PSEUDO-HYPERBOLIC CURVES IN E_ν^5

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Abstract

In this paper we characterize all spacelike, timelike and null curves lying on the pseudohyperbolic space $H_{\nu-1}^4$ in Minkowski space E_ν^5. Moreover, we prove that there are no timelike and no null curves lying on the pseudohyperbolic space $H_{\nu-1}^4$ in E_ν^5.

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1 Introduction

A necessary and sufficient conditions for a curve to be a spherical curve in the Euclidean space E^3 are given in [5, 9] and [10]. In [11, 12] and [13], the authors have characterized the Lorentzian spherical curves in the Minkowski 3-space. On the other hand, a similar characterizations of a spacelike, a timelike and a null curves lying on the pseudohyperbolic spaces H_0^2 and H_0^3 in the Minkowski space are obtained in [2, 7].

The Minkowski space E_ν^5 is the Euclidean space E^5 provided with the standard flat metric given by

$$g = \varepsilon_0 dx_1^2 + \varepsilon_1 dx_2^2 + \varepsilon_2 dx_3^2 + \varepsilon_3 dx_4^2 + \varepsilon_4 dx_5^2,$$

where $(x_1, x_2, x_3, x_4, x_5)$ is a rectangular coordinate system of E_ν^5.

Since g is indefinite metric, recall that a vector $v \in E_\nu^5$ can have one of three causal characters: it can be *spacelike* if $g(v,v) > 0$ or $v = 0$, *timelike* if $g(v,v) < 0$ and *null (lightlike)* if $g(v,v) = 0$ and $v \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in E_ν^5 can locally be *spacelike, timelike* or *null (lightlike)*, if all of its velocity vectors $\alpha'(s)$ are respectively *spacelike, timelike* or *null.* Besides, recall that the norm of a vector v is given by $\| v \| = \sqrt{| g(v,v) |}$. Therefore, v is a unit vector if $g(v,v) = \pm 1$. Next, vectors v, w in E_ν^5 are said to be orthogonal if $g(v,w) = 0$. The velocity of the curve $\alpha(s)$ is given by $\|\alpha'(s)\|$.

The pseudohyperbolic space with center $m \in E_\nu^5$ and radius $r \in R^+$ in the space E_ν^5 is the hyperquadric

$$H_{\nu-1}^4(r) = \{a \in E_\nu^5 \mid g(a-m, a-m) = -r^2\}.$$

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with dimension 4 and index $\nu - 1$ [6].

Denote by $\{T(s), N(s), B_1(s), B_2(s), B_3(s)\}$ the moving Frenet frame along the curve $\alpha(s)$ in the space E^5_ν. Then $T(s)$,$N(s)$,$B_1(s)$,$B_2(s)$, $B_3(s)$ are respectively the tangent, the principal normal, the first binormal, the second binormal and the third binormal vector. Spacelike or timelike curve $\alpha(s)$ is said to be parameterized by a arclength function s, if $g(\alpha'(s), \alpha'(s)) = \pm 1$. In particular, a null curve $\alpha(s)$ is said to be parameterized by a arclength function s if $g(\alpha''(s), \alpha''(s)) = 1$ [1, 3].

Consider $\alpha = \alpha(s)$ be a unit speed non-null curve in Minkowski space E^5_ν, parametrized by arclength function s. Let $\{T(s), N(s), B_1(s), B_2(s), B_3(s)\}$ be the moving Frenet frame along α which is satisfying $\langle T, T\rangle = \varepsilon_0$, $\langle N, N\rangle = \varepsilon_1$, $\langle B_1, B_1\rangle = \varepsilon_2$, $\langle B_2, B_2\rangle = \varepsilon_3$, $\langle B_3, B_3\rangle = \varepsilon_4$.The Frenet equations for α given by [4] are following: $T' = \kappa_1 N$, $N' = -\varepsilon_0\varepsilon_1\kappa_1 T + \kappa_2 B_1$, $B_1' = -\varepsilon_1\varepsilon_2\kappa_2 N + \kappa_3 B_2$, $B_2' = -\varepsilon_2\varepsilon_3\kappa_3 B_1 + \kappa_4 B_3$, $B_3' = -\varepsilon_3\varepsilon_4\kappa_3 B_2$. By using these equations, in this paper we give some necessary and sufficient conditions for a spacelike curve to lie on the pseudohyperbolic space $H^4_{\nu-1}$ in the Minkowski space. Also we show that there are no timelike and no null curves lying on the pseudohyperbolic space $H^4_{\nu-1}$ in E^5_ν.

2 The spacelike, timelike and null curves lying on $H^4_{\nu-1}$

In this section, under assumption that spacelike, timelike and null curves lie on the pseudohyperbolic space $H^4_{\nu-1}(r)$, we give some characterization theorems for these curves in terms of their curvatures.

Theorem 1. *Let $\alpha(s)$ be a unit speed spacelike curve in E^5_ν with curvatures $k_1(s) \neq 0$, $k_2(s) \neq 0$, $k_3(s) \neq 0, k_4(s) \neq 0$ for each $s \in I \subset R$. Then α lies on $H^4_{\nu-1}(r)$ if and only if*

$$\begin{aligned}&\varepsilon_1(\tfrac{1}{k_1})^2+\varepsilon_2(\tfrac{1}{k_2}(\tfrac{1}{k_1})')^2+\varepsilon_3[\tfrac{1}{k_3}(\varepsilon_1\varepsilon_2\tfrac{k_2}{k_1}+(\tfrac{1}{k_2}(\tfrac{1}{k_1})')')]^2\\ &+\varepsilon_4[\tfrac{1}{k_4}(\varepsilon_2\tfrac{k_3}{k_2}(\tfrac{1}{k_1})')+\varepsilon_3(\tfrac{1}{k_3}(\varepsilon_1\varepsilon_2\tfrac{k_2}{k_1}+(\tfrac{1}{k_2}(\tfrac{1}{k_1})')'))']^2 \quad = -r^2\end{aligned} \tag{2.1}$$

Proof. Let us first suppose that α lies on $H^4_{\nu-1}(r)$ with center m. Then $g(\alpha - m, \alpha - m) = -r^2$, for each $s \in I \subset R$. Differentiating the previous equation four times with respect to s and by applying Frenet equations, we get

$$\begin{aligned}&g(T, \alpha - m) = 0, \quad g(N, \alpha - m) = -\tfrac{1}{k_1},\\ &g(B_1, \alpha - m) = -(\tfrac{1}{k_2})(\tfrac{1}{k_1})', \quad g(B_2, \alpha - m) = -(\tfrac{1}{k_3})[\varepsilon_1\varepsilon_2\tfrac{k_2}{k_1} + (\tfrac{1}{k_2}(\tfrac{1}{k_1})')'],\\ &g(B_3, \alpha - m) = -(\tfrac{1}{k_4})[\varepsilon_2\varepsilon_3\tfrac{k_3}{k_2}(\tfrac{1}{k_1})' + \varepsilon_3(\tfrac{1}{k_3}(\varepsilon_1\varepsilon_2\tfrac{k_2}{k_1} + (\tfrac{1}{k_2}(\tfrac{1}{k_1})')'))'].\end{aligned} \tag{2.2}$$

Let us decompose the vector $\alpha - m$ with respect to pseudo–orthonormal basis $\{T, N, B_1, B_2, B_3\}$ by

$$\alpha(s) - m = a(s)T(s) + b(s)N(s) + c(s)B_1(s) + d(s)B_2(s) + e(s)B_3(s) \tag{2.3}$$

where $a(s), b(s), c(s), d(s), e(s)$ are arbitrary functions. Therefore, using (2.2) and (2.3) we obtain (2.1).

Conversely, let us assume that the relation (2.1) holds. By taking the derivative of (2.1) with respect to s, we find

$$\begin{aligned} &\tfrac{1}{k_4}[\varepsilon_2\tfrac{k_3}{k_2}(\tfrac{1}{k_1})' + \varepsilon_3(\tfrac{1}{k_3}(\varepsilon_1\varepsilon_2\tfrac{k_2}{k_1} + (\tfrac{1}{k_2}(\tfrac{1}{k_1})')'))'] \\ &\times[\tfrac{k_4}{k_3}(\varepsilon_1\varepsilon_2\tfrac{k_2}{k_1} + (\tfrac{1}{k_2}(\tfrac{1}{k_1})')') + \varepsilon_4\mathcal{A}'] = 0 \end{aligned} \tag{2.4}$$

where

$$\mathcal{A} = \frac{1}{k_4}[\varepsilon_2\tfrac{k_3}{k_2}(\tfrac{1}{k_1})' + \varepsilon_3(\tfrac{1}{k_3}(\varepsilon_1\varepsilon_2\tfrac{k_2}{k_1} + (\tfrac{1}{k_2}(\tfrac{1}{k_1})')'))'].$$

If $\varepsilon_2\frac{k_3}{k_2}(\frac{1}{k_1})' + \varepsilon_3(\frac{1}{k_3}(\varepsilon_1\varepsilon_2\frac{k_2}{k_1} + (\frac{1}{k_2}(\frac{1}{k_1})')'))' = 0$, then substituting this into (2.2) we get $g(B_3, \alpha - m) = 0$ which means that α lies in a plane spanned by $\{N, B_1, B_2\}$. This result is a contradiction with our assumption. Therefore, it follows that

$$\tfrac{k_4}{k_3}(\varepsilon_1\varepsilon_2\tfrac{k_2}{k_1} + (\tfrac{1}{k_2}(\tfrac{1}{k_1})')') + \varepsilon_4\mathcal{A}' = 0. \tag{2.5}$$

Next, we may consider the vector $m \in E_v^5$ given by

$$\begin{aligned} m = \alpha + &\tfrac{\varepsilon_1}{k_1}N + \tfrac{\varepsilon_2}{k_2}(\tfrac{1}{k_1})'B_1 + \tfrac{\varepsilon_3}{k_3}[\varepsilon_1\varepsilon_2\tfrac{k_2}{k_1} + (\tfrac{1}{k_2}(\tfrac{1}{k_1})')']B_2 \\ &+ \tfrac{\varepsilon_3\varepsilon_4}{k_4}[\varepsilon_2\tfrac{k_3}{k_2}(\tfrac{1}{k_1})' + \varepsilon_3(\tfrac{1}{k_3}(\varepsilon_1\varepsilon_2\tfrac{k_2}{k_1} + (\tfrac{1}{k_2}(\tfrac{1}{k_1})')'))']B_3. \end{aligned} \tag{2.6}$$

Differentiating (2.6) with respect to s and by using Frenet formulae, we obtain

$$m' = [\tfrac{k_4}{k_3}(\varepsilon_1\varepsilon_2\tfrac{k_2}{k_1} + (\tfrac{1}{k_2}(\tfrac{1}{k_1})')') + \varepsilon_4\mathcal{A}']B_3. \tag{2.7}$$

Then substitution of (2.5) into (2.7) implies $m' = 0$ and thus m =constant. Finally, from (2.6) we find $g(\alpha - m, \alpha - m) = -r^2$, so α lies on $H_{v-1}^4(r)$. □

Theorem 2. *Let $\alpha(s)$ be a unit speed spacelike curve $\alpha(s)$ in E_v^5 with curvatures $k_1(s) \neq 0$, $k_2(s) \neq 0$, $k_3(s) \neq 0, k_4(s) \neq 0$ for each $s \in I \subset R$. Then α lies on $H_{v-1}^4(r)$ if and only if*

$$\tfrac{k_4}{k_3}(\varepsilon_1\varepsilon_2\tfrac{k_2}{k_1} + (\tfrac{1}{k_2}(\tfrac{1}{k_1})')') = -\varepsilon_4\mathcal{A}' \tag{2.8}$$

where

$$\mathcal{A} = \tfrac{1}{k_4}[\varepsilon_2\tfrac{k_3}{k_2}(\tfrac{1}{k_1})' + \varepsilon_3(\tfrac{1}{k_3}(\varepsilon_1\varepsilon_2\tfrac{k_2}{k_1} + (\tfrac{1}{k_2}(\tfrac{1}{k_1})')'))']$$

and

$$\varepsilon_1(\tfrac{1}{k_1})^2 + \varepsilon_2(\tfrac{1}{k_2}(\tfrac{1}{k_1})')^2 + \varepsilon_3[\tfrac{1}{k_3}(\varepsilon_1\varepsilon_2\tfrac{k_2}{k_1} + (\tfrac{1}{k_2}(\tfrac{1}{k_1})')')]^2 + \varepsilon_4\mathcal{A}^2 < 0. \tag{2.9}$$

Proof. Let us first assume that $\alpha(s)$ lies on $H_{v-1}^4(r)$. Then from Theorem 1. it follows that the relation (2.1) holds. Differentiating (2.1) with respect to s, we find that the equation in (2.8) is satisfied. Moreover, by using (2.1) we immediately get that the inequality in (2.9) holds.

Conversely, let us suppose that (2.8) and (2.9) holds. It can be easily seen that (2.9) is the differential of the equation

$$\varepsilon_1(\tfrac{1}{k_1})^2 + \varepsilon_2(\tfrac{1}{k_2}(\tfrac{1}{k_1})')^2 + \varepsilon_3[\tfrac{1}{k_3}(\varepsilon_1\varepsilon_2\tfrac{k_2}{k_1} + (\tfrac{1}{k_2}(\tfrac{1}{k_1})')')]^2 + \varepsilon_4 A^2 = c < 0.$$

Finally, we may take $c = -r^2$, $r \in R^+$, so Theorem 1 implies that α lies on $H^4_{\nu-1}(r)$. □

Theorem 3. *A unit speed spacelike curve $\alpha(s)$ in E^5_ν with curvatures $k_1(s) \neq 0$, $k_2(s) \neq 0$, $k_3(s) \neq 0, k_4(s) \neq 0$ for each $s \in I \subset R$. Then α lies on $H^4_{\nu-1}(r)$ if and only if there exist a differentiable function $f(s)$ such that*

$$\begin{aligned} fk_4 &= \varepsilon_2\tfrac{k_3}{k_2}(\tfrac{1}{k_1})' + \varepsilon_3(\tfrac{1}{k_3}(\tfrac{k_2}{k_1} + (\tfrac{1}{k_2}(\tfrac{1}{k_1})')'))', \\ f' &= -\varepsilon_4\tfrac{k_4}{k_3}(\varepsilon_1\varepsilon_2\tfrac{k_2}{k_1} + (\tfrac{1}{k_2}(\tfrac{1}{k_1})')'), \\ \varepsilon_1(\tfrac{1}{k_1})^2 &+ \varepsilon_2(\tfrac{1}{k_2}(\tfrac{1}{k_1})')^2 + \varepsilon_3(\tfrac{f'}{k_4})^2 + \varepsilon_4 f^2 < 0. \end{aligned} \tag{2.10}$$

Proof. Let us first assume that $\alpha(s)$ lies on $H^4_0(r)$. Then Theorems 1 and 2 imply that respectively relations (2.1), (2.8), (2.9) hold. Next, let us define the differentiable function $f(s)$ by

$$f(s) = \tfrac{1}{k_4}[\varepsilon_2\tfrac{k_3}{k_2}(\tfrac{1}{k_1})' + \varepsilon_3(\tfrac{1}{k_3}(\tfrac{k_2}{k_1} + (\tfrac{1}{k_2}(\tfrac{1}{k_1})')'))']. \tag{2.11}$$

Consequently, by using (2.1), (2.8), (2.9) and (2.11) we easily find that the relations in (2.10) are satisfied.

Conversely, let us assume that there exist a differentiable function $f(s)$ such that the relations in (2.10) hold for each $s \in I \subset R$. By using relations in (2.10), we easily find that the relations in (2.8), (2.9) are satisfied. Finally, by Theorem 2 it follows that α lies on $H^4_{\nu-1}(r)$. □

Theorem 4. *A unit speed spacelike curve $\alpha(s)$ in E^5_ν with curvatures $k_1(s) \neq 0$, $k_2(s) \neq 0$, $k_3(s) \neq 0, k_4(s) \neq 0$ for each $s \in I \subset R$. Then α lies on $H^4_{\nu-1}(r)$ if and only if there exist constants $A, B \in R$ such that the following relations hold:*
***a.** If $\varepsilon_3 = \varepsilon_4$ then*

$$\begin{aligned} -\tfrac{1}{k_3}(\varepsilon_1\varepsilon_2\tfrac{k_2}{k_1} + (\tfrac{1}{k_2}(\tfrac{1}{k_1})')') &= [A + \varepsilon_2\varepsilon_3\textstyle\int_0^s \tfrac{k_3}{k_2}(\tfrac{1}{k_1})'\sin(\int_0^s k_4ds)ds]\sin(\int_0^s k_4ds) \\ &\quad -[B - \varepsilon_2\varepsilon_3\textstyle\int_0^s \tfrac{k_3}{k_2}(\tfrac{1}{k_1})'(\int_0^s k_4ds)ds]\cos(\int_0^s k_4ds) \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} &[A + \varepsilon_2\varepsilon_3\textstyle\int_0^s \tfrac{k_3}{k_2}(\tfrac{1}{k_1})'\sin(\int_0^s k_4ds)ds]^2 + [B - \varepsilon_2\varepsilon_3\int_0^s \tfrac{k_3}{k_2}(\tfrac{1}{k_1})'\cos(\int_0^s k_4ds)ds]^2 \\ &\quad < -\varepsilon_1(\tfrac{1}{k_1})^2 - \varepsilon_2(\tfrac{1}{k_2}(\tfrac{1}{k_1})')^2. \end{aligned} \tag{2.13}$$

***b.** If $\varepsilon_3 = -\varepsilon_4$ then*

$$\begin{aligned} \tfrac{1}{k_3}(\varepsilon_1\varepsilon_2\tfrac{k_2}{k_1} + (\tfrac{1}{k_2}(\tfrac{1}{k_1})')') &= \varepsilon_3[A + \textstyle\int_0^s \varepsilon_2\tfrac{k_3}{k_2}(\tfrac{1}{k_1})'\sinh(\int_0^s k_4ds)ds]\sinh(\int_0^s k_4ds) \\ &\quad -\varepsilon_3[B + \textstyle\int_0^s \varepsilon_2\tfrac{k_3}{k_2}(\tfrac{1}{k_1})'\cosh(\int_0^s k_4ds)ds]\cosh(\int_0^s k_4ds) \end{aligned} \tag{2.14}$$

and

$$\begin{aligned}&-\varepsilon_3[A+\int_0^s \varepsilon_2(\tfrac{k_3}{k_2})(\tfrac{1}{k_1})'\sinh(\int_0^s k_4 ds)ds]^2\\ &+\varepsilon_3[B+\int_0^s \varepsilon_2(\tfrac{k_3}{k_2})(\tfrac{1}{k_1})'\cosh(\int_0^s k_4 ds)ds]^2 < -\varepsilon_1(\tfrac{1}{k_1})^2-\varepsilon_2(\tfrac{1}{k_2}(\tfrac{1}{k_1})')^2.\end{aligned} \tag{2.15}$$

Proof. **a.** Let us first suppose that α lies on $H^4_{\nu-1}(r)$. By Theorem 3 there exist a differentiable function $f(s)$ such that (2.10) hold. Next, let us define the C^2 function $\theta(s)$ by $\theta(s)=\int_0^s k_4 ds$. Moreover, let us define the C^1 functions $g(s)$ and $h(s)$ by

$$\begin{aligned}g(s)&=\omega\sin\theta+f(s)\cos\theta-\varepsilon_2\varepsilon_3\int_0^s \tfrac{k_3}{k_2}(\tfrac{1}{k_1})'\sin\theta ds,\\ h(s)&=-\omega\cos\theta+f(s)\sin\theta-\varepsilon_2\varepsilon_3\int_0^s \tfrac{k_3}{k_2}(\tfrac{1}{k_1})'\sin\theta ds\end{aligned} \tag{2.16}$$

where $\omega=\frac{\varepsilon_4}{k_3}(\varepsilon_1\varepsilon_2(\frac{k_2}{k_1})+(\frac{1}{k_2}(\frac{1}{k_1})')')$. Differentiating functions $\theta(s)$, $g(s)$ and $h(s)$ with respect to s, we find $\theta'(s)=k_4$, $g'(s)=h'(s)=0$. Hence $g(s)=A$, $h(s)=B$, $A,B\in R$. Substituting this into (2.16) yields

$$\begin{aligned}\omega\sin\theta+f(s)\cos\theta-\varepsilon_2\varepsilon_3\int_0^s \tfrac{k_3}{k_2}(\tfrac{1}{k_1})'\sin\theta ds&=A,\\ -\omega\cos\theta+f(s)\sin\theta-\varepsilon_2\varepsilon_3\int_0^s \tfrac{k_3}{k_2}(\tfrac{1}{k_1})'\sin\theta ds&=B.\end{aligned} \tag{2.17}$$

By multiplying the first of the equations in (2.16) with $\sin\theta$ and the second with $-\cos\theta$ and adding, we find that the equation in (2.12) holds. Next, by multiplying the first of the equations in (2.16) with $\cos\theta$ and the second with $\sin\theta$ and adding, we get

$$f(s)=[A+\varepsilon_2\varepsilon_3\int_0^s \tfrac{k_3}{k_2}(\tfrac{1}{k_1})'\sin\theta ds]\cos\theta+[B-\varepsilon_2\varepsilon_3\int_0^s \tfrac{k_3}{k_2}(\tfrac{1}{k_1})'\cos\theta ds]\sin\theta. \tag{2.18}$$

Then the relations (2.10) and (2.18) imply that inequality in (2.13) holds.

Conversely, let us suppose that there exist constants $A,B\in R$ such that (2.12) and (2.13) hold for each $s\in I\subset R$. Differentiating the equation in (2.12) with respect to s yields

$$\begin{aligned}&\tfrac{\varepsilon_4}{k_3}(\varepsilon_1\varepsilon_2(\tfrac{k_2}{k_1})+(\tfrac{1}{k_2}(\tfrac{1}{k_1})')')\\ &=[A+\varepsilon_2\varepsilon_3\int_0^s \tfrac{k_3}{k_2}(\tfrac{1}{k_1})'\sin(\int_0^s k_4 ds)ds]\cos(\int_0^s k_4 ds)ds\\ &+[B-\varepsilon_2\varepsilon_3\int_0^s \tfrac{k_3}{k_2}(\tfrac{1}{k_1})'\cos(\int_0^s k_4 ds)ds]\sin(\int_0^s k_4 ds)ds\end{aligned} \tag{2.19}$$

Let us define the differentiable function $f(s)$ by (2.13). Then by (2.13) and (2.19) it follows that (2.18) holds. Differentiating (2.18) with respect to s and using (2.12) we obtain $f'=\frac{k_4}{k_3}(\varepsilon_1\varepsilon_2\frac{k_2}{k_1}+(\frac{1}{k_2}(\frac{1}{k_1})')')$. Moreover, by using (2.18), inequality in (2.13) and by taking the derivative of (2.18) with respect to s, we find $\varepsilon_3(\frac{f'}{k_4})^2+\varepsilon_4 f^2<-\varepsilon_1(\frac{1}{k_1})^2-\varepsilon_2(\frac{1}{k_2}(\frac{1}{k_1})')^2$. Finally, Theorem 3 implies that α lies on $H^4_{\nu-1}(r)$.

The second case can be obtained similarly. □

Theorem 5. *There are no timelike and no null curves $\alpha(s)$ lying on $H^4_{\nu-1}(r)$ in E^5_ν.*

Proof. If $\alpha(s)$ is timelike unit speed vector lying on $H^4_{\nu-1}(r)$ with center m, then$g(\alpha - m, \alpha - m) = -r^2$. Differentiating the previous equation respect to s, we get $g(T, \alpha - m) = 0$. It follows that T and $\alpha - m$ are two timelike mutually ortogonal vectors in E^5_ν, which is contradiction. If $\alpha(s)$ is null curve lying on $H^4_{\nu-1}(r)$, then in a similar way it follows that null vector T and timelike vector $\alpha - m$ are orthogonal vectors in E^5_ν, which is a contradiction. □

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