + \frac{a-1}{a^2}((a-1)m + va).$$

If we can show that using only the value $2a-2$ yields a sum that is at least this large, and using only the value 1 yields a sum that is at most this large, then there is a unique solution that uses values in one of the sets $\{j,\ j+1\}$, where $j \in [2a-3]$.

Since $v \leq a-1$, we can achieve our first objective by showing that

$$(2a-2)\left(\frac{m}{a} + a - 3\right) \geq \frac{a-1}{a}(m-a) + \frac{a-1}{a^2}((a-1)m + (a-1)a),$$

which simplifies to

$$2a^2 - 8a + 7 - \frac{1}{a} \geq m\left(\frac{1-a}{a^2}\right),$$

and this is easily seen to be true for $a \geq 3$, since the right-hand side is negative.

Since $v \geq 0$, we can achieve our second objective by showing that

$$\left(\frac{m}{a} + a - 3\right) \leq \frac{a-1}{a}(m-a) + \frac{a-1}{a^2}((a-1)m).$$

But this simplifies to $2a^3 - 4a^2 \leq m(2a^2 - 4a + 1)$, which is true for all $a \geq 3$ and $m \geq a$.

Case 3: $m \equiv c \pmod{a}$, $2 \leq c \leq a-1$

In this case we have $\frac{m+a-c}{a} \in B$ by Lemma 9. Choosing s such that $s \in R$, $s+1 \in B$, and $s+1 \leq \frac{m+a-c}{a}$, we consider the element

$$\beta = \left(\frac{m-c}{a}\right)(s+1) + \frac{(c-1)m + c - c^2 - vac + va + ta^2}{a^2},$$

where t is as in Lemma 10 and the second term in the sum is

$$C(m, a) - \left(\frac{m-c}{a}\right)\left(\frac{m+a-c}{a}\right),$$

according to the expression for $C(m,a)$ in Lemma 10. Then

$$\beta \le \left(\frac{m-c}{a}\right)\left(\frac{m+a-c}{a}\right) + \frac{(c-1)m+c-c^2-vac+va+ta^2}{a^2},$$

so $\beta \le C(m,a)$. In order to work with β, it will be helpful to have bounds on the quantity $-vac+va+ta^2$.

Lemma 11. We have $ac-a \le -vac+va+ta^2 \le ac-a+a^2$.

Proof. By the definition of t,

$$\frac{(c-1)(v+1)}{a} \le t \le \frac{(c-1)(v+1)}{a} + 1.$$

We obtain the lemma by multiplying by a^2 and then adding $-vac+va$. □

We can now show that $\beta \in R$. If we let

$$\gamma = \frac{(c-a)(c-1)+c-c^2-vac+va+ta^2}{a},$$

then by Lemma 11 we have $0 \le \gamma \le a$. By Lemma 9 we therefore have $\frac{m+a-c}{a} + \gamma \in B$, since $m+a-c+a^2 \le 2m-2$ because $m \ge a+a^2$. Thus the solution

$$\left[m-c \to s+1;\ c-2 \to \frac{m+a-c}{a};\ 1 \to \frac{m+a-c}{a}+\gamma;\ 1 \to \beta\right]$$

shows that $\beta \in R$.

To obtain our final contradiction, we construct a red solution of $L(m,a)$ by assigning the value β to x_{m-1} and x_m and the value s to $(a-1)(\frac{m-c}{a})$ other variables, and showing that we can assign values in R to the remaining $\frac{m-c}{a}+c-2$ variables to complete the solution. We again use values in the set $[2a-2]$.

The values assigned to the remaining $\frac{m-c}{a}+c-2$ variables must add up to

$$\frac{a-1}{a}(m-c) + \frac{a-1}{a^2}((c-1)m+c-c^2-vac+va+ta^2),$$

which can be written as

$$m\left(1+\frac{c-2}{a}-\frac{c-1}{a^2}\right) + \frac{a-1}{a^2}(-ac+c-c^2-vac+va+ta^2). \qquad (9)$$

If we can show that using only the value $2a-2$ (respectively, 1) yields a sum that is at least (respectively, at most) this large, then, as before, there must be a solution that uses values in one of the sets $\{j,\ j+1\}$, where $j \in [2a-3]$.

Using the upper bound on $-vac+va+ta^2$ from Lemma 11, we can achieve our first objective by showing that

$$(2a-2)\left(\frac{m-c}{a}+c-2\right) \geq m\left(1+\frac{c-2}{a}-\frac{c-1}{a^2}\right)+\frac{a-1}{a^2}(c-c^2-a+a^2),$$

which simplifies to

$$c^2(a-1)+c(2a^3-4a^2+a+1)+(-5a^3+6a^2-a) \geq m(-a^2+1+c(a-1)).$$

If we regard a as a constant and denote the quantity on the left-hand side of this inequality by $f(c)$, then the derivative

$$f'(c) = 2c(a-1)+(2a^3-4a^2+a+1)$$

is easily seen to be positive for $c \geq 1$ and $a \geq 3$, so the minimum value of $f(c)$ for $2 \leq c \leq a-1$ occurs at $c=2$. Since

$$m(-a^2+1+c(a-1)) \leq m(-a^2+1+(a-1)^2) = m(2-2a),$$

we only need to verify that $f(2) \geq m(2-2a)$, and this simplifies to

$$2m \geq a^2+3a-2,$$

which is clearly true for $a \geq 3$ and $m \geq a^2$.

To achieve our second objective, it will suffice, by using expression (9) and the lower bound on $-vac+va+ta^2$ from Lemma 11, to show that

$$\left(\frac{m-c}{a}+c-2\right) \leq m\left(1+\frac{c-2}{a}-\frac{c-1}{a^2}\right)+\frac{a-1}{a^2}(c-c^2-a).$$

This inequality simplifies to

$$c^2(a-1)+c(a^2-2a+1)-a^2-a \leq m(a^2-3a+1+c(a-1)).$$

Denoting the quantity on the left-hand side by $g(c)$, we have

$$g'(c) = 2c(a-1)+(a^2-2a+1),$$

so $g'(c) > 0$ for $c \geq 1$ and $a \geq 3$. Therefore the maximum value of $g(c)$ for $2 \leq c \leq a-1$ occurs at $c=a-1$. Since

$$m(a^2-3a+1+c(a-1)) \geq m(a^2-3a+1+2(a-1)) = m(a^2-a-1),$$

we need only verify that $g(a-1) \leq m(a^2-a-1)$, i.e., that

$$2a^3-7a^2+5a-2 \leq m(a^2-a-1).$$

This is easily verified for $a \geq 3$ and $m \geq 2a$.

We have proved the following proposition, which completes the proof of Theorem 1.

Proposition 3. If $a \geq 3, m \geq 2a^2 - a + 2$, and $a - 2 \in R$, then every 2-coloring of $[C(m,a)]$ with $a - 1 \in R$ yields a monochromatic solution of $L(m,a)$.

6. The case $a = 3$.

We now turn to the proof of Theorem 2.

We have determined the 2-color Rado number of $L(m,3)$ for all $m \geq 17$, and must consider $16 \geq m \geq 3$. We continue our convention that $a-2 \in R$, so $1 \in R$.

Case 1: $16 \geq m \geq 8$.

In this case we want to show that the Rado number is $C(m,3)$. By Proposition 1, the Rado number is at least $C(m,3)$.

We now suppose that we have a 2-coloring of $[C(m,3)]$ that yields no monochromatic solution of $L(m,3)$, and seek a contradiction. The values of $C(m,3)$ for $16 \geq m \geq 8$ are, respectively, $25, 24, 22, 16, 15, 14, 9, 8, 7$. In each of these cases, both $a+1$ and $m-1$ are in $[C(m,a)]$, and an inspection of the arguments in Section 3 reveals that this is all we need to obtain a contradiction when $a - 1 \in B$, i.e., $2 \in B$. So we assume that $2 \in R$, and note that then the proof of Lemma 9 is still valid, and the proof of Lemma 4 still shows that $a \in R$, so $3 \in R$.

If $16 \geq m \geq 14$, then $2m - 8 \leq C(m,3)$, and the solution

$$[m - 5 \to 2;\; 2 \to 1;\; 3 \to 2m - 8]$$

shows that $2m - 8 \in B$. From the solution

$$[m - 10 \to 2;\; 8 \to 1;\; 2 \to m - 6]$$

we see that $m - 6 \in B$, and then from the solution

$$[m - 3 \to 4;\; 2 \to m - 6;\; 1 \to 2m - 8]$$

we see that $4 \in R$. On the other hand, by Lemma 9, we have $5 \in B$ and $6 \in B$. Now if $m = 16$ then the solution $[15 \to 5;\; 1 \to 25]$ shows that $25 \in R$, while the solution $[6 \to 3;\; 8 \to 4;\; 2 \to 25]$ shows that $25 \in B$. If $m = 15$ then the solution $[12 \to 5;\; 2 \to 6;\; 1 \to 24]$ shows that $24 \in R$, while the solution $[4 \to 3;\ 9 \to 4;\ 2 \to 24]$ shows that $24 \in B$. If $m = 14$, the solution $[12 \to 5;\ 1 \to 6;\ 1 \to 22]$ shows that $22 \in R$, while the solution $[4 \to 3;\ 8 \to 4;\ 2 \to 22]$ shows that $22 \in B$.

If $13 \geq m \geq 11$, then by Lemma 9 we have $4 \in B$ and $5 \in B$. If $m = 13$, then from the solutions $[12 \to 4;\ 1 \to 16]$ and $[10 \to 3;\ 1 \to 2;\ 2 \to 16]$ we see that $16 \in R$ and $16 \in B$. If $m = 12$, then we see from the solutions $[10 \to 4;\ 1 \to 5;\ 1 \to 15]$ and $[10 \to 3;\ 2 \to 15]$ that $15 \in R$ and $15 \in B$. If $m = 11$ then we use the solutions $[8 \to 4;\ 2 \to 5;\ 1 \to 14]$ and $[6 \to 2;\ 2 \to 1;\ 3 \to 14]$ to see that $14 \in R$ and $14 \in B$.

If $10 \geq m \geq 8$ then Lemma 9 shows that $3 \in B$, contradicting $3 \in R$.

Case 2: $m = 7$

In this case we again want to show that the Rado number is $C(m, 3)$, which is now 4. We know by Proposition 1 that $C(m, 3)$ is a lower bound. We no longer have have $m - 1 \leq C(m, a)$, however, so we cannot rely on the results of Sections 3 and 4 in showing that every 2-coloring of [4] yields a monochromatic solution of $L(m, 3)$. But suppose we have a 2-coloring that yields no such solution. Then the solution $[6 \to 1;\ 1 \to 2]$ shows that $2 \in B$, and doubling this solution shows that $4 \in R$. But the solution $[4 \to 1;\ 3 \to 4]$ shows that $4 \in B$.

Case 3: $6 \geq m \geq 5$

We have $C(6, 3) = 4$, and it is easy to check that the coloring $R = \{1, 4\}, B = \{2, 3\}$ of [4] yields no monochromatic solution of $L(6, 3)$. To see that the Rado number is 5, suppose we have a 2-coloring of [5] that yields no monochromatic solution of $L(6, 3)$. Then the solution $[4 \to 1;\ 2 \to 2]$ shows that $2 \in B$, and doubling this solution shows that $4 \in R$. The solution $[3 \to 1;\ 3 \to 3]$ shows that $3 \in B$. Then the solution $[2 \to 2;\ 2 \to 3;\ 2 \to 5]$ shows that $5 \in R$, while the solution $[3 \to 1;\ 1 \to 5;\ 2 \to 4]$ shows that $5 \in B$.

To deal with $m = 5$, note that we have $C(5, 3) = 3$, and the coloring $R = \{1, 3\}, B = \{2\}$ of [3] yields no monochromatic solution of $L(5, 3)$. To see that the Rado number is 4, suppose we have a 2-coloring of [4] that yields no monochromatic solution of $L(5, 3)$. The solution $[2 \to 1;\ 3 \to 2]$ shows that $2 \in B$, and doubling this solution shows that $4 \in R$. Then the solution $[4 \to 3;\ 1 \to 4]$ shows that $3 \in B$, while the solution $[3 \to 2;\ 2 \to 3]$ shows that $3 \in R$.

Case 4: $4 \geq m \geq 3$.

If $m = 4$ then the Rado number is 1 because $[4 \to 1]$ is a monochromatic solution.

If $m = 3$ then arguments similar to the above show that the Rado number is 9, but this result is also proved in [**3**], where the 2-color Rado number of $L(3, a)$ is determined for all a.

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