ON B_5-GROUPS

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ABSTRACT. A group G is said to be a B_k-group if for any k-subset $\{a_1, \cdots, a_k\}$ of G, $|\{a_i a_j | 1 \leq i, j \leq k\}| \leq \frac{k(k+1)}{2}$. In this paper a complete classification of B_5-groups is given.

Key Words: B_k-Groups: Quaternion Group; Dihedral Groups.

1. Introduction and Main Result

A group G is called a B_k-group if it satisfies the condition that for any k-subset $S = \{a_1, \cdots, a_k\}$ of G, $|S^2| \leq \frac{k(k+1)}{2}$, where $S^2 = \{a_i a_j | 1 \leq i, j \leq k\}$ (see [1, 7]).

Clearly nonabelian groups of order $\leq \frac{k(k+1)}{2}$ and all abelian groups are B_k-groups. A B_k-group of order $\leq \frac{k(k+1)}{2}$ is referred to as trivial and an interesting problem is to determine all nontrivial nonabelian B_k-groups.

Freiman [5] showed that a nonabelian group is a B_2-group if and only if it is a Hamiltonian 2-group. Parmenter [7] gave a complete characterization of B_k-groups in the cases $k = 3$ or $k = 4$, by proving that all nonabelian B_k-groups are trivial. However, this trend does not continue as Parmenter [7] provided an example of a nontrivial nonabelian B_5-group, and in fact, he showed that $Q_8 \times C_2$ is such a group.

In this paper, we will show that the above mentioned group is the only nontrivial nonabelian B_5-group, and therefore, give a complete classification of B_5-groups. Our main theorem is as follows:

2000 *Mathematics Subject Classification.* Primary 20D60; Secondary 20A05, 20E34.
This research was supported in part by a Discovery Grant from the Natural Sciences and Engineering Research Council of Canada.
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Main Theorem
A group G is a B_5-group if and only if one of the following holds:
(1) G is abelian.
(2) G is a trivial nonabelian B_5-group.
(3) $G = Q_8 \times C_2$.

Proof. To prove the sufficiency, we need only consider that $G = Q_8 \times C_2$. It follows from Lemma 2.1 that G is a B_5-group.

To prove the necessity, we may assume that G is a nontrivial nonabelian B_5-group. It follows from Theorem 2.8 and Theorem 3.7 that $G = Q_8 \times C_2$ and we are done. □

We note that it was proved in [3] that when $k > 2$, any nonabelian B_k-group must be finite of order $\leq 2(k^3 - k)$. Throughout the paper, all nonabelian groups are assumed to be finite and our notation for groups is standard. In particular, we denote by Q_8 and D_{2n}, the quaternion group of order 8 and the dihedral group of order $2n$ respectively with the following representations:
$Q_8=\langle a, b \mid a^4 = 1, a^2 = b^2, a^b = a^{-1}\rangle$ and $D_{2n}=\langle a, b \mid a^n = b^2 = 1, a^b = a^{-1}\rangle$.

2. B_5 2-GROUPS

This section deals with nonabelian B_5 2-groups. We will show that $Q_8 \times C_2$ is the only nontrivial nonabelian B_5 2-group. We first give a few useful results which will be used to prove the main result.

Lemma 2.1. *$Q_8 \times C_2$ is the only nonabelian B_5 2-group of order 16.*

Proof. It was proved in [7] that $Q_8 \times C_2$ is a B_5 2-group. We note that in addition to $Q_8 \times C_2$, there are 8 nonabelian 2-groups of order 16. It was proved in [4] that for each such group G, there exists a subset S of 5 elements of G such that $|S^2| = 16$, and thus G is not a B_5-group. For example, if $G = D_{16}$, then let $S = \{a, b, ab, a^3, a^3b\}$, and then a straightforward computation shows that $|S^2| = 16$. Thus D_{16} is not a B_5-group. □

Corollary 2.2. *If G is a 2-group with a proper subgroup $H \cong D_8$, then G is not a B_5-group.*

Proof. It is easy to see that G has a subgroup K of order 16 containing H and $K \ncong Q_8 \times C_2$. By Lemma 2.1, K is not a B_5-group, neither is G. □

Remark 2.3. *We note that the observation given in the end of* [7]*, which claimed that 14 is the best bound of $|A^2|$ for any 5-element set A in $Q_8 \times$*

C_2, is incorrect. In fact, let $G = Q_8 \times C_2 = \langle a,b\rangle \times \langle c\rangle$ and $A = \{a, a^3c, b, bc, ab\}$. Then $A^2 = G - \{1\}$, and thus $|A^2| = 15$. This together with Lemma 2.1 proves that the bound of 15 is best possible.

Although the Hamiltonian 2-group $Q_8 \times C_2$ of order 16 is a B_5-group, the Hamiltonian 2-group of order 32 is no longer a B_5-group. The following two lemmas were proved in [4]. We include outlines of the proofs here for the convenience of a reader.

Lemma 2.4. *$Q_8 \times C_2 \times C_2$ is not a B_5-group.*

Proof. Let $G = Q_8 \times C_2 \times C_2 = \langle a, b, c, d \mid a^4 = c^2 = d^2 = 1, a^2 = b^2, a^b = a^{-1}, [a, c] = [b, c] = [a, d] = [b, d] = [c, d] = 1\rangle$, where $[x, y]$ denotes the commutator of $x, y \in G$. Let $S = \{a, bc, abd, a^3d, a^2bd\}$. A straightforward computation shows that $|S^2| \geq 16$ and thus G is not a B_5-group. □

Lemma 2.5. *For any integer $n \geq 3$, $Q_8 \times C_n$ is not a B_5-group. In particular, $Q_8 \times C_4$ is not a B_5-group.*

Proof. Let $G = Q_8 \times C_n = \langle a, b, c \mid a^4 = c^n = 1, a^2 = b^2, a^b = a^{-1}, [a, c] = [b, c] = 1\rangle$. Let $S = \{a, bc, abc, a^3b, a^3c\}$. A straightforward computation shows that $|S^2| \geq 16$, and thus G is not a B_5-group. □

Proposition 2.6. *Let G be a nonabelian B_5 2-group such that every proper subgroup of G is abelian. Then G is either D_8 or Q_8.*

Proof. Let G be a 2-group as assumed in the proposition. Clearly, $|G| \geq 8$. We first assume that $|G| \geq 16$. Since G is a minimal nonabelian 2-group, it follows from [6, P. 309] that
$G = G_1 = \langle a, b \mid a^{2^m} = b^{2^n} = 1,\ b^{-1}ab = a^{1+2^{m-1}}\rangle$ with $m \geq 2$ and $|G| = 2^{m+n}$, or $G = G_2 = \langle a, b \mid a^{2^m} = b^{2^n} = 1,\ [a, b]^2 = 1\rangle$ with $m \geq 2$ and $|G| = 2^{m+n+1}$.

Suppose that $G = G_1 = \{b^i a^j \mid 0 \leq i \leq 2^n - 1,\ 0 \leq j \leq 2^m - 1\}$. It is easy to see that $Z(G) = \langle a^2, b^2\rangle$. We divide the proof into 3 cases according to $m > 3$, $m = 3$ or $m = 2$.

Case 1. $m > 3$. Let $S = \{a,\ b,\ ba,\ ba^2,\ a^5\}$. Note that S^2 contains a subset $B = \{b^2,\ ba,\ b^2a,\ a^2,\ ba^2,\ ba^3,\ b^2a^3,\ b^2a^4,\ ba^5,\ a^6,\ ba^{1+2^{m-1}},\ b^2a^{1+2^{m-1}},\ ba^{2+2^{m-1}},\ b^2a^{2+2^{m-1}},\ ba^{3+2^{m-1}},\ b^2a^{3+2^{m-1}}\}$ of 16 distinct elements since $1 + 2^{m-1} \geq 9$. So $|S^2| \geq 16$ and thus G is not a B_5-group, giving a contradiction.

Case 2. $m = 3$. If $n = 1$, we know that $|G| = 16$. It is easy to see that $G \not\cong Q_8 \times C_2$ since G has an element of order 8. It follows from Lemma 2.1 that G is not a B_5-group. Next assume that $n \geq 2$, that is, $o(b) \geq 4$. Let $S = \{a,\ b,\ ba,\ ba^2,\ 1\}$. Note that S^2 contains a subset $C = \{1,\ b,\ b^2,\ ba,\ b^2a,\ a^2,\ ba^2,\ ba^3,\ b^2a^3,\ b^2a^4,\ ba^5,\ b^2a^5,\ ba^6,\ b^2a^6,\ ba^7,\ b^2a^7\}$

of 16 distinct elements. So $|S^2| \geq 16$ and thus G is not a B_5-group, giving a contradiction.

Case 3. $m = 2$. Recall that $|G| \geq 16$. So $n \geq 2$. Assume that $n \geq 3$, that is, $o(b) \geq 8$. Let $S = \{a,\ b,\ ab^2,\ ab^3, 1\}$. Note that S^2 contains a subset $D = \{a,\ a^2,\ ba^3,\ ba,\ b^2,\ b^2a^2,\ b^3a^2,\ b^3,\ b^3a^3,\ b^3a,\ b^4a,\ b^4a^2,\ b^4a^3,\ b^5a^2,\ b^5,\ b^6\}$ of 16 distinct elements. So $|S^2| \geq 16$ and thus G is not a B_5-group, giving a contradiction. Thus, $n = 2$ and then $|G| = 16$. Note that $G = \langle a, b \mid a^4{=}b^4{=}1,\ b^{-1}ab{=}a^3\rangle$, which is not isomorphic to $Q_8 \times C_2$. It follows from Lemma 2.1 that G is not a B_5-group, giving a contradiction again.

Next consider that $G = G_2$. Let $c = [a, b]$. Since all the proper subgroups of G are abelian and $c^2 = 1$, we obtain $c \in Z(G)$. Therefore, each element of G can be written uniquely as $a^ib^jc^k$, where $0 \leq i \leq 2^m - 1$, $0 \leq j \leq 2^n - 1$ and $0 \leq k \leq 1$. Let $S = \{a,\ b,\ ab,\ a^3b,\ a^2\}$. Note that S^2 contains a subset $E = \{ab,\ ab^2,\ abc,\ ab^2c,\ a^2,\ a^2b,\ a^2bc,\ a^2b^2c,\ a^3b,\ a^3b^2,\ a^3b^2c,\ a^3,\ a^4b,\ a^4bc,\ a^4b^2c,\ b^2\}$ of 16 distinct elements. So $|S^2| \geq 16$ and thus G is not a B_5-group, giving a contradiction.

Thus, $|G| = 8$. Since G is nonabelian, it is either D_8 or Q_8 as desired. □

Proposition 2.7. *There is no nonabelian B_5 2-group of order ≥ 32.*

Proof. Assume that G is a nonabelian B_5 2-group of order ≥ 32. We will show that no such a group exists. We first assume that $|G| = 32$. By Proposition 2.6, we know that G has a nonabelian maximal subgroup H of order 16. It follows from Lemma 2.1 that $H \cong Q_8 \times C_2$. Assume that $H = \langle a, b, c \mid a^4 = c^2 = 1, a^2 = b^2, a^b = a^{-1}, [a, c] = 1, [b, c] = 1\rangle$. Since H is maximal of index 2 in G, $H \lhd G$. Let $x \in G - H$. Then $G = \langle H, x\rangle$.

Next, we will show that $a^2 \in Z(G)$. Since $H \lhd G$, $a^x \in H$. Let $a^x = a^ub^vc^w$. $(a^2)^x = (a^x)^2 = (a^ub^v)^2c^{2w} = a^2$, so a^2 commutes with x. Since $a^2 \in Z(H)$, $a^2 \in Z(G)$.

We now show that $G/\langle a^2\rangle$ is abelian. Since $H/\langle a^2\rangle \cong C_2 \times C_2 \times C_2$ is a subgroup of $G/\langle a^2\rangle$, there are at least 7 elements of order 2 in $G/\langle a^2\rangle$. But there are only 3 elements of order 2 in $Q_8 \times C_2$. So $G/\langle a^2\rangle$ is not isomorphic to $Q_8 \times C_2$. It follows from Lemma 2.1 that $G/\langle a^2\rangle$ is an abelian group.

Let $\overline{x}, \overline{y}$ be the images of x, y in $G/\langle a^2\rangle$ for $y \in G$ and $x \in G - H$. Then we have

$$(2.1) \qquad [\overline{y}, \overline{x}] = 1 \Rightarrow [y, x] = 1 \text{ or } [y, x] = a^2.$$

If $y = a$, we get either $x^{-1}ax = a$ or $x^{-1}ax = a^{-1}$. In both cases we have $x^2a = ax^2$. Recall that $a^2 = b^2$. Similarly, we get $x^2b = bx^2$. Since $|G/H| = 2$, $x^2 \in H$, and thus x^2 commutes with c. Therefore, $x^2 \in Z(H)$, and thus $x^4 = 1$. This says that $o(x) = 2$ or $o(x) = 4$. We next divide the proof into the following two cases.

Case 1. $o(x) = 2$. We first show that $x \in Z(G)$. By (2.1), $[a, x] = 1$ or $[a, x] = a^2$. If $[a, x] = a^2$, then $\langle a, x\rangle \cong D_8$, contradicting Corollary 2.2. Therefore, $[a, x] = 1$. Similarly, by using the the symmetry of a and b, we have $[b, x] = 1$. If $[c, x] = a^2$, then $(cx)^2 = [c, x] = a^2$. So $o(cx) = 4$ and thus $\langle c, x\rangle = \langle cx, x\rangle \cong D_8$, again contradicting Corollary 2.2. Thus $x \in Z(G)$. Therefore, $G = H \times \langle x\rangle$ is a Hamiltonian 2-group of order 32. It follows from Lemma 2.4 that G is not a B_5-group, giving a contradiction.

Case 2. $o(x) = 4$. Let $K = \langle a, b\rangle \cong Q_8$ be a subgroup of H. We first show that x does not centralize K. If x centralizes K, then $K\langle x\rangle = \langle K, x\rangle$ is a subgroup of G. It is easy to see that $|K\langle x\rangle| \geq 16$. If $|K\langle x\rangle| = 32$, we know that $K \bigcap \langle x\rangle = 1$, and then $G = K\langle x\rangle = K \times \langle x\rangle \cong Q_8 \times C_4$. It follows from Lemma 2.5 that G is not a B_5-group. Thus $|K\langle x\rangle| = 16$. Since x is a central element of order 4 in $K < x >$ and $Q_8 \times C_2$ has no central elements of order 4, we have $K\langle x\rangle \ncong Q_8 \times C_2$. Therefore, $K\langle x\rangle$ is not a B_5-group, and so, G also is not a B_5-group, giving a contradiction. So, we have x does not centralize K.

Without loss of generality, we may assume that $[a, x] \neq 1$. Thus $[a, x] = a^2$ by (2.1), so $a^x = a^{-1}$. By replacing b by ab if necessary, we may assume that $[b, x] = 1$. If $a^2 \neq x^2$, then $\langle a, x\rangle$ is a nonabelian subgroup of G of order 16. Since $\langle a, x\rangle$is not isomorphic to $Q_8 \times C_2$, it follows from Lemma 2.1 that $\langle a, x\rangle$ is not a B_5-group, neither is G, giving a contradiction. Therefore, $a^2 = x^2$, and thus $(bx)^2 = b^2x^2 = 1$. We now show that $\langle a, b, x\rangle = \langle K, x\rangle \cong Q_8 \times C_2$. Since $a^{bx} = (a^{-1})^x = a$ and $b^{bx} = b^x = b$, we know that bx centralizes K. Thus $\langle a, b, x\rangle = \langle a, b, bx\rangle = K \times \langle bx\rangle \cong Q_8 \times C_2$.

Next, we show that c commutes with x. Assume to the contrary that c does not commute with x. By (2.1), $[c, x] = a^2$. Note that $a^2 = x^2$. Thus $x^c = xa^2 = x^3 = x^{-1}$. This implies that $\langle c, x\rangle \cong D_8$, giving a contradiction to Corollary 2.2.

So, the only possible structure for G is $G = K \times \langle c\rangle \cdot \langle x\rangle \cong Q_8 \times C_2 \times C_2$. It follows from Lemma 2.4 that G is not a B_5-group, again a contradiction.

Next, we consider the case that $|G| = 2^n > 32$. If there exists a non-abelian subgroup H of order 2^{n-1} of G, then by what we just proved, we know that $|H| > 32$. By keeping replacing G by H if necessary, we may assume that all subgroups of order 2^{n-1} of G are abelian. It now follows from Proposition 2.6 that G is either Q_8 or D_8, giving a contradiction to the order of G.

In all cases, we found contradictions. This completes the proof. □

As a consequence of Lemma 2.1 and Proposition 2.7, we obtain the main result in this section.

Theorem 2.8. *G is a B_5 2-group if and only if one of the following holds:*

(1) *G is abelian.*
(2) *G is a trivial nonabelian B_5-group, that is, G is either D_8 or Q_8.*
(3) *$G \cong Q_8 \times C_2$.*

3. B_5-GROUPS OTHER THAN 2-GROUPS

In this section, we consider B_5-groups that are not 2-groups. We will show that there are no nontrivial nonabelian B_5-groups other than 2-groups. In what follows, we always assume that G is nonabelian and is not a 2-group. We first show that a Sylow 2-subgroup P of a B_5-group G must be abelian.

Lemma 3.1. *Let G be a B_5-group with a nontrivial Sylow 2-subgroup P. Then P is abelian.*

Proof. We prove by contradiction. Suppose that P is nonabelian. It follows from Theorem 2.8 that $P \cong D_8$, Q_8 or $Q_8 \times C_2$. We divide the proof into three cases.

Case 1. $P \cong D_8$.

Let $P=\langle a, b \mid a^4 = b^2 = 1,\ a^b = a^{-1}\rangle$ and $S=\{a,\ b,\ ab,\ c,\ c^2\}$, where c is any element of odd order in G. Note that S^2 contains a subset $A=\{1,\ a,\ a^2,\ a^3,\ b,\ ab, a^2b,\ a^3b,\ ac,\ bc,\ abc,\ ac^2,\ bc^2,\ abc^2,\ c^2,\ c^4\}$ of 16 distinct elements. So $|S^2| \geq 16$ and thus G is not a B_5-group, giving a contradiction.

Case 2. $P \cong Q_8$.

Let $P=\langle a, b \mid a^4 = 1,\ a^2 = b^2,\ a^b = a^{-1}\rangle$ and $S=\{a,\ b,\ ab,\ c,\ c^2\}$, where c is any element of odd order in G. Note that S^2 contains a subset $B=\{a,\ a^2,\ a^3,\ ab,\ a^2b,\ a^3b,\ b,\ ac, bc, abc, ac^2, bc^2,\ abc^2,\ c^2, c^3, c^4\}$ of 16 distinct elements. So $|S^2| \geq 16$ and thus G is not a B_5-group, giving a contradiction.

Case 3. $P \cong Q_8 \times C_2$. By using the same set S as used in the case $P \cong Q_8$, we can easily get a contradiction. □

The following lemmas are similar to those in [2]. However, we will be dealing with 5-sets while [2] dealt with 3-sets. Our proofs are much more difficult since there are more hurdles to overcome.

Lemma 3.2. *Each Sylow subgroup P of a B_5-group G of odd order is abelian.*

Proof. Assume to the contrary that P is nonabelian for some Sylow p-subgroup P of G. Then P has two maximal subgroups M and N which contain $Z(P)$. Let $L = M\bigcap N$. It was proved in [2] that there exist $a \in M - L$, and $b \in N - L$ such that $ab \neq ba$. Let $S = \{a,\ ab,\ b^2,\ ab^2,\ 1\}$. Then S^2 contains a subset $A = \{b^4, ab^2,\ ab^3, b^2a, b^2ab, b^2ab^2, ab^4, a^2, a^2b,$ $a^2b^2, aba, abab,\ abab^2,\ ab^2a,\ ab^2ab,\ ab^2ab^2\}$. Note that A is a disjoint union of three subsets; namely, $A = (A\bigcap N)\bigcup(A\bigcap aN)\bigcup(A\bigcap a^2N)$, where $A\bigcap N = \{b^4\}$, $A\bigcap aN = \{ab^2,\ ab^3, b^2a,\ b^2ab,\ b^2ab^2, ab^4\}$ and $A\bigcap a^2N$ $= \{a^2,\ a^2b,\ a^2b^2,\ aba,\ abab,\ abab^2,\ ab^2a,\ ab^2ab,\ ab^2ab^2\}$. We next show that the elements in each of the above subsets are distinct. Since cosets b^iM, $2 \leq i \leq 4$, are disjoint and $A\bigcap aN$ can be written as a disjoint union of 3 subsets, it is not hard to show that the elements of $A\bigcap aN$ are all distinct. Similarly, since subsets b^2M, $M\bigcup b^3M$ and $bM\bigcup b^4M$ are disjoint, it is not hard to show that the elements of $A\bigcap a^2N$ are all distinct. Hence $|A| = 1+6+9 = 16$. So $|S^2| \geq 16$, and thus G is not a B_5-group, giving a contradiction. Therefore, P must be abelian. □

Lemma 3.3. *A finite B_5-group of odd order is abelian.*

Proof. Suppose on the contrary that there exists a finite nonabelian B_5-group of odd order and let G be such a group with the least possible order. Since G is nonabelian, it follows from Lemma 3.2 that G is not nilpotent. Note that G is not a nilpotent group, but all proper subgroups of G are abelian. It follows from [8, 9.1.9] that $|G| = p^u q^v$ where p and q are distinct primes. Moreover, G has a normal Sylow q-subgroup Q and a non-normal cyclic Sylow p-subgroup P. Let $P = \langle a \rangle$. Since P is not a normal subgroup of G, there exists $b \in Q$ such that $a^b \notin \langle a \rangle$; in particular, $ab \neq ba$. We next divide the proof into the two cases according to $|P| > 3$ or $|P| = 3$.

Case 1. $|P| > 3$.

Let $S = \{b, a, ba^2, a^2, b^2\}$. Note that S^2 has a subset $A = \{ab,\ ba,\ a^2,\ ba^2,$ $b^2a^2,\ ba^2b,\ a^3,\ ba^3,\ aba^2,\ a^4,\ ba^4,\ a^2ba^2,\ ba^2ba^2,\ b^2,\ b^3,\ b^4\}$. Since $Q \lhd G$ and cosets a^iQ, $0 \leq i \leq 4$, are disjoint, it not difficult to prove that the 16 elements of A are all distinct. So $|S^2| \geq 16$ and thus G is not a B_5-group, giving a contradiction.

Case 2. $|P| = 3$.

We first assume that $ba = ab^2$. Since $b = a^{-3}ba^3 = b^8$, $o(b) = 7$. Let $S =$ $\{a,\ b^2,\ ab,\ a^2b^3,\ b^3\}$. Note that S^2 has a subset $B = \{1,\ b^3,\ b^4,\ b^5,\ b^6,\ ab,$ $ab^2,\ ab^3,\ ab^4,\ ab^5,\ a^2,\ a^2b,\ a^2b^2,\ a^2b^3,\ a^2b^4,\ a^2b^5\}$ of 16 distinct elements. So $|S^2| \geq 16$ and thus G is not a B_5-group, giving a contradiction. Thus $ba \neq ab^2$. By replacing a with a^2 in the above argument, we can show that $ba^2 \neq a^2b^2$.

Let $S = \{a,\ b,\ ab,\ ab^2,\ b^2\}$. Note that S^2 contains a subset $B = \{b^2,\ b^3,\ b^4,\ ab,\ ba,\ bab,\ bab^2,\ ab^2,\ a^2,\ a^2b,\ a^2b^2,\ aba,\ abab^2,\ abab,\ ab^2a,\ ab^2ab\}$. As in Case 1, since $q \neq 3$ and cosets a^iQ, $0 \leq i \leq 2$, are disjoint, it is not difficult to prove that the 16 elements of B are all distinct. Therefore, $|S^2| \geq 16$, and thus G is not a B_5-group, giving a contradiction.

This proves that a B_5-group G of odd order is abelian as desired. □

Lemma 3.4. *If G is a nontrivial B_5-group having a nontrivial Sylow 2-subgroup P, then G has a normal 2-complement T such that $G = TP$.*

Proof. Assume to the contrary that G is a nontrivial nonabelian B_5-group such that G does not have a normal subgroup of odd order with 2-power index. Let H be a subgroup of G with minimal order such that it does not have a normal subgroup of odd order with 2-power index. Then every proper subgroup of H has a normal subgroup of odd order with 2-power index. By [6, IV. 5.4], we know that H has a normal Sylow 2-subgroup P_1 with exponent at most 4. Moreover, $|H/P_1| = q^v$ for some odd prime q and a Sylow q-subgroup T of H is cyclic, say $T = \langle a \rangle$. Since T is not normal in H, it is easy to see that there exists an element $b \in P_1$ such that $a^b \notin \langle a \rangle$, especially, $ab \neq ba$.

We first assume that $|H| \leq 15$. By checking all the groups of order ≤ 15 which satisfy the above mentioned properties, we know that $H \cong A_4$. We may choose a and b to be the elements of H corresponding to the elements (123) and (12)(34) of A_4 respectively. Since $|G| \geq 16$, there exists an element $c \in G - H$. Let $S = \{a, b, ab, c, ac\}$. Since $ab \neq ba$, by replacing c with ac if necessary we can assume $bc \neq cb$. Note that $S^2 = \{a^2, ab, a^2b, ac, a^2c, ba, 1, bab, bc, bac, aba, a, abab, abc, abac, ca, cb, cab, c^2, cac, aca, acb, acab, ac^2, acac\}$. $\{1, ab, ba, bab, a, a^2, a^2b, aba, abab\} \subseteq S^2 \bigcap H$ and $\{ac, bc, abc, cb, a^2c, bac, abac\} \subseteq S^2 \bigcap (G - H)$. A straightforward computation shows that the above elements in $S^2 \bigcap H$ and $S^2 \bigcap (G - H)$ are distinct. Therefore, $|S^2| \geq 16$, and thus G is not a B_5-group, giving a contradiction.

Next, assume that $|H| \geq 16$. Without loss of generality, we may assume that $H = G$. Let b be an element of maximal order in P such that $ab \neq ba$. Since P is abelian, we know that the order of b is equal to the exponent of P. We divide the proof into 2 cases.

Case 1: $o(a) > 3$.

Let $S = \{a, b, ab, a^{-1}b, a^2\}$. Note that S^2 contains a subset $A = \{b,\ b^2,\ a^{-1}ba,\ a^{-1}bab,\ ab,\ ba,\ bab,\ ab^2,\ a^2,\ a^2b,\ aba,\ a^{-1}ba^{-1}b,\ a^3,\ a^3b,\ ba^{-1}b,\ a^{-1}b^2\}$. Recall that $P \lhd G$. Since subsets P, aP, a^2P, $a^3P \bigcup a^{-2}P$ and $a^{-1}P$ are disjoint, it is not hard to show that the 16 elements of A are distinct. Therefore, we have $|S^2| \geq 16$, and thus G is not a B_5-group, giving a contradiction.

Case 2: $o(a) = 3$.

Suppose first that $o(b) = 4$. Let $S = \{a,\ b,\ ab,\ ab^{-1},\ a^2\}$. Note that S^2 contains a subset $B = \{1,\ b,\ b^2,\ a,\ ab,\ ba,\ bab,\ ab^2,\ bab^{-1},\ a^2,\ a^2b,\ aba,\ abab,\ a^2b^{-1},\ abab^{-1},\ ab^{-1}a\}$. Since cosets a^iP, $0 \leq i \leq 2$, are disjoint, it is not hard to show that the 16 elements of B are distinct. Therefore, $|S^2| \geq 16$ and thus G is not a B_5-group, giving a contradiction.

Thus $o(b) = 2$, and then P is elementary abelian. Recall that $|G| \geq 16$ and $|T| = 3$. We have $|P| \geq 8$, and then we can choose an element $c \in P$ such that $c \notin \langle b^a, b\rangle = K$. Let $S = \{a,\ b,\ ab,\ ac,\ bca^2\}$. Note that S^2 contains a subset $C = \{1,\ b,\ c,\ bc,\ a,\ ab,\ ba,\ bab,\ bac,\ abc,\ a^2,\ a^2b,\ a^2c,\ aba,\ abab,\ abac\}$. As before, it is not hard to show that the 16 elements of C are distinct. Therefore, $|S^2| \geq 16$ and thus G is not a B_5-group, giving a contradiction.

In all cases, we have found contradictions. Thus G must have a normal 2-complement. □

In what follows, we assume that G is a nontrivial B_5-group having a Sylow 2-subgroup P and the normal 2-complement T.

Lemma 3.5. *T is abelian and is not centralized by P.*

Proof. It follows from Lemma 3.3 that T is abelian. Suppose that P centralizes T. Then $G = P \times T$. It follows from Lemma 3.1 that P is abelian, so G is abelian, giving a contradiction. □

Lemma 3.6. *The subgroup P has a subgroup Q of index 2 which centralizes T and every element of $P - Q$ inverts T. Moreover, the exponent of Q is at most 2.*

Proof. We first show that for each $b \in P$ either b centralizes T or b inverts T. Assume that $b \in P$ does not centralize T. Then there exists $a \in T$ such that $ba \neq ab$. We claim that $ab^2=b^2a$. Suppose on the contrary that $ab^2 \neq b^2a$. Then $o(b) \geq 4$. Let $S=\{a,\ ab,\ ab^2,\ ab^3,\ 1\}$. Note that S^2 contains a subset $A = \{a,\ abab^3,\ ab^2ab^2,\ ab^3ab,\ a^2b,\ aba,\ ab,\ a^2b^2,\ abab,\ ab^2a,\ ab^2,\ a^2b^3,\ abab^2,\ ab^2ab,\ ab^3a,\ ab^3\}$. Since subsets $T \bigcup b^4T$, bT, b^2T and b^3T are distinct, it is not hard to show that the 16 elements of A are all distinct. Therefore, $|S^2| \geq 16$, and then G is not a B_5-group, giving a contradiction. So $ab^2 = b^2a$.

We now show that $b^{-1}ab=a^{-1}$. Suppose on the contrary that $b^{-1}ab \neq a^{-1}$. We first assume that $o(b) \geq 4$. Let $S = \{a,\ ab,\ a^2b,\ ab^2,\ 1\}$. Note that S^2 contains a subset $B = \{a,\ a^2,\ ab,\ a^2b,\ a^3b,\ aba,\ a^2ba,\ a^2b^2,\ abab,\ aba^2b,\ a^2bab,\ a^2ba^2b,\ ab^2ab,\ abab^2,\ a^2bab^2,\ a^3b^3\}$. As above, it is not difficult to show that the 16 elements of B are distinct. Therefore, $|S^2| \geq 16$, and thus G is not a B_5-group, giving a contradiction.

Thus, $o(b) = 2$. Let $A = \{a, ab, a^2b, b\}$. Then $A^2 = \{1,\ a,\ a^2,\ bab,\ abab,\ a^2bab,\ ba^2b,\ aba^2b,\ a^2ba^2b,\ ab,\ a^2b,\ a^3b,\ aba,\ a^2ba,\ ba\}$. Since T and bT are two disjoint cosets, it is not hard to see that the 15 elements of A^2 are distinct. Since $|G| \geq 16$, there exists an element $c \in G - A^2$. Let $S = \{a, ab, a^2b, b, bc\}$. Since $c = b \cdot bc \in S^2 - A^2$, $|S^2| \geq 16$, and thus G is not a B_5-group, again giving a contradiction.

Thus we must have $b^{-1}ab = a^{-1}$. Note that we just showed that for each $y \in T$ either $y^b = y$ or $y^b = y^{-1}$.

Next we show that b inverts T. Suppose that there exists $x \in T - \{1\}$ such that $x^b = x$. Since $xa \in T$, we have either $(xa)^b = xa$ or $(xa)^b = (xa)^{-1}$. The former gives that $xa^{-1} = (xa)^b = xa$, so $a^2 = 1$, a contradiction. The latter gives that $xa^{-1} = (xa)^b = (xa)^{-1} = a^{-1}x^{-1} = x^{-1}a^{-1}$, so $x^2 = 1$, again giving a contradiction. Therefore, b inverts T.

We now show that P has a subgroup Q of index 2 which centralizes T and every element of $P - Q$ inverts T. Let $Q = \{g \in P | t^g = t \text{ for all } t \in T\} = C_P(T)$. Clearly Q is a subgroup of P which centralizes T and every element b of $P - Q$ does not centralize T. So by what we just proved, b inverts T. It remains to show that $[P : Q] = 2$. It follows from Lemma 3.5 that $P \neq Q$, so there exists $b \in P - Q$. Since for every element $b' \in P - Q$ b' inverts T, we have $b'b \in Q$. Thus $b' \in Qb^{-1}$, proving $[P : Q] = 2$.

Finally, we show that the exponent of Q is at most 2. Assume to the contrary that Q contains an element g of order 4. Let $b \in P - Q$. If $o(b) \geq 4$, let $S = \{b,\ a,\ bag,\ ag,\ 1\}$. Note that S^2 contains a subset $C = \{b^2, b^2g^2, ab^2g, a^2, a^2g,\ a^2g^2, bg, bg^2, ab, abg, a^{-2}bg^2,\ a^{-1}b,\ a^{-1}bg\}$ of 13 distinct elements. If $b^2g \neq 1$, a straightforward computation shows that b, a and ag are three elements in $S^2 - C$. If $b^2g = 1$, then b, 1 and ag are three elements in $S^2 - C$. So in both cases we have $|S^2| \geq 16$ and thus G is not a B_5-group, giving a contradiction.

Thus, $o(b) = 2$. Let $S = \{b,\ a,\ a^{-1}g^2,\ a^{-1}bg^3, 1\}$. Note that S^2 contains a subset $D = \{1,\ g^2,\ ag^3,\ a^{-2},\ a^2,\ a^{-1}g^2,\ a^{-1}g^3,\ g,\ bg,\ bg^3,\ ab,\ a^{-2}bg,\ abg^2,\ a^{-2}bg^3,\ a^{-1}b,\ a^{-1}bg^2\}$ of 16 distinct elements. So $|S^2| \geq 16$ and thus G is not a B_5-group, again giving a contradiction. □

Theorem 3.7. *There are no nontrivial nonabelian B_5-groups other than 2-groups.*

Proof. Assume that G is a nonabelian nontrivial B_5-group. By Lemmas 3.4 and 3.5, $G = TP$ and T is abelian. We consider two cases according to $|P| = 2$ or $|P| \geq 4$.

Case 1. $|P| = |\langle x \rangle| = 2$. If the exponent of T is greater than 9, there exists an element $t \in T$ such that $o(t) \geq 9$. Let $S = \{t^2,\ t^3,\ t^5x,\ t^7x,\ t^5\}$. Note that S^2 contains a subset $A = \{t^{-2},\ t^2,\ t^4,\ t^5,\ t^6,\ t^8,\ t^{10},\ 1,\ t^2x,\ t^3x,\ t^4x,\ t^5x,\ t^7x,\ t^8x,\ t^9x,\ t^{10}x\}$ of 16 distinct elements. So $|S^2| \geq 16$ and thus G is not a B_5-group, giving a contradiction.

Thus the exponent of T is 3, 5 or 7. Since $|G| \geq 16$ and $|P| = 2$, $|T| \geq 9$. So T must have a subgroup $H = \langle a \rangle \times \langle b \rangle$, where $o(a) = o(b) = 3, 5$ or 7. Let $S = \{a,\ ax,\ abx,\ b^2x,\ 1\}$. Note that S^2 contains a subset $B = \{\ 1,\ b,\ b^{-1},\ a,\ ab^{-2},\ ab^{-1},\ a^2,\ a^{-1}b,\ a^{-1}b^2, x,\ bx,\ ax,\ ab^2x,\ a^2x,\ a^2bx,\ a^{-1}b^2x\}$ of 16 distinct elements. So $|S^2| \geq 16$ and thus G is not a B_5-group, giving a contradiction.

Case 2. $|P| \geq 4$. Let $x \in P - Q$. We first show that the exponent of T is 3. Assume that there exists $a \in T$ such that $o(a) \geq 5$. Recall from Lemma 3.6 that Q has index 2 in P and it centralizes T. Then there exists $y \in Q$ such that $o(ay) \geq 10$. Let $t = ay$ and $S = \{t, t^2, t^4, t^5, t^5x\}$. Note that S^2 contains a subset $C = \{t^2,\ t^3,\ t^4,\ t^5,\ t^6,\ t^7,\ t^8,\ t^9,\ t^{10}, tx,\ t^3x,\ t^4x,\ t^6x,\ t^7x,\ t^9x,\ t^{10}x\}$ of 16 distinct elements. So $|S^2| \geq 16$ and thus G is not a B_5-group, giving a contradiction. Therefore, the exponent of T is 3.

Next, we show that $T = C_3$. If T has a subgroup $H = \langle a \rangle \times \langle b \rangle$, let $S = \{1, a, b, ab, x\}$. Note that S^2 contains a subset $C = \{1,\ a,\ b,\ ab,\ a^2,\ a^2b,\ b^2, ab^2,\ a^2b^2,\ x,\ ax,\ bx,\ abx,\ a^2x,\ b^2x,\ a^2b^2x\}$ of 16 distinct elements. So $|S^2| \geq 16$ and thus G is not a B_5-group, giving a contradiction. Therefore, $T = C_3$.

Since $|G| \geq 16$, $|P| \geq 8$, so $|Q| \geq 4$. Thus there exist $y,\ z \in Q - \{1\}$ such that $x^2 \neq y$ and $x^2 \neq z\}$. Let $S = \{a,\ x,\ a^2y,\ azx,\ 1\}$. Note that S^2 contains a subset $D = \{y,\ x^2,\ a,\ azx^2,\ a^2,\ a^2zx^2, x, yzx,\ zx,\ ax,\ ayx, azx,\ a^2zx,\ a^2x,\ a^2yx,\ a^2yzx\}$ of 16 distinct elements. So $|S^2| \geq 16$ and thus G is not a B_5-group, giving a contradiction.

This completes the proof. □

ACKNOWLEDGMENTS.

The authors would like to thank the referee for many helpful suggestions.

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