Local-restricted-edge-connectivity of graphs*

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Abstract

The local-restricted-edge-connectivity $\lambda'(e, f)$ of two nonadjacent edges e and f in graph G is the maximum number of edge-disjoint e-f paths in G. It is clear that $\lambda'(G) = \min\{\lambda'(e, f)|\ e \text{ and } f \text{ are nonadjacent edges in } G\}$, and $\lambda'(e, f) \leq \min\{\xi(e), \xi(f)\}$ for all pairs e and f of nonadjacent edges in G, where $\lambda'(G)$, $\xi(e)$ and $\xi(f)$ denote the restricted-edge-connectivity of G, the edge-degree of edges e and f, respectively. Let $\xi(G)$ be the minimum edge-degree of G. We call a graph G optimally restricted-edge-connected when $\lambda'(G) = \xi(G)$ and optimally local-restricted-edge-connected if $\lambda'(e, f) = \min\{\xi(e), \xi(f)\}$ for all pairs e and f of nonadjacent edges in G. In this paper we show that some known sufficient conditions that guarantee that a graph is optimally restricted-edge-connected also guarantee that it is optimally local-restricted-edge-connected.

Keywords: Local-restricted-edge-connectivity; Edge-degree; Restricted-edge-connectivity

1 Introduction

We consider finite, undirected, and simple graphs G with the vertex set $V(G)$ and the edge set $E(G)$. For each vertex $v \in V(G)$, the neighborhood $N(v)$ of v is defined as the set of all vertices adjacent to v, and $d(v) = |N(v)|$ is the degree of v. We denote by $\delta(G)$ the minimum degree. For two vertex subsets X and Y of a graph, let (X, Y) be the set of edges with one endpoint

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in X and the other in Y, and $|(X, Y)|$ denotes the cardinality of (X, Y). Let X-Y path be the path having first vertex in X, last vertex in Y, and no other vertex in $X \cup Y$. If $X \subseteq V(G)$, then let $G[X]$ be the subgraph induced by X, and let $\bar{X} = V(G) - X$. The clique number $\varpi(G)$ of a graph G is the maximum cardinality of a complete subgraph of G. For $e = u_1u_2 \in E(G)$, let $\xi(e) = d(u_1) + d(u_2) - 2$ be the edge-degree of e, and let $\xi(G) = \min\{\xi(e) : e \in E(G)\}$ be the minimum edge-degree of G.

The underlying topology of an interconnection network is usually modeled by a graph in which the vertices and edges represent the nodes and links, respectively. A classic measure of network reliability is the edge-connectivity $\lambda(G)$. In general, the larger $\lambda(G)$ is, the more reliable the network is. For $\lambda(G) \leq \delta(G)$, a graph G with $\lambda(G) = \delta(G)$ is naturally said to be maximally edge-connected, or λ-optimal for simplicity. For further study, Esfahanian and Hakimi proposed the concept of restricted-edge-connectivity [2, 3]. An edge set $S \subset E$ is said to be a restricted-edge-cut if $G \backslash S$ is disconnected and every component of $G \backslash S$ has at least two vertices. The restricted-edge-connectivity of G, denoted by $\lambda'(G)$, is the cardinality of a minimum restricted-edge-cut of G. A restricted-edge-cut S is called a λ'-cut if $|S| = \lambda'(G)$. Clearly, for any λ'-cut, the graph $G \backslash S$ consists of exactly two components. A connected graph G is called λ'-connected, if λ'-cut exists. It is shown by Wang and Li that the larger $\lambda'(G)$ is, the more reliable the network is [10]. In [3], the authors proved that $\lambda'(G) \leq \xi(G)$ holds for any graph G of order at least 4 which is not isomorphic to the star $K_{1,n-1}$. Hence, a graph G with $\lambda'(G) = \xi(G)$ is said to be optimally restricted-edge-connected. A graph G is said to be super restricted-edge-connected, if every λ'-cut of G isolates an edge of G. Obviously, every super restricted-edge-connected graph is also optimally restricted-edge-connected. Some classes of optimally restricted-edge-connected graphs were studied in [5, 8, 11, 12].

The local-edge-connectivity $\lambda(u, v)$ of two vertices u and v in a graph G is the maximum number of edge-disjoint u-v paths in G, and $\lambda(G) = \min\{\lambda(u, v) | u, v \in V(G); u \neq v\}$. Clearly, $\lambda(u, v) \leq \min\{d(u), d(v)\}$ for all pairs u and v of vertices in G. A graph G is said to be maximally local-edge-connected when $\lambda(u, v) = \min\{d(u), d(v)\}$ for all pairs u and v of vertices in G. It is clear that if a graph is maximally local-edge-connected, then it is maximally edge-connected. Fricke et al.[4], Hellwig and Volkmann [6], and Volkmann [9] have shown that some known sufficient conditions that guarantee that a graph is maximally edge-connected also guarantee that it is maximally local-edge-connected.

In this paper, we define the locally restricted-edge-connectivity as follows.

Definition 1.1. *The locally restricted-edge-connectivity $\lambda'(e, f)$ of two non-adjacent edges e and f in graph G is the maximum number of edge-disjoint e-f paths in G.*

By the definition, we immediately obtain the following observation.

Observation 1.2. *If G is a λ'-connected graph, then $\lambda'(G) = \min\{\lambda'(e,f)|$ e and f are nonadjacent edges in $G\}$.*

Proof. It is clear that $\lambda'(G) \leq \min\{\lambda'(e,f)|\, e$ and f are nonadjacent edges in $G\}$. On the other hand, let S be a λ'-cut such that $|(X,\bar{X})| = \lambda'(G)$, then there exist two edges $e' \in E(G[X])$ and $f' \in E(G[\bar{X}])$ satisfying $\lambda'(e',f') \leq \lambda'(G)$. Therefore, $\lambda'(G) \geq \min\{\lambda'(e,f)|\, e$ and f are nonadjacent edges in $G\}$. The proof is complete. □

Clearly, $\lambda'(e,f) \leq \min\{\xi(e), \xi(f)\}$ for all pairs e and f of nonadjacent edges in G. We call a graph G optimally local-restricted-edge-connected if $\lambda'(e,f) = \min\{\xi(e), \xi(f)\}$ for all pairs of e and f of nonadjacent edges in G. In this paper we show that some known sufficient conditions that guarantee that a graph is optimally restricted-edge-connected also guarantee that it is optimally local-restricted-edge-connected.

Our proofs are based on the following consequence of Menger's [7] theorem.

Lemma 1.3. *Let e and f be a pair of nonadjacent edges in graph G. Then $\lambda'(e,f) \geq q$ if and only if $|(X,\bar{X})| \geq q$ for all subsets $X \subset V(G)$ such that $e \in E(G[X])$ and $f \in E(G[\bar{X}])$.*

Terminologies not given here are referred to [1].

2 Arbitrary graphs

If a graph G is optimally restricted-edge-connected rather than super restricted-edge-connected, then there exists a λ'-cut $S = (X,\bar{X})$ with $|X|, |\bar{X}| \geq 3$. It is easy to see that if there exist two edges $e \in E(G[X])$ and $f \in E(G[\bar{X}])$ such that $\xi(e), \xi(f) > \xi(G)$, then $\lambda'(e,f) = \xi(G) < \min\{\xi(e), \xi(f)\}$. Hence, a sufficient condition that guarantee that a graph is optimally restricted-edge-connected graph may not guarantee that it is optimally local-restricted-edge-connected.

Our first observation demonstrates that an optimally local-restricted-edge-connected graph is also optimally restricted-edge-connected.

Observation 2.1. *If a graph G is optimally local-restricted-edge-connected, then it is optimally restricted-edge-connected.*

Proof. Note that if $\xi(e) = \xi(G)$ for $e \in E(G)$, then there exists an edge $f \in E(G)$ such that e and f are nonadjacent. Since G is optimally local-restricted-edge-connected, we have $\lambda'(e,f) = \min\{\xi(e), \xi(f)\}$ for all pairs

e and f of nonadjacent edges in G. This implies

$$\begin{aligned}\lambda'(G) &= \min\{\lambda'(e,f) | e \text{ and } f \text{ are nonadjacent edges in } G\} \\ &= \min\{\min\{\xi(e),\xi(f)\} | e \text{ and } f \text{ are nonadjacent edges in } G\} \\ &= \xi(G).\end{aligned}$$

Hence, G is optimally restricted-edge-connected. □

Our second observation demonstrates that a r-regular and optimally restricted-edge-connected graph is also optimally local-restricted-edge-connected

Observation 2.2. *If a graph G is r-regular and optimally restricted-edge-connected, then it is optimally local-restricted-edge-connected.*

Proof. Since G is r-regular and optimally restricted-edge-connected, we have $\lambda'(G) = \xi(G) = \xi(e)$ for any edge $e \in E(G)$. This implies $\lambda'(G) \leq \lambda'(e,f) \leq \min\{\xi(e),\xi(f)\} = \xi(G)$ for all pairs e and f of nonadjacent edges in G. Therefore, G is optimally local-restricted-edge-connected. □

Theorem 2.3. *Let G be a λ'-connected graph with order n and minimum degree δ. If $\delta \geq \lfloor \frac{n}{2} \rfloor + 1$, then G is optimally local-restricted-edge-connected.*

Proof. Let e and f be any two nonadjacent edges of G, $e = u_1u_2, f = v_1v_2$. As noted above, $\lambda'(e,f) \leq \min\{\xi(e),\xi(f)\}$. In view of Lemma 1.3, it remains to show that $|(X,\bar{X})| \geq \min\{\xi(e),\xi(f)\}$ for all subsets $X \subset V(G)$ with the property that $e \in E(G[X])$ and $f \in E(G[\bar{X}])$. Let X be such a set.

Case 1: $|X| \leq \lfloor \frac{n}{2} \rfloor$. Then the condition $\delta \geq \lfloor \frac{n}{2} \rfloor + 1$ implies $|X| \leq \delta - 1$. Since G is a simple graph, we have $E(G[X]) \leq |X|(|X|-1)/2$ and hence

$$\begin{aligned}|(X,\bar{X})| &= \sum_{x\in X} d(x) - 2E(G[X]) \\ &\geq \sum_{x\in X} d(x) - |X|(|X|-1) \\ &= d(u_1) + d(u_2) + (|X|-2)\delta - |X|(|X|-1) \\ &= \xi(e) + 2 + (|X|-2)\delta - |X|(|X|-1) \\ &= \xi(e) + (|X|-2)(\delta - 1 - |X|) \\ &\geq \xi(e) + (|X|-2)(\lfloor \frac{n}{2} \rfloor - |X|) \\ &\geq \xi(e).\end{aligned}$$

This yields the desired inequality $|(X,\bar{X})| \geq \xi(e) \geq \min\{\xi(e),\xi(f)\}$.

Case 2: $|\bar{X}| \leq \lfloor \frac{n}{2} \rfloor$. Analogously to Case 1, we then obtain $|(X,\bar{X})| \geq \xi(f) \geq \min\{\xi(e),\xi(f)\}$, and the proof is complete. □

Using Theorem 2.3 and Observation 2.1, we immediately obtain the following sufficient condition for graphs to be optimally restricted-edge-connected.

Corollary 2.4. *[3] Let G be a λ'-connected graph with order n and minimum degree δ. If $\delta \geq \lfloor \frac{n}{2} \rfloor + 1$, then G is optimally restricted-edge-connected.*

3 Graphs with given clique number

Using Turán's bound $2|E(G)| \leq \frac{p-1}{p}|V(G)|^2$ for graphs G satisfying $\omega(G) \leq p$, we obtain the following result.

Theorem 3.1. *Let $p \geq 3$ be an integer, and let G be a λ'-connected graph of order n with clique number $\omega(G) \leq p$ and degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n = \delta$. If*

$$\sum_{i=1}^{\max\{1,\delta-2\}} d_{n-i+1} \geq \max\{1,\delta-2\}\frac{p-1}{p}(\lfloor \frac{n}{2} \rfloor + 2 + \frac{p-3}{(p-1)\max\{1,\delta-2\}}),$$

then G is optimally local-restricted-edge-connected.

Proof. Let e and f be any two nonadjacent edges of G, $e = u_1u_2, f = v_1v_2$. As noted above, $\lambda'(e,f) \leq \min\{\xi(e), \xi(f)\}$. In view of Lemma 1.3, it remains to show that $|(X,\bar{X})| \geq \min\{\xi(e), \xi(f)\}$ for all subsets $X \subset V(G)$ with the property that $e \in E(G[X])$ and $f \in E(G[\bar{X}])$. Let X be such a set.

Case 1: $|X| \leq \lfloor \frac{n}{2} \rfloor$. If $|X| = 2$, then we are done. Now let $|X| \geq 3$. If $\delta \geq |X| + 1$, then it is analogous to the proof of Case 1 of Theorem 2.3. If $\delta \leq |X|$, using Turán's bound and the inequality $1 \leq \max\{1, \delta - 2\} \leq |X| - 2$, we have

$$|(X,\bar{X})| = \sum_{x\in X} d(x) - 2E(G[X]) \geq \sum_{x\in X} d(x) - \tfrac{p-1}{p}|X|^2$$
$$= \sum_{x\in X} d(x) - \tfrac{p-1}{p}[(|X|-2)(|X|+2)+4]$$
$$= d(u_1) + d(u_2) - 2 + 2 + \sum_{x\in X\setminus\{u_1,u_2\}} d(x) - \tfrac{p-1}{p}[(|X|-2)(|X|+2)+4]$$
$$\geq \xi(e) + 2 + \sum_{i=1}^{\max\{1,\delta-2\}} d_{n-i+1} + \sum_{i=\max\{2,\delta-1\}}^{|X|-2} d_{n-i+1} - \tfrac{p-1}{p}[(|X|-2)(|X|+2)+4]$$
$$\geq \xi(e) + 2 + (|X|-2)\tfrac{p-1}{p}(\lfloor \tfrac{n}{2} \rfloor + 2 + \tfrac{p-3}{(p-1)\max\{1,\delta-2\}}) - \tfrac{p-1}{p}[(|X|-2)(|X|+2)+4]$$
$$= \xi(e) + 2 + (|X|-2)\tfrac{p-1}{p}(\lfloor \tfrac{n}{2} \rfloor + 2 + \tfrac{p-3}{(p-1)\max\{1,\delta-2\}} - (|X|+2)) - 4\tfrac{p-1}{p}$$
$$\geq \xi(e) + 2 + (|X|-2)\tfrac{p-3}{p\max\{1,\delta-2\}} - 4\tfrac{p-1}{p}$$
$$\geq \xi(e) + 2 + \tfrac{p-3}{p} - 4\tfrac{p-1}{p} = \xi(e) - 1 + \tfrac{1}{p}.$$

Since $|(X,\bar{X})|$ and $\xi(e)$ are integers and $0 < 1/p < 1$, it follows that $|(X,\bar{X})| \geq \xi(e) \geq \min\{\xi(e),\xi(f)\}$.

Case 2: $|\bar{X}| \leq \lfloor \frac{n}{2} \rfloor$. Analogously to Case 1, we then obtain $|(X,\bar{X})| \geq \xi(f) \geq \min\{\xi(e),\xi(f)\}$, and the proof is complete. □

Corollary 3.2. *Let $p \geq 3$ be an integer, and let G be a λ'-connected graph of order n with clique number $\omega(G) \leq p$ and minimum degree δ. If*

$$\lfloor \frac{n}{2} \rfloor \geq \lfloor \frac{p\delta}{p-1} - \frac{p-3}{(p-1)\max\{1,\delta-2\}} \rfloor - 2, \qquad (1)$$

then G is optimally local-restricted-edge-connected.

Proof. If the inequality (1) holds, then

$$\delta \geq \frac{p-1}{p}(\lfloor \frac{n}{2} \rfloor + 2 + \frac{p-3}{(p-1)\max\{1,\delta-2\}}).$$

Therefore, the degree sequence condition of Theorem 3.1 holds. The proof is complete. □

The following result of Hellwig and Volkmann is an easy consequence of Corollary 3.2.

Corollary 3.3. *[5] Let $p \geq 3$ be an integer, and let G be a λ'-connected graph of order n with clique number $\omega(G) \leq p$ and minimum degree δ. If*

$$\lfloor \frac{n}{2} \rfloor \geq \lfloor \frac{p\delta}{p-1} - \frac{p-3}{(p-1)\max\{1,\delta-2\}} \rfloor - 2,$$

then G is optimally restricted-edge-connected.

4 Triangle-free graphs

Using Turán bound $2|E(G)| \leq |V(G)|^2/2$ for triangle-free graphs G, we obtain the following result.

Theorem 4.1. *Let G be a λ'-connected triangle-free graph of order $n \geq 4$ and degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n = \delta$. If*

$$\sum_{i=1}^{\max\{1,\delta-2\}} d_{n-i+1} \geq \max\{1,\delta-2\}\frac{1}{2}(\lfloor \frac{n}{2} \rfloor + 2 - \frac{4}{n-3}),$$

then G is optimally local-restricted-edge-connected.

Proof. Let e and f be any two nonadjacent edges of G, $e = u_1u_2, f = v_1v_2$. As noted above, $\lambda'(e,f) \leq \min\{\xi(e), \xi(f)\}$. In view of Lemma 1.3, it remains to show that $|(X, \bar{X})| \geq \min\{\xi(e), \xi(f)\}$ for all subsets $X \subset V(G)$ with the property that $e \in E(G[X])$ and $f \in E(G[\bar{X}])$. Let X be such a set.

Case 1: $|X| \leq \lfloor \frac{n}{2} \rfloor$. If $|X| = 2$, then we are done. Now let $|X| \geq 3$. If $\delta \geq |X| + 1$, then it is analogous to the proof of Case 1 of Theorem 2.3. If $\delta \leq |X|$, using Turán's bound and the inequality $1 \leq \max\{1, \delta - 2\} \leq |X| - 2$, we have

$$
\begin{aligned}
|(X,\bar{X})| &= \sum_{x \in X} d(x) - 2E(G[X]) \\
&\geq \sum_{x \in X} d(x) - \frac{|X|^2}{2} \\
&= d(u_1) + d(u_2) - 2 + 2 + \sum_{x \in X \setminus \{u_1,u_2\}} d(x) - \frac{(|X|-2)(|X|+2)+4}{2} \\
&\geq \xi(e) + 2 + \sum_{i=1}^{\max\{1,\delta-2\}} d_{n-i+1} + \sum_{i=\max\{2,\delta-1\}}^{|X|-2} d_{n-i+1} - \frac{(|X|-2)(|X|+2)+4}{2} \\
&\geq \xi(e) + 2 + (|X|-2)\frac{1}{2}(\lfloor \frac{n}{2} \rfloor + 2 - \frac{4}{n-3} - |X| - 2) - 2 \\
&\geq \xi(e) - \frac{2(|X|-2)}{n-3}.
\end{aligned}
$$

Since $|(X, \bar{X})|$ and $\xi(e)$ are integers and $0 \leq \frac{2(|X|-2)}{n-3} < 1$, it follows that $|(X, \bar{X})| \geq \xi(e) \geq \min\{\xi(e), \xi(f)\}$.

Case 2: $|\bar{X}| \leq \lfloor \frac{n}{2} \rfloor$. Analogously to Case 1, we then obtain $|(X, \bar{X})| \geq \xi(f) \geq \min\{\xi(e), \xi(f)\}$, and the proof is complete. □

Corollary 4.2. *Let G be a λ'-connected triangle-free graph of order n. If*

$$d(x) \geq \lceil \frac{1}{2} \lfloor \frac{n}{2} \rfloor + 1 \rceil = \lfloor \frac{n+2}{4} \rfloor + 1$$

for all vertices x in G, then G is optimally local-restricted-edge-connected.

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